Recent developments in very high order Residual Distribution Schemes for inviscid and viscous problems.

R. Abgrall

Team Bacchus
INRIA Bordeaux Sud Ouest and Université de Bordeaux Talence, France

Roscoff, september 19th 2011

Thanks

Financing and Numerous discussions, coding, tests with

- M. Ricchiuto (former VKI, now INRIA)
- H. Deconinck (VKI)
- Z.J. Wang (Iowa state), C.W. Shu (Brown)& T. Barth (Nasa Ames Rc)
- former students : M. Mezine, A. Larat, J. Trefilick
- current students : G. Baurin, A. Krutz, A. Froehly, D. de Santis
- various contracts + EC : funding.

Outline

TYPICAL PROBLEM TO SOLVE

IN $\Omega \subset \mathbb{R}^2, \mathbb{R}^3$,

$$\frac{\partial W}{\partial t} + \text{div } F_e(W) = \frac{1}{Re}\text{div} F_v(W, \nabla W)$$

- with initial and boundary conditions,
- *Re* very large.

Typical problem to solve

In $\Omega \subset \mathbb{R}^2, \mathbb{R}^3$,

$$\frac{\partial W}{\partial t} + \text{div } F_e(W) = \frac{1}{Re}\text{div} F_v(W, \nabla W)$$

- with initial and boundary conditions,
- *Re* very large.

Steady version

$$\text{div } F_e(W) = \frac{1}{Re}\text{div} F_v(W, \nabla W)$$

with BCs.

Typical problem to solve

In $\Omega \subset \mathbb{R}^2, \mathbb{R}^3$,

$$\frac{\partial W}{\partial t} + \text{div } F_e(W) = \frac{1}{Re} \text{div} F_v(W, \nabla W)$$

- with initial and boundary conditions,
- *Re* very large.

Steady version

$$\text{div } F_e(W) = \frac{1}{Re} \text{div} F_v(W, \nabla W)$$

with BCs.

This talk:

1. Simplify to scalar
2. First: foccus on non viscous problems, then modifications for viscous ones
3. Second : go to steady to unsteady.

Overview

Variational formulation of convected dominated problems,1

Continuous finite elements

Streamline diffusion

- Choose $V^h = U^h = \bigoplus\{u^h_{|K} \in \mathbb{P}^k(K)$ and globally continuous$\}$
- $$\sum_K \int_K \Big(-\int_K \nabla v^h \cdot f(u^h) dx + h_K \int_K (\nabla f_u(u^h) \cdot \nabla v^h dl) \mathcal{T} (\nabla f_u(u^h) \cdot \nabla u^h) dx \Big) = 0$$

  with $\mathcal{T} \geq 0$.

2 interpretations

- Petrov Galerkin on the original PDE with same $U^h = \text{span}\{\varphi_i\}$ and test functions
  $$V^h = \text{span}\Big\{\varphi_i + h\,\mathcal{T} \times \nabla f_u(u^h) \cdot \nabla \varphi_i\Big\}.$$
- Or Galerkin method applied to the (formal) PDE
  $$\text{div } f(u) - h \,\text{div}\, \Big(\mathcal{T} \times \text{ div } f(u)\Big) = 0$$

Variational formulation of convected dominated problems ,2

Discontinuous finite elements

Discontinuous Galerkin methods

- Choose $V^h = U^h = \bigoplus \{u^h_{|K} \in \mathbb{P}^k(K)\}$.No continuity requirement
- Variational formulation :

$$\sum_K \int_K \Big(- \int_K \nabla v^h \cdot f(u^h) dx + \int_{\partial K} \hat{f}(u^h_+, u^h_-, \vec{n}) v^h dl \Big) = 0$$

- Choice of numerical flux $\hat{f}$: E-scheme implies entropy stability.

Variational formulation of convected dominated problems,2

Discontinuous finite elements

Discontinuous Galerkin methods

- Choose $V^h = U^h = \bigoplus \{ u^h_{|K} \in \mathbb{P}^k(K) \}$. No continuity requirement
- Variational formulation :

$$\sum_K \int_K \left(- \int_K \nabla v^h \cdot f(u^h) dx + \int_{\partial K} \hat{f}(u^h_+, u^h_-, \vec{n}) v^h dl \right.$$
$$\left. + h_K \int_K \left(\nabla f_u(u^h) \cdot \nabla v^h \right) \tau \left(\nabla f_u(u^h) \cdot \nabla u^h \right) dx \right) = 0$$

- Choice of numerical flux $\hat{f}$: E-scheme implies entropy stability.

Variational formulation of convected dominated problems,3

On $\Omega = \cup_{j=1,n_e} K_j \subset \mathbb{R}^d$, scalar problem :

$$\text{div } f(u) = 0 \qquad + \text{BCs.}$$

Multiply by test function $v^h \in V^h$, seek for $v^h \in U^h$, rearrange

$$\sum_K \int_K v^h \text{div } f(u^h) dx = \sum_K \left(- \int_K \nabla v^h \cdot f(u^h) dx + \int_{\partial K} v^h \hat{f}(u^h) dl \right) = 0$$

Choices of V^h and U^h :

A priori independant choices

Variational formulation of convected dominated problems,3

On $\Omega = \cup_{j=1,n_e} K_j \subset \mathbb{R}^d$, scalar problem :

$$\text{div } f(u) = 0 \qquad + \text{BCs.}$$

Multiply by test function $v^h \in V^h$, seek for $v^h \in U^h$, rearrange

$$\sum_K \int_K v^h \text{div } f(u^h) dx = \sum_K \Big(- \int_K \nabla v^h \cdot f(u^h) dx + \int_{\partial K} v^h \hat{f}(u^h) dl$$

$$+ h_K \int_K (\nabla f_u(u^h) \cdot \nabla v^h) \tau (\nabla f_u(u^h) \cdot \nabla u^h) dx \Big) = 0$$

Choices of V^h and U^h :

A priori independant choices

Variational formulation of convected dominated problems ,3

On $\Omega = \cup_{j=1,n_e} K_j \subset \mathbb{R}^d$, scalar problem :

$$\text{div } f(u) = 0 \qquad + \text{BCs.}$$

Multiply by test function $v^h \in V^h$, seek for $v^h \in U^h$, rearrange

$$\sum_K \int_K v^h \text{div } f(u^h) dx = \sum_K \Big(- \int_K \nabla v^h \cdot f(u^h) dx + \int_{\partial K} v^h \hat{f}(u^h) dl$$
$$+ h_K \int_K (\nabla f_u(u^h) \cdot \nabla v^h) \tau (\nabla f_u(u^h) \cdot \nabla u^h) dx \Big) = 0$$

Choices of V^h and U^h :

A priori independant choices, let us us this fact. . .

Variational formulation of convected dominated problems,3

In between : continuous and discontinuous Residual distribution scheme

- Choose $U^h = \bigoplus \{u^h_{|K} \in \mathbb{P}^k(K)\}$.
  - Version with continuity requirement,
  - Version without continuity requirement
- Variational formulation :

$$\sum_K \Bigg(-\int_K \nabla \ell(v^h) \cdot f(u^h) dx + \int_{\partial K} \ell(v^h) \hat{f}(u^h_+, u^h_-, \vec{n}) dl \\ + h_K \int_K (\nabla f_u(u^h) \cdot \nabla v^h) \mathcal{T} (\nabla f_u(u^h) \cdot \nabla u^h) dx \Bigg) = 0$$

- Construct mapping $\ell : U^h \rightarrow L^2$ to ensure non oscillatory properties,

Variational formulation of convected dominated problems ,3

In between : continuous and discontinuous Residual distribution scheme

- Choose $U^h = \bigoplus \{u^h_{|K} \in \mathbb{P}^k(K)\}$.
  - Version with continuity requirement,
  - Version without continuity requirement
- Variational formulation :

$$\sum_K \Bigg(- \int_K \nabla \ell(v^h) \cdot f(u^h) dx + \int_{\partial K} \ell(v^h) \hat{f}(u^h_+, u^h_-, \vec{n}) dl + h_K \int_K (\nabla f_u(u^h) \cdot \nabla v^h) \mathcal{T} (\nabla f_u(u^h) \cdot \nabla u^h) dx \Bigg) = 0$$

- Construct mapping $\ell : U^h \to L^2$ to ensure non oscillatory properties,

How ?

Efficiency issues: why continuous FEMs
What about the number of DOFs?

Euler's formula gives:

$$2D : \begin{cases} n_t \approx 2n_v \\ n_e \approx 3n_v \end{cases} \qquad 3D : \begin{cases} n_t \approx 6n_v \\ n_f \approx 10n_v \\ n_e \approx 7n_v \end{cases}$$

vertices, triangles (tetrahedrons), edges, faces (3D)

Order	$2D$		$3D$	
	Discontinuous	Continuous	Discontinuous	Continuous
2	$6n_v$	n_v	$24n_v$	n_v
3	$12n_v$	$4n_v$	$40n_v$	$8n_v$
4	$20n_v$	$9n_v$	$80n_v$	$27n_v$

Overview

2 ways of writing schemes $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$, 2nd order

Finite volumes 1D: $u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\hat{f}_{i+1/2} - \hat{f}_{i-1/2}\right)$

- flux : $\hat{f}_{i+1/2}$
- Conservation: $\pm$

RDS: $u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\phi^-_{i+1/2} + \phi^+_{i-1/2}\right).$

- Residuals $\phi^-_{i+1/2} = \hat{f}_{i+1/2} - f(u_i)$, $\phi^+_{i-1/2} = f(u_i) - \hat{f}_{i-1/2}$
- Conservation :

$$\phi^-_{i+1/2} + \phi^+_{i+1/2} = f(u_{i+1}) - f(u_i) = \int_{x_i}^{x_{i+1}} \frac{\partial f(u)}{\partial x} dx$$

Non oscillatory properties

- either : inputs in $\hat{f}$,
- or tuning of numerical dissipation : symmetric TVD schemes

