Coupling of Hyperbolic PDEs : Thin versus Thick Coupling Interfaces.

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The present lecture gives an overview of some results obtained by the author with various collaborators on the mathematical coupling of nonlinear hyperbolic PDEs. Such an issue is motivated by the modeling of complex industrial setups under the form of a partition of components. The partition under consideration results from the fact that the phenomena taking place in a given component can be characterized by a typical range of physical time and space scales. To the resulting hierarchy of characteristic scales is associated a corresponding hierarchy of PDEs models that are naturally arranged from the coarsest to the nest ones according to the characteristic scales they are supposed to resolve. The large size of the complete device generally prevents from modeling each component using the nest model, the CPU e ort would be indeed too large. The simulation of the whole operating system therefore requires to solve a Cauchy problem for a collection of hyperbolic PDEs formulated on a given partition of the physical domain, separated by interfaces at which transient exchange conditions have to be prescribed. For simplicity, we will only address the case of given fixed interfaces, whose locations are decided *ab initio* by the user.

We will present a mathematical formalism for handling the coupling of hyperbolic systems based on an augmented PDEs formulation. It natural supports two distinct and complementary frameworks : namely a first one using singular or say infinitely thin coupling interfaces and a second one based on regularized interfaces or handshake coupling zones. The first framework relies on the use of infinitely thin interfaces that allows to model the coupling on the ground of coupled boundary conditions. Such conditions are formulated so as to promote some continuity properties for the unknown across the coupling interface, the choice of which is left to the user. Importantly, resonance phenomena may occur at the expense of loosing the expected continuity properties and generally yields multiple discontinuous solutions. This rises the natural question of designing suitable selection criteria.

The second framework intends to tackle this issue. Roughly speaking, it consists in modeling the existence of the coupling interfaces under the form of standing waves for an augmented PDEs model which is set over the entire physical domain. Each standing wave is designed so as to exhibit a complete set of Riemann invariants (away from resonance) in agreement with the expected continuity properties in the coupled solutions. The coupling problem thus takes the form of a standard Cauchy problem which can in turn support various regularization mechanisms. In a single space dimension, we adopt the viscous regularization *á la Dafermos*. Existence of self-similar weak solutions for the coupling of two hyperbolic systems can be obtained under fairly general conditions. However, in the limit of a vanishing viscosity, the lack of uniqueness of resonant solutions can be still observed. We will explain in which sense multiplicity in the resonant solutions sounds natural. This will naturally lead us to promote another regularization mechanism based on thickened coupling interfaces resulting from the use of smooth transition profiles in between two domains. The proposed framework naturally allows for the definition of multidimensional and multicomponent couplings with possible recovering. Within this frame, existence and uniqueness of the coupled solution is proved in the setting of scalar conservations laws thanks to a convenient well-balanced finite volume strategy in several space variables. Numerical illustrations will be given all along the course of the lecture.

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