Well-balanced schemes for linear models of Boltzmann equation: the legacy of Case, Cercignani and Siewert

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A difficult task suggested by G. Toscani in 2003 was to extend the WB and AP results of [6] to linearized Boltzmann-type kinetic equations,

$$\partial_t h + \xi \partial_x h = \nu \left(\sum_{i=0}^{4} \int_{\mathbb{R}^3} M(\mathbf{v}')\psi_i(\mathbf{v}')h(t,x,\mathbf{v}')d\mathbf{v}' - h \right) := -\nu(Id - \mathcal{P})h,$$

with ψ_i, $i = 0, ..., 4$ being the orthonormal basis functions of the vector space spanned by the 5 collisional invariants. The integral term reads like:

$$\mathcal{P}h(t,x,\mathbf{v}) = \int_{\mathbb{R}^3} M(\mathbf{v}') \left[1 + 2\mathbf{v}.\mathbf{v}' + \frac{2}{3}\left(|\mathbf{v}|^2 - \frac{3}{2}\right)\left(|\mathbf{v}'|^2 - \frac{3}{2}\right)\right] h(t,x,\mathbf{v}')d\mathbf{v}',$$

and $\mathbf{v} = (\xi, \mathbf{v}_2, \mathbf{v}_3)$. Classically, one goes to convert this non-homogeneous equation into a more singular one, for which the collision term is *localized* onto a discrete lattice corresponding to the computational grid's interfaces:

$$\partial_t h + \xi \partial_x h = -\nu \Delta x \sum_{j \in \mathbb{Z}} (Id - \mathcal{P})h \ \delta\left(x - (j - \frac{1}{2})\Delta x \right),$$

thus yielding stationary discontinuities resoved by means of appropriate Rankine-Hugoniot jump relations. The main stepping stone in carrying out such a program is to derive, ideally in an explicit manner, the solutions of the *forward-backward problem* for the steady state-equations. Obviously, there is an intrinsic complexity because they are integro-differential and $\mathbf{v} \in \mathbb{R}^3$: it is at this level that the breakthrough originally due to K. Case [2] and C. Cercignani [3] in the 60's (the so–called "elementary solutions"), later developed in [7], suggests a feasible method of solution. Within the deterministic framework of the "discrete ordinates", these theoretical results can be translated into a powerful numerical method where only the ξ variable needs to be discretized (as explained by

Siewert and co-workers [9, 1]). With the stationary solutions at hand, essentially computed by inverting a well-conditioned matrix of eigenfunctions, the WB scheme follows from similar calculations as in [6]: see [5]. The radiative transfer equation is an interesting special case which is treated in detail in [4]; extensions to *e.g.* linear chemotaxis kinetic models [8] are in preparation.

References

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