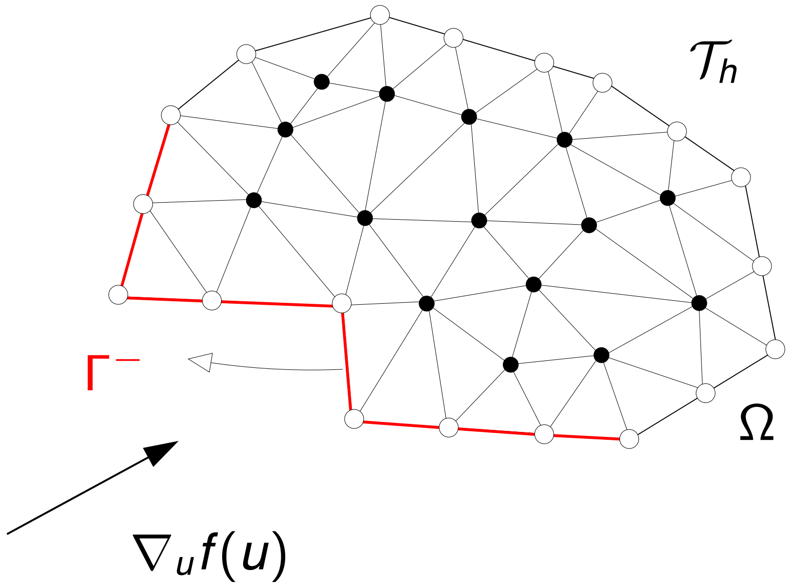
Aim of the talk

- This simple trick can ge generalised in multi dimension (2, 3),
- Allow to construct high order schemes (≥ 2) using only their immediate neighbors, easy parallelisation.
- Provable non oscillatory

Overview

Model problem, Framework for scalar conservation laws.

$$\operatorname{div} f(u) = 0 \quad \text{in} \quad \Omega$$
$$u = g \quad \text{on} \quad \Gamma^-$$



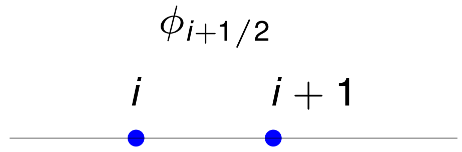
Some notations...

- Consider $\mathcal{T}_h$ triangulation of Ω (can do with quads...)
- Unknowns (Degrees of Freedom, DoF) : $u_i \approx u(M_i)$
- $M_i \in \mathcal{T}_h$ a given set of nodes (vertices +other dofs)
- Denote by u_h continuous piecewise approximation (e.g. P^k Lagrange triangles/quads, Bézier, NURBS, etc) : $u_h = \sum_i \psi_i u_i$

Principle for higher order

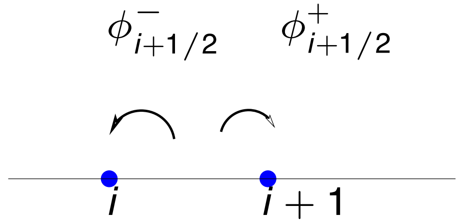
Back to 1D for 1 second.

1. $\forall\ [x_i, x_{i+1}],\ \phi_{i+1/2}(u^h) = \int_{x_i}^{x_{i+1}} \frac{\partial f}{\partial x}(u^h)dx$



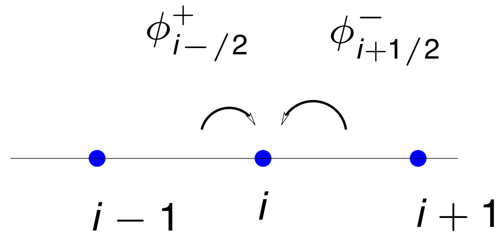
2. Distribution : $\phi^T(u^h) = \phi^+_{i+1/2}(u^h) + \phi^-_{i+1/2}(u^h)$

   Distribution coeff.s : $\phi^{\pm}_{i+1/2}(u^h) = \pm\hat{f}_{i+1/2} \mp f(u_i)$



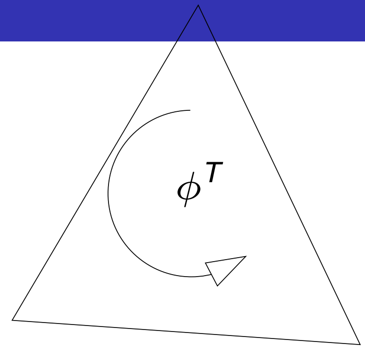
3. Compute nodal values : solve algebraic system

$$\phi^-_{i+1/2} + \phi^+_{i-1/2} = 0 \quad \forall i$$



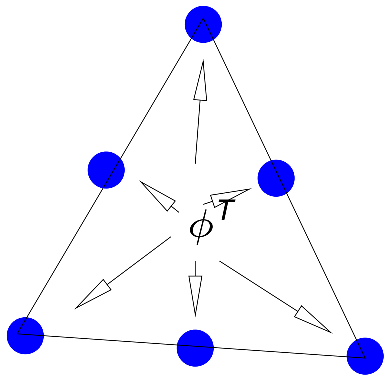
Principle for higher order

1. $\forall T \in \mathcal{T}_h$ compute : $\phi^T = \int_{\partial T} f_h(u_h)\cdot \vec{n}$



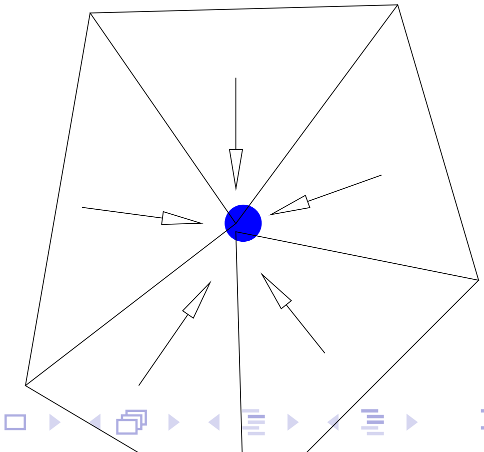
2. Distribution : $\phi^T(u^h) = \sum_{i\in T} \phi_i^T$

   Distribution coeff.s : $\phi_i^T(u^h) =$ sub-residuals



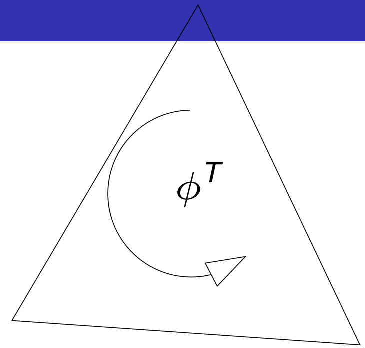
3. Compute nodal values : solve algebraic system

$$\sum_{T|i\in T} \phi_i^T(u^h) = 0, \quad \forall i \in \mathcal{T}_h$$



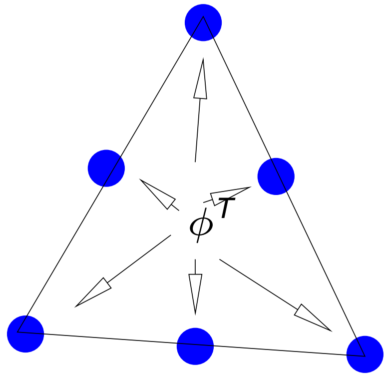
PRINCIPLE FOR HIGHER ORDER

1. $\forall T \in \mathcal{T}_h$ compute : $\phi^T = \int_{\partial T} f_h(u_h) \cdot \vec{n}$



2. Distribution : $\phi^T(u^h) = \sum_{i \in T} \phi_i^T$

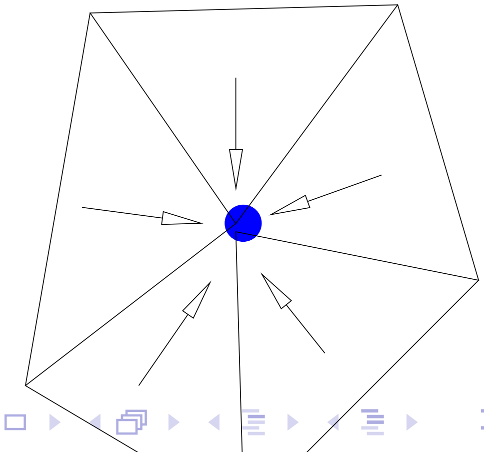
   Distribution coeff.s : $\phi_i^T(u^h) = \text{sub-residuals}$



3. Compute nodal values : solve algebraic system

$$\sum_{T|i \in T} \phi_i^T(u^h) = 0, \quad \forall i \in \mathcal{T}_h$$

$$u_i^{n+1} = u_i^n - \omega_i \left(\sum_{T|i \in T} \phi_i^T \left((u^h)^n \right) \right), \quad \forall i \in \mathcal{T}_h$$



Design properties

Structural conditions, basic properties

Under which conditions on the ϕ_i^Ts we get

- Correct weak solutions (if convergent with h)
- Formal k^{th} order of accuracy
- Monotonicity (discrete max principle)
- Convergence (with h, and with n !)

Condition 1 : conservation

Conservation principle

If there is a f_h, continuous approximation of f such that

$$\phi^T = \sum_{j\in T} \phi_j^T = \oint_{\partial T} f_h \cdot \hat{n}$$

example: $f_h = f(u^h)$ or Lagrange interp. of $f(u_i)$ or ...

Basic relation

- Scheme : for all dof i,

$$\sum_{T\ni i} \phi_i^T(u^h) = 0 \tag{1}$$

- introduce $\phi_i^{Gal,T} = \int_T \psi_i \text{div } f(u^h)dx = \int_T \nabla\psi_i \cdot f(u^h)dx - \int_{\partial T} \psi_i f(u^h)\cdot \hat{n}d\sigma$
- multiply (1) by test function v evaluated at i

$$0 = \sum_i v_i \Big(\sum_{T\ni i} \phi_i^T(u^h) \Big) = \sum_T \sum_{i\in T} v_i \phi_i^T = \sum_T \Big(\sum_{i\in T} v_i \phi_i^{Gal,T} + \sum_{i\in T} v_i(\phi_i^T - \phi_i^{Gal,T}) \Big)$$

$$= \int_\Omega \nabla v^h \cdot f_h(u^h)dx + \Big(\sum_T \frac{1}{N_T!} \sum_{i,j\in T} (v_i - v_j)(\phi_i^T - \phi_i^{Gal,T}) \Big)$$

CONDITION 2 : ACCURACY.

$u^{ex,h}$ interpolant of exact sol. assumed smooth

Truncation error

$$\mathcal{E}(u^{ex,h}) := \sum_{i \in \mathcal{T}_h} v_i \Big(\sum_{T |\, i \in T} \phi_i^T (u^{ex,h}) \Big)$$

GUIDING PRINCIPLE

$$\mathcal{E}(u^{ex,h}) = \overbrace{\int_\Omega \nabla v_h \cdot f_h(u^{ex,h})}^{I \equiv \mathcal{E}^{\text{Galerkin}}} + \overbrace{\sum_{T \in \mathcal{T}_h} \frac{1}{N_T!} \sum_{i,j \in T} (v_i - v_j)(\phi_i^T - \phi_i^{\text{Gal}})(u^{ex,h})}^{II}$$

Condition 2 : accuracy.

Truncation error

$u^{ex,h}$ interpolant of exact sol. assumed smooth

$$\mathcal{E}(u^{ex,h}) := \sum_{i\in\mathcal{T}_h} v_i\Big(\sum_{T|\, i\in T}\phi_i^T(u^{ex,h})\Big)$$

Guiding principle

$$\mathcal{E}(u^{ex,h}) = \overbrace{\int_\Omega \nabla v_h \cdot f_h(u^{ex,h})}^{I\equiv\mathcal{E}^{\text{Galerkin}}} + \overbrace{\sum_{T\in\mathcal{T}_h}\frac{1}{N_T!}\sum_{i,j\in T}(v_i - v_j)(\phi_i^T - \phi_i^{\text{Gal}})(u^{ex,h})}^{II}$$

Key remark & final result

- div $f(w) = 0 \Longrightarrow \phi_i^{Gal,T}(u^{ex,h}) = \int_T \nabla\psi_i \cdot f_h(u^{ex,h})dx - \int_{\partial T}\psi_i f_h(u^{ex,h})\cdot\hat{n}d\sigma = O(h^{k+d})$
- Truncation error : $|\mathcal{E}(u^{ex,h})| \leq C'(\mathcal{T}_h, u^{ex})\|\nabla v\|_\infty\, h^{k+1}$

if (in d-D) $\quad |\phi_i^T(u^{ex,h})| \leq C''(\mathcal{T}_h, u^{ex})h^{k+d} = \mathcal{O}(h^{k+d})$

Condition 2 : accuracy

Linearity (Accuracy) preserving schemes

Since $\phi^T(w_h) = \int_{\partial T} f^h(u^h) \cdot \hat{n} dl = \mathcal{O}(h^{k+d})$ schemes for which

$$\phi_i^T = \beta_i^T \phi^T \qquad \text{with } \beta_i^T \text{ uniformly bounded distribution coeff.s}$$

are formally $k+1^{\text{th}}$ order accurate (for $k+1^{\text{th}}$ order spatial interpolation)

However: Godunov's theorem

The β_i^T must depend on the solution : *A scheme cannot be both high order accurate and linear for a linear problem.*

Condition 2 : accuracy

Linearity (Accuracy) preserving schemes

Since $\phi^T(w_h) = \int_{\partial T} f^h(u^h) \cdot \hat{n} dl = \mathcal{O}(h^{k+d})$ schemes for which

$$\phi_i^T = \beta_i^T \phi^T \qquad \text{with } \beta_i^T \text{ uniformly bounded distribution coeff.s}$$

are formally $k+1^{\text{th}}$ order accurate (for $k+1^{\text{th}}$ order spatial interpolation)

However: Godunov's theorem

The β_i^T must depend on the solution : *A scheme cannot be both high order accurate and linear for a linear problem.*

Fundemental assumption in all this business:

$$\sum_{T|i \in T} \phi_i^T(u^h) = 0, \quad \forall i \in \mathcal{T}_h \qquad \text{has a unique solution}$$

i.e. $u_i^{n+1} = u_i^n - \omega_i \left(\sum_{T|i\in T} \phi_i^T\Big((u^h)^n\Big)\right), \quad \forall i \in \mathcal{T}_h$ **must converges**

CONDITION 3: PRESERVATION OF MONOTONY + ACCURACY

GOAL

Given any element T, a set of residuals $\{\phi_i^M(u^h)\}_{i\in T}$, construct a set of residuals $\{\phi_i^H(u^h)\}_{i\in T}$ with $\phi_i^H(u^{ex,h}) = O(h^{k+d})$.

IDEA

- Known residuals $\phi_i^T = \sum\limits_{\substack{j\in T\\ j\neq i}} c_{ij}(u_i - u_j)$
- If $c_{ij} \geq 0$: local maximum principle
- Remark: start from $\phi_i^M = \sum_{i,j} c_{ij}^M(u_i - u_j)$

$$\phi_i^H = \left(\frac{\phi_i^H}{\phi_i^M}\right)\phi_i^M = \sum_{\substack{j\in T\\ j\neq i}} \underbrace{\left(\frac{\phi_i^H}{\phi_i^M}\right) c_{ij}^M}_{c_{ij}^H}(u_i - u_j)$$

- $c_{ij}^H = \left(\frac{\phi_i^H}{\phi_i^M}\right) c_{ij}^M \geq 0$. Since $c_{ij}^M \geq 0$, need $\phi_i^M \times \phi_i^H \geq 0$.

Condition 3: Preservation of monotony + accuracy

Goal

Given any element T, a set of residuals $\{\phi_i^M(u^h)\}_{i\in T}$, construct a set of residuals $\{\phi_i^H(u^h)\}_{i\in T}$ with $\phi_i^H(u^{ex,h}) = O(h^{k+d})$.

Example: Struijs' "limiter"

$$\beta_i^H = \frac{\max(0, \phi_i^M/\phi^T)}{\sum_{j\in T} \max(0, \phi_j^M/\phi^T)}$$

- $\{\phi_i^M(u^h)\}_{i\in T}$, $\sum_{i\in T} \phi_i^M(u^h) = \phi^T$
- $\phi_i^H = \beta_i^H \phi^T$.

Condition 3: Preservation of monotony + accuracy

Goal

Given any element T, a set of residuals $\{\phi_i^M(u^h)\}_{i\in T}$, construct a set of residuals $\{\phi_i^H(u^h)\}_{i\in T}$ with $\phi_i^H(u^{ex,h}) = O(h^{k+d})$.

Example: Struijs' "limiter"

$$\beta_i^H = \frac{\max(0, \phi_i^M/\phi^T)}{\sum_{j\in T}\max(0, \phi_j^M/\phi^T)}$$

- $\{\phi_i^M(u^h)\}_{i\in T}$, $\sum_{i\in T}\phi_i^M(u^h) = \phi^T$
- $\phi_i^H = \beta_i^H\phi^T + h_K\int_K \left(\nabla f_u(u^h)\cdot\nabla v^h\right)\mathcal{T}\left(\nabla f_u(u^h)\cdot\nabla u^h\right)dx$ Questions

Examples of monotone schemes

Monotone schemes : the Rusanov scheme (Local Lax Friedrichs)

Choice of Rusanov : not essential at all !

Examples of monotone schemes

Monotone schemes : the Rusanov scheme (Local Lax Friedrichs)

Centered linear first order distribution :

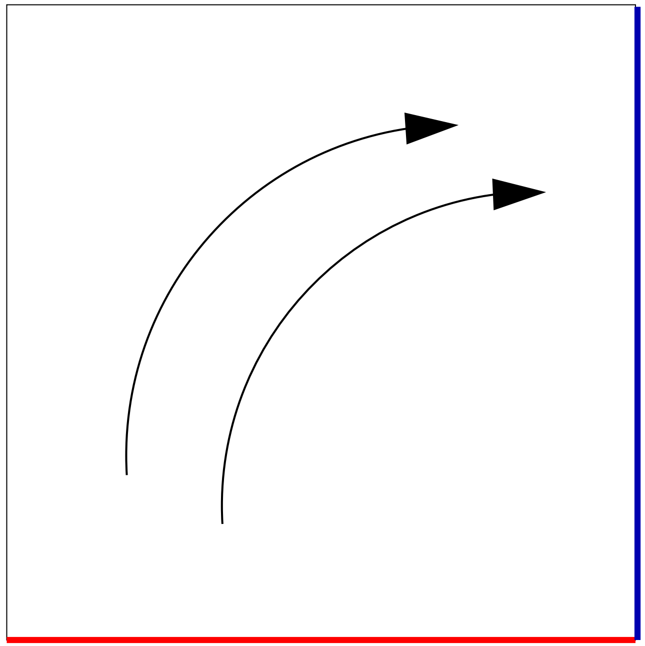
$$\phi_i^{\text{Rv}} = \frac{1}{K}\phi^T + \frac{\alpha}{K}\sum_{\substack{j\in T\\ j\neq i}}(u_i - u_j), \quad \alpha \geq \max_{j\in T}\left|\int_T \nabla_u f(u^h)\cdot\nabla\psi_j\right|$$

- K number of DoF per element
- ψ_j Lagrange basis fcn. relative to node j

Why this scheme ?

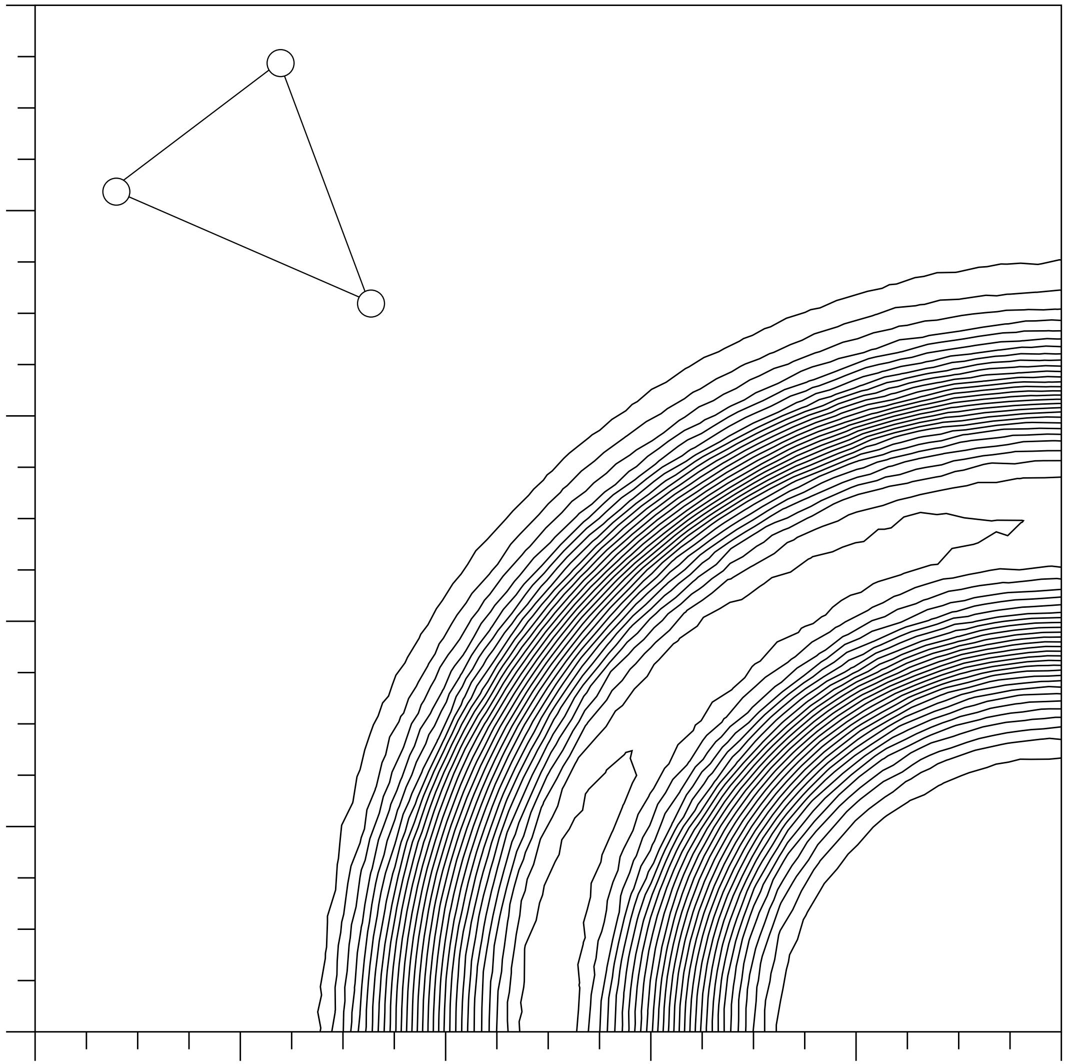
1. The Rv scheme is cheap and has general formulation
2. The Rv scheme is monotone and energy stable in the P^1 case.
3. By far one of the most dissipative ones

Numerical example : rotation

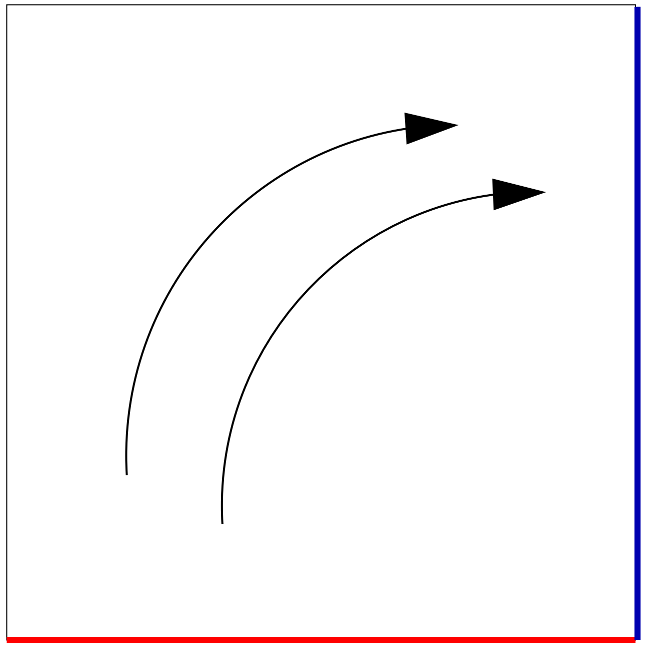


$u_{\text{inlet}} = \cos^2(2\pi x)$

$0.25 \leq x \leq 0.75$

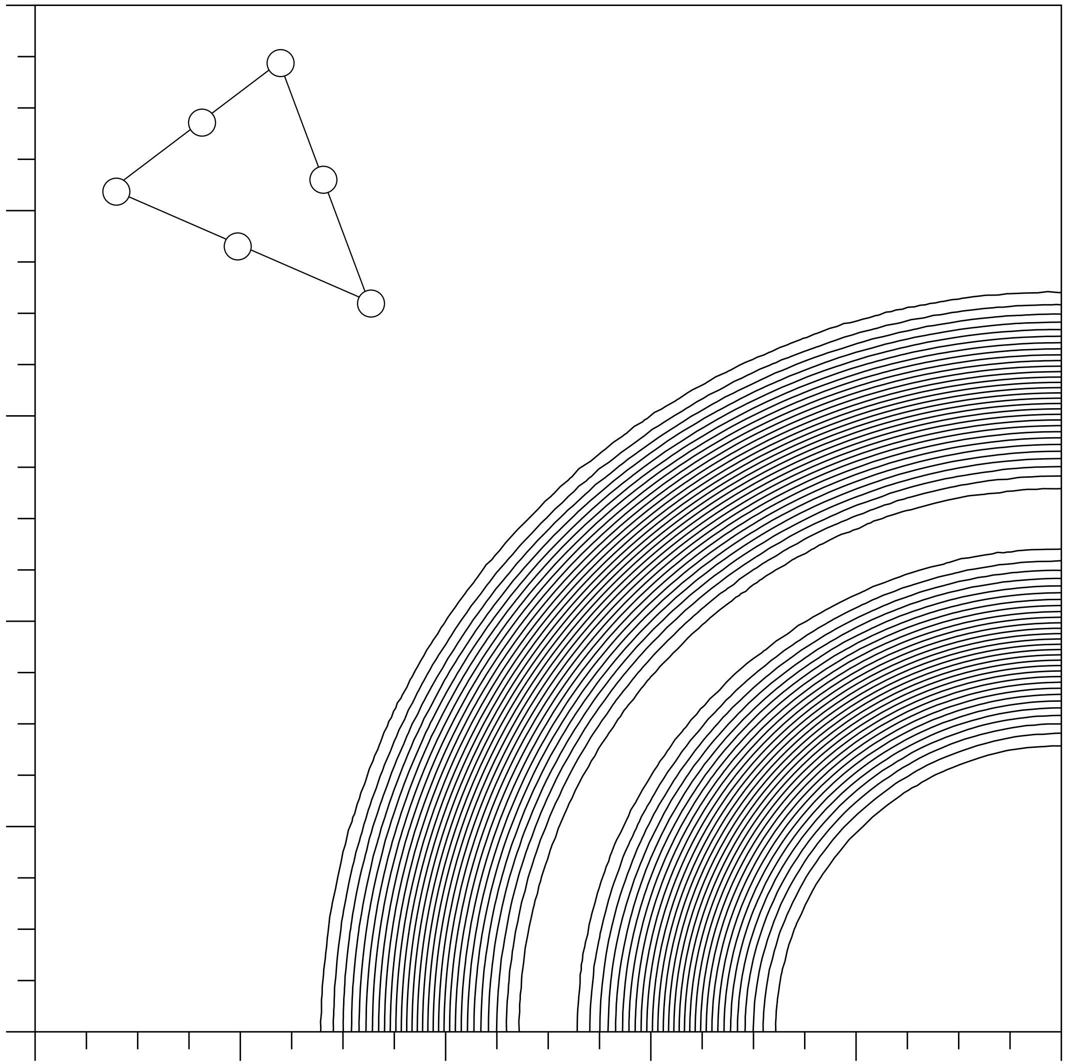


NUMERICAL EXAMPLE : ROTATION

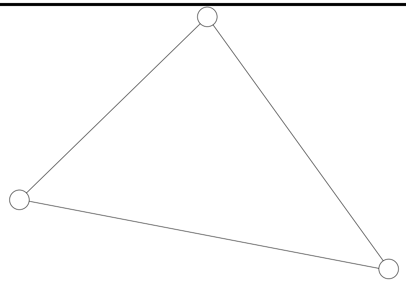
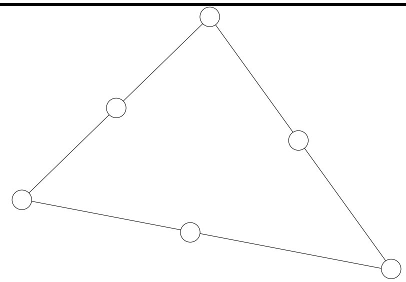
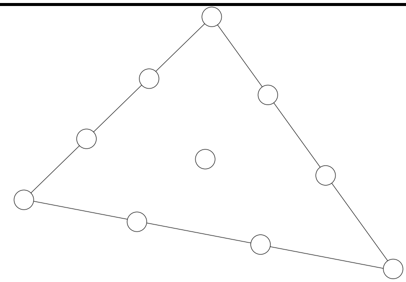


$u_{\text{inlet}} = \cos^2(2\pi x)$

$0.25 \leq x \leq 0.75$



Grid convergence

			
h	$\epsilon_{L^2}(P^1)$	$\epsilon_{L^2}(P^2)$	$\epsilon_{L^2}(P^3)$
1/25	0.50493E-02	0.32612E-04	0.12071E-05
1/50	0.14684E-02	0.48741E-05	0.90642E-07
1/75	0.74684E-03	0.13334E-05	0.16245E-07
1/100	0.41019E-03	0.66019E-06	0.53860E-08
	$\mathcal{O}^{\text{ls}}_{L^2}$ =**1.790**	$\mathcal{O}^{\text{ls}}_{L^2}$ =**2.848**	$\mathcal{O}^{\text{ls}}_{L^2}$ =**3.920**

ALGORITHM

The scheme consists in 4 steps :

1. Evaluate the total residual, local (continuous interpolant)
2. Evaluate monotone residual (Rusanov) : local,
3. Evaluate high order residual : local
4. Gather residual : indirections, importance of good numering of the degrees of freedom

The scheme is local and easy to parallelise

BACK TO THE VARIATIONAL FORMULATION :

$$\sum_K \Big(-\int_K \nabla \ell(v^h)\cdot f(u^h)dx + \int_{\partial K} \ell(v^h)\hat{f}(u_+^h, u_-^h, \vec{n})dl + h_K \int_K (\nabla f_u(u^h)\cdot \nabla v^h)\mathcal{T}(\nabla f_u(u^h)\cdot \nabla u^h)dx \Big) = 0$$

WHAT IS ℓ ?

- Multiply by test function v^h, rearrange

$$\sum_K \Big(\overbrace{\Big(\sum_{i\in K} \beta_i^K v_i\Big)}^{\ell(v_h)} \int_{\partial K} f(u^h)\cdot \vec{n}dl + h_K \int_K (\nabla f_u(u^h)\cdot \nabla v^h)\mathcal{T}(\nabla f_u(u^h)\cdot \nabla u^h)dx \Big) = 0$$

- $\ell(v_h)$ constant in each T, and

$$\begin{aligned} v_h \in V_h \quad &\mapsto \quad \pi_h(v_h) \in \widetilde{V}_h \\ \pi_h(v_h) \quad &= \quad \textstyle\sum_{i\in K} \beta_i^K(u^h) v_i \end{aligned}$$

Overview

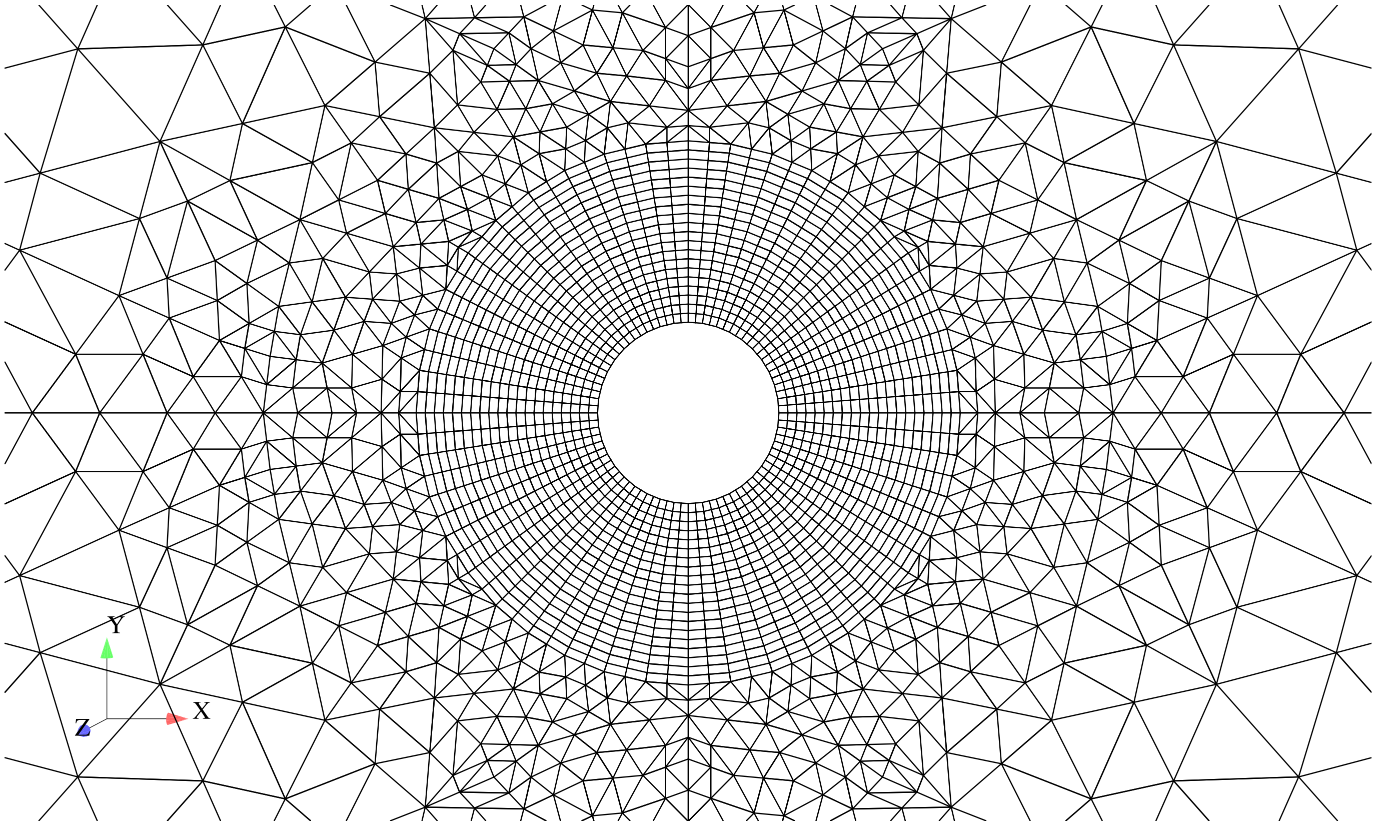
EXTENSION TO SYSTEMS

$$\nabla \cdot f(\mathbf{u}) = 0$$

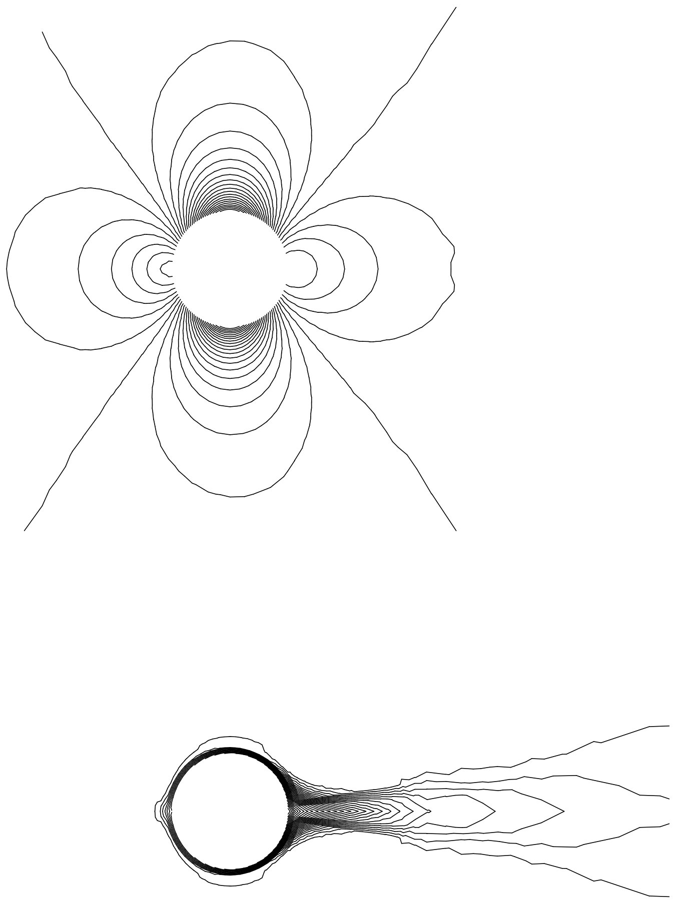
- Schemes formally identical to scalar case
- Nonlinear mapping on scalar residuals obtained by locally projecting on Eigenvector basis
- Stabilization : same as in the scalar case with matrix notation

EULER EQ.S : $Ma = 0.35$ CYLINDER FLOW

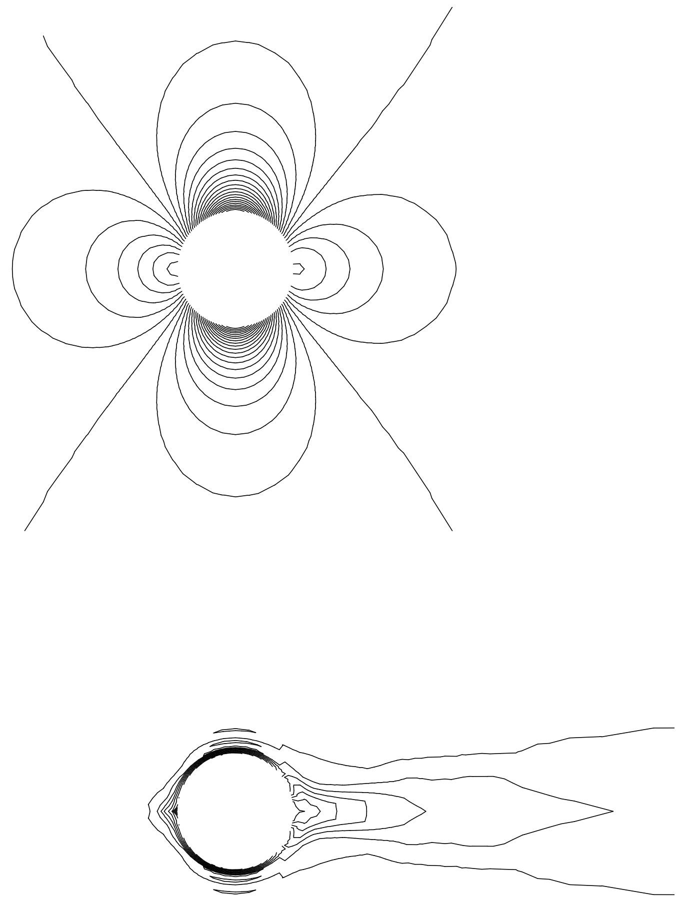
$Ma = 0.35$
flow on cylinder
Mesh :
1536 nodes
2912 elements
Hybrid mesh
on cylinder



Pressure

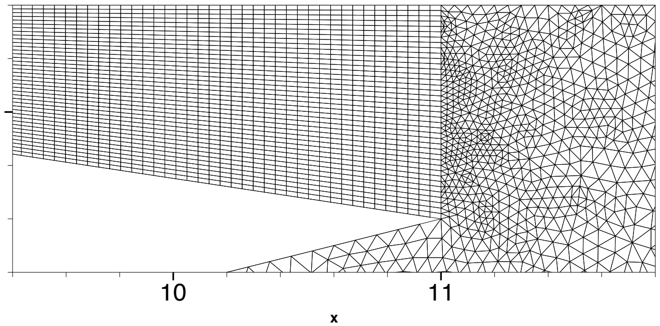


2nd order

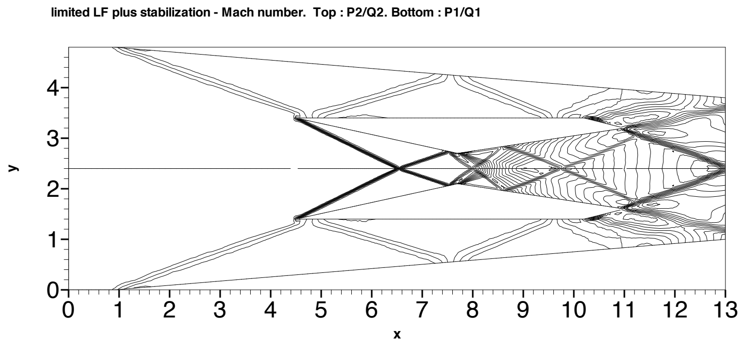


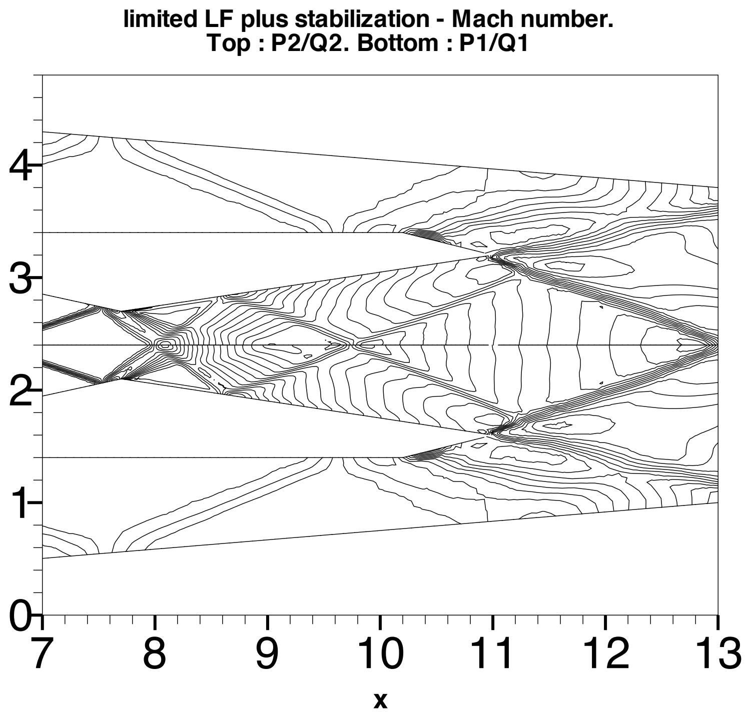
3rd order

SCRAMJET LIKE, HYBRID MESH



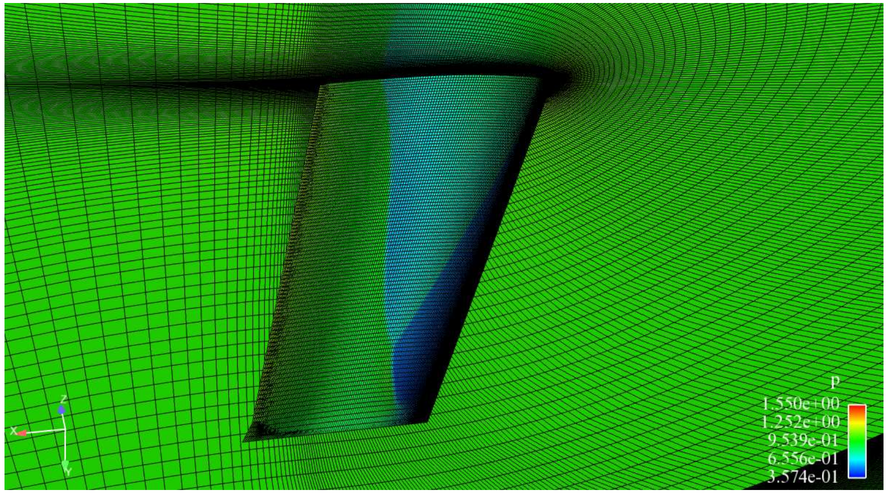
MACH NUMBER, 3RD ORDER



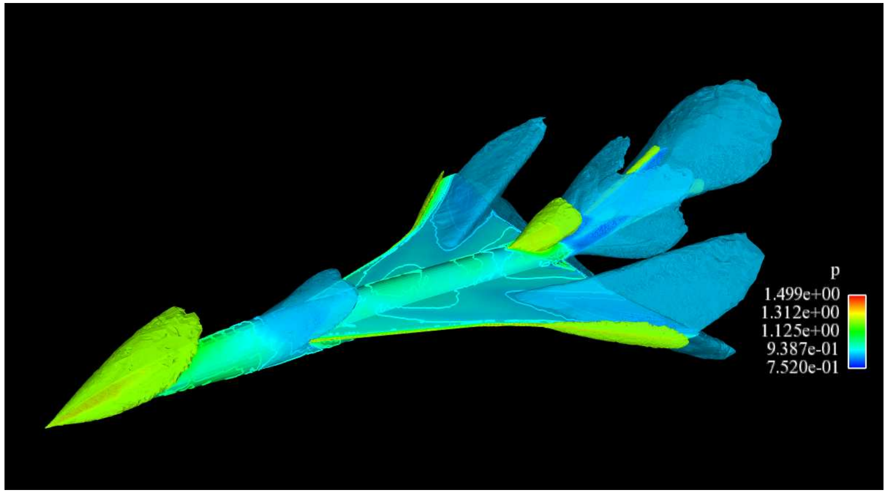


zoom

3D SOLNS



M6 wing
H1



supersonic business jet
P2

Overview

Viscous problems
Using the same variational formulation

$$\text{div } f(u) - \text{div}(\varepsilon \nabla u) = 0 + \text{ BCs}$$

Use the variational formulation and $u \in H^2$

$$\sum_K \left(\int_{\partial K} \left[\widetilde{\varepsilon \nabla u \cdot \vec{n}} + \hat{f}(u) \right] \ell(v) dl - \int_K \nabla \ell(v) \cdot (\varepsilon \nabla u \cdot \vec{n} + f(u)) dx \right.$$
$$\left. + h_K \int_K \Big(\nabla f_u(u) \nabla v - \varepsilon \Delta v \Big) \mathcal{T} \Big(\nabla f_u(u) \nabla u - \varepsilon \Delta u \Big) \right) = 0$$

With

- Cell residual: $\oint_T \Big(\mathbf{f}(\mathbf{u}^h) - \{\nu \nabla \mathbf{u}\} \Big) \cdot \boldsymbol{n}$
- Average gradients: $\{\nabla \mathbf{u}^h\}_i = \dfrac{\sum_{T \ni i} |T| \nabla \mathbf{u}^h}{\sum_{T \ni i} |T|}$
- Correct order approximation: $\nu \nabla \mathbf{u}^h \simeq \sum_{i \in T} \{\nu \nabla u\}_i \psi_i$

Accuracy tests

Heat equation

$$\frac{\partial u}{\partial y} - \varepsilon \frac{\partial^2 u}{\partial x^2} = 0$$

on $[0, 1]^2$ with the boundary conditions

$$u(x, 0) = \sin(\pi x) \text{ on } y = 0$$
$$u(x, y) = \varphi(x, y) \text{ on } x = 0 \text{ and } x = 1$$

RESULTS FOR THE CONVERGENCE STUDY OF HEAT EQUATION.

Δx	L^∞		L^2	
-0.532115666963180	-2.41783092916874	-	-2.42918055471673	
-0.846872634669396	-3.22731516724477	2.57	-3.15553327436338	2.30
-1.08957273264021	-4.05793545206366	3.42	-3.87000533630969	2.94
-1.36540918681519	-4.90199016882381	3.06	-4.60138273684662	2.65
$\varepsilon = 0.0001$				

Δx	L^∞		L^2	
-0.532115666963180	-2.42235466229356	-	-2.43644370369152	-
-0.846872634669396	-3.24877046688954	2.62	-3.21509140129168 2.47	
-1.08957273264021	-4.09492244395854	3.48	-3.95823335106917	3.06
-1.36540918681519	-4.99047469215026	3.24	-4.85559507238436	3.25
$\varepsilon = 0.001$				

Δx	L^∞		L^2	
-0.532115666963180	-2.45230965825349	-	-2.52191658082643	-
-0.846872634669396	-3.29453851242374	2.67	-3.26021775685192	2.34
-1.08957273264021	-4.01681756317218	2.97	-3.74468087319104	1.99
-1.36540918681519	-4.71151297471185	2.51	-4.48933815669847	2.7
$\varepsilon = 0.01$				

Δx	L^∞		L^2	
-0.532115666963180	-2.12079249189368	-	-2.07369114240901	-
-0.846872634669396	-2.56866661478255	1.42	-2.55012986275973	1.51
-1.08957273264021	-3.19486137685157	2.58	-3.19339332748624	2.65
-1.36540918681519	-4.13416777580946	3.40	-3.82076591805969	2.27
$\varepsilon = 0.1$				

MultiD case

Smith and Hutton case

- Problem : in $[-1,1]\times[0,1]$

$$\lambda_x \frac{\partial u}{\partial x} + \lambda_y \frac{\partial u}{\partial y} = \varepsilon \nabla u$$

with

$$\lambda_x = -\frac{\partial \psi}{\partial y}, \lambda_y = \frac{\partial \psi}{\partial y}, \qquad \psi = -(1-x^2)(1-y^2).$$

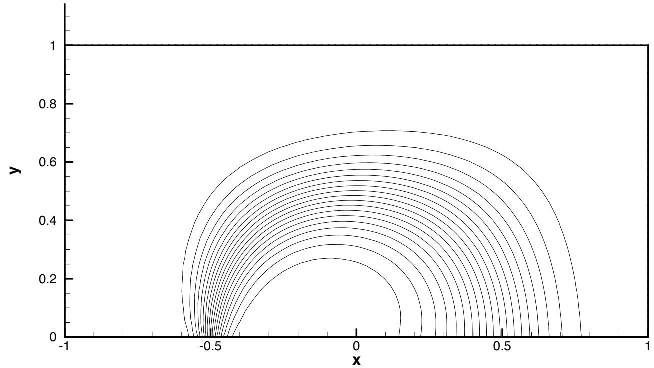
- Boundary conditions such that for $\varepsilon = 0$ the solution is

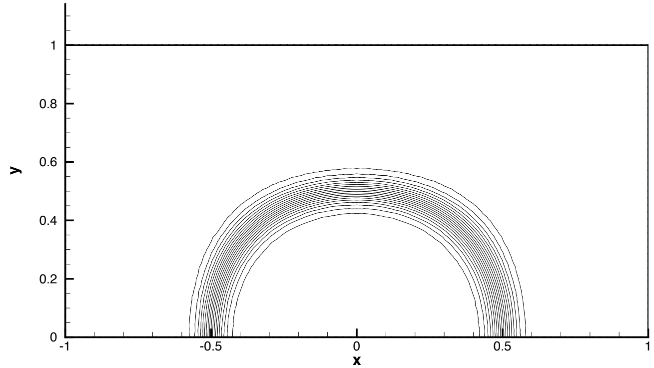
$$u(x,y) = 1 + \tanh\Big(\theta(1-2\sqrt{1+\psi})\Big)$$

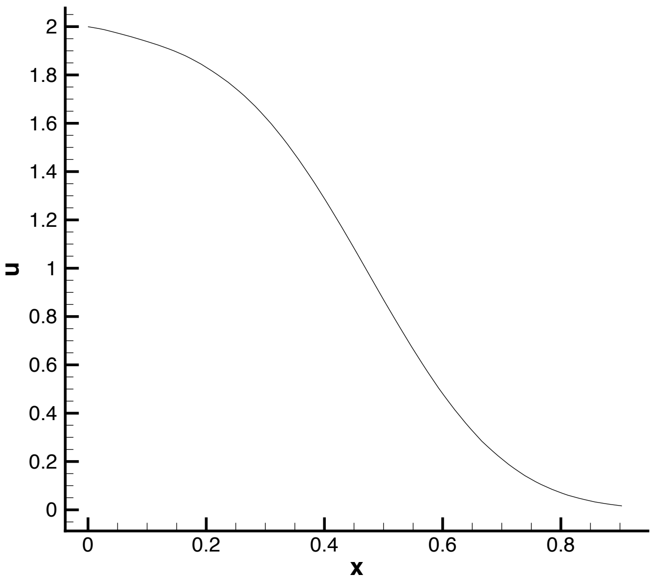
- solutions for $\alpha = 100$: very sharp.

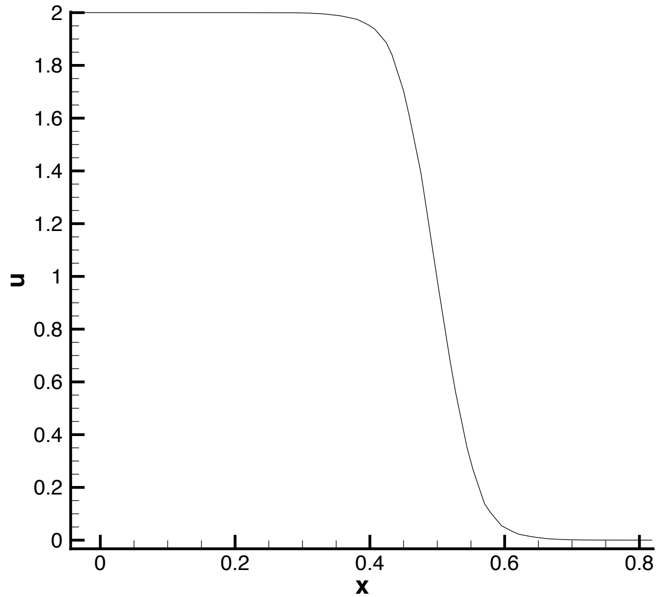
SMITH-HUTTON PROBLEMS

$\nu = 10^{-2}$ $\nu = 10^{-4}$



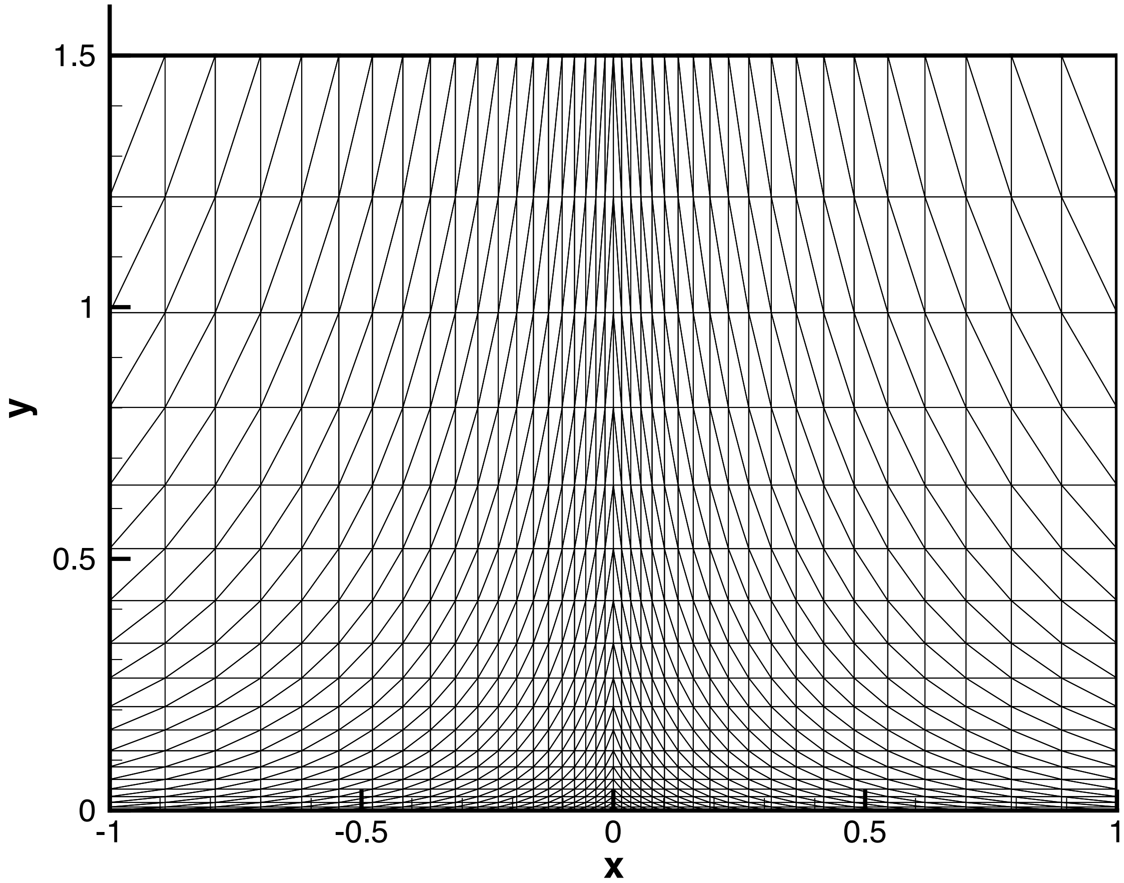






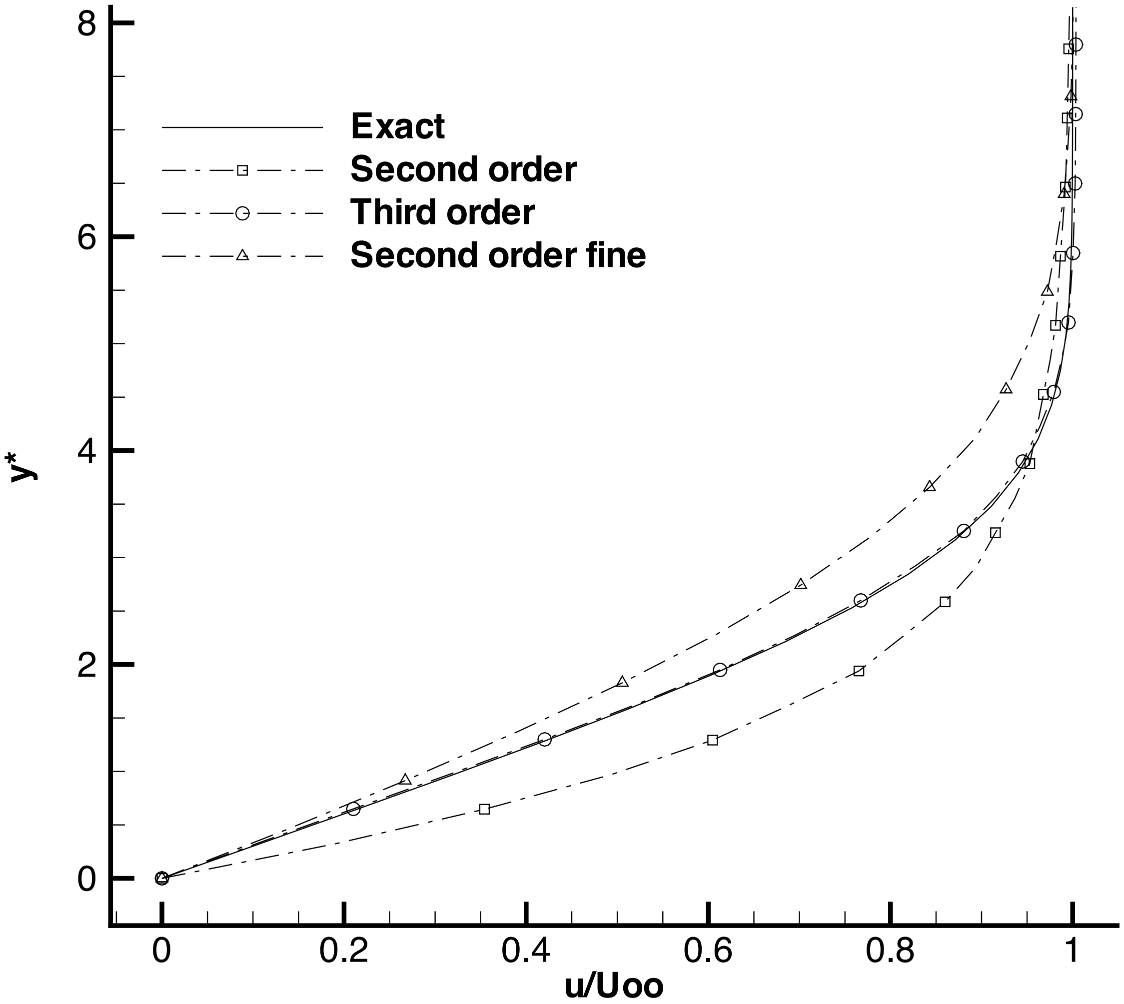
Navier-Stokes

Blasius boundary layer. $M = 0.3$, $Re = 1000$



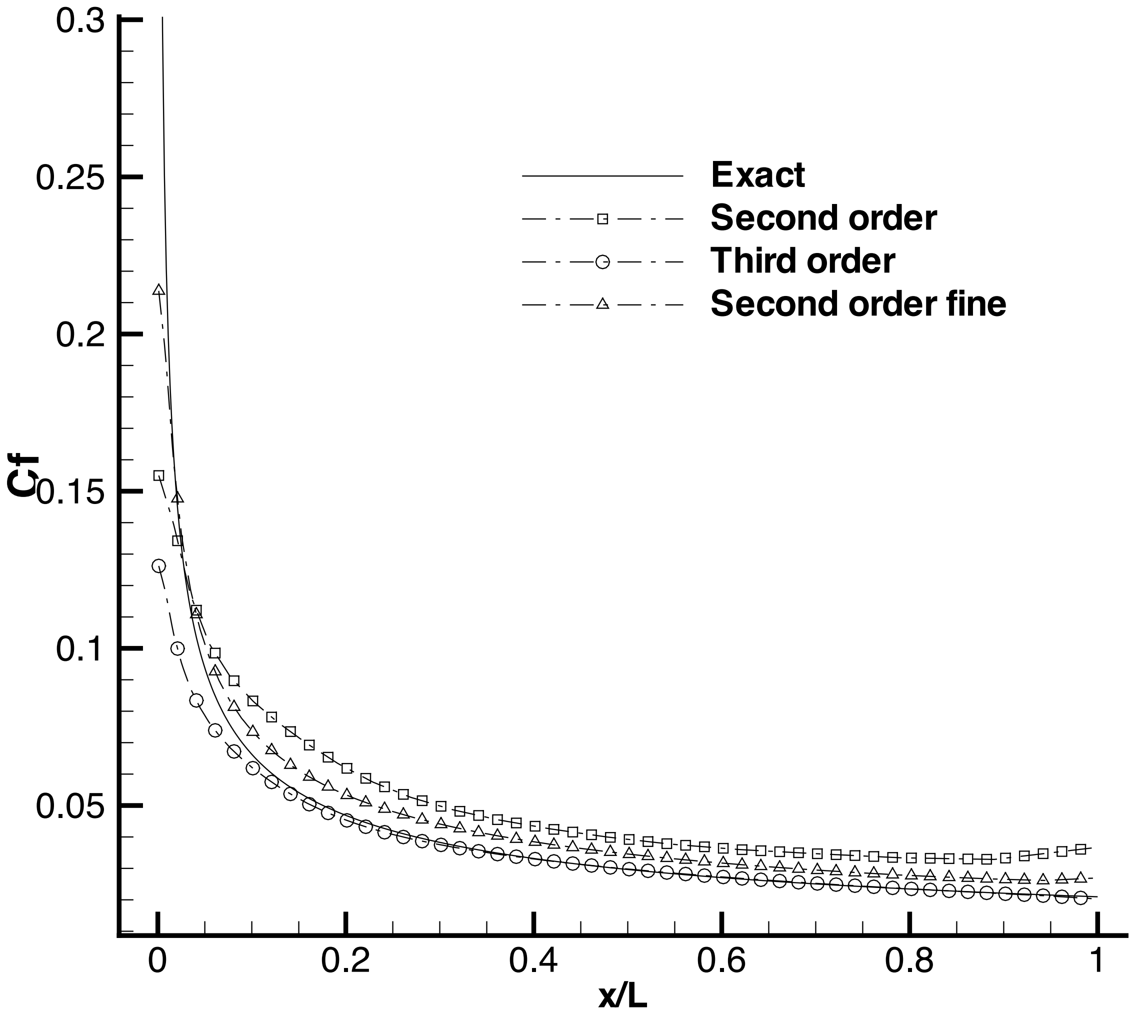
Navier-Stokes

Blasius boundary layer. $M = 0.3$, $Re = 1000$



Navier-Stokes

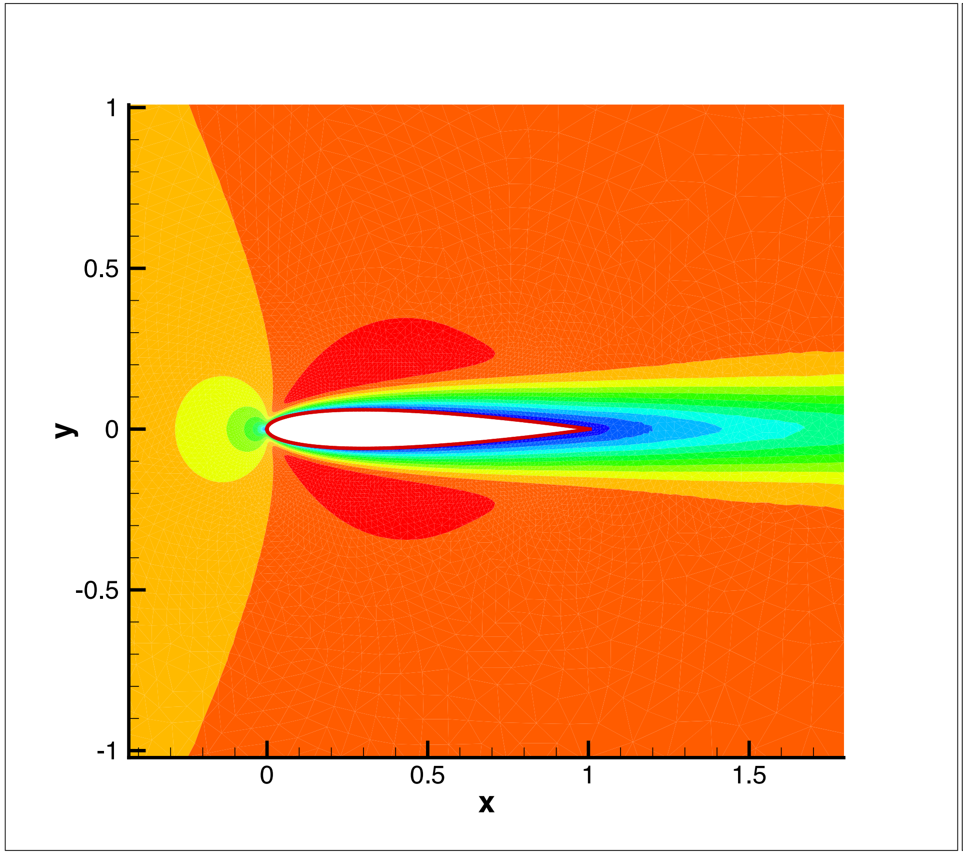
Blasius boundary layer. $M = 0.3$, $Re = 1000$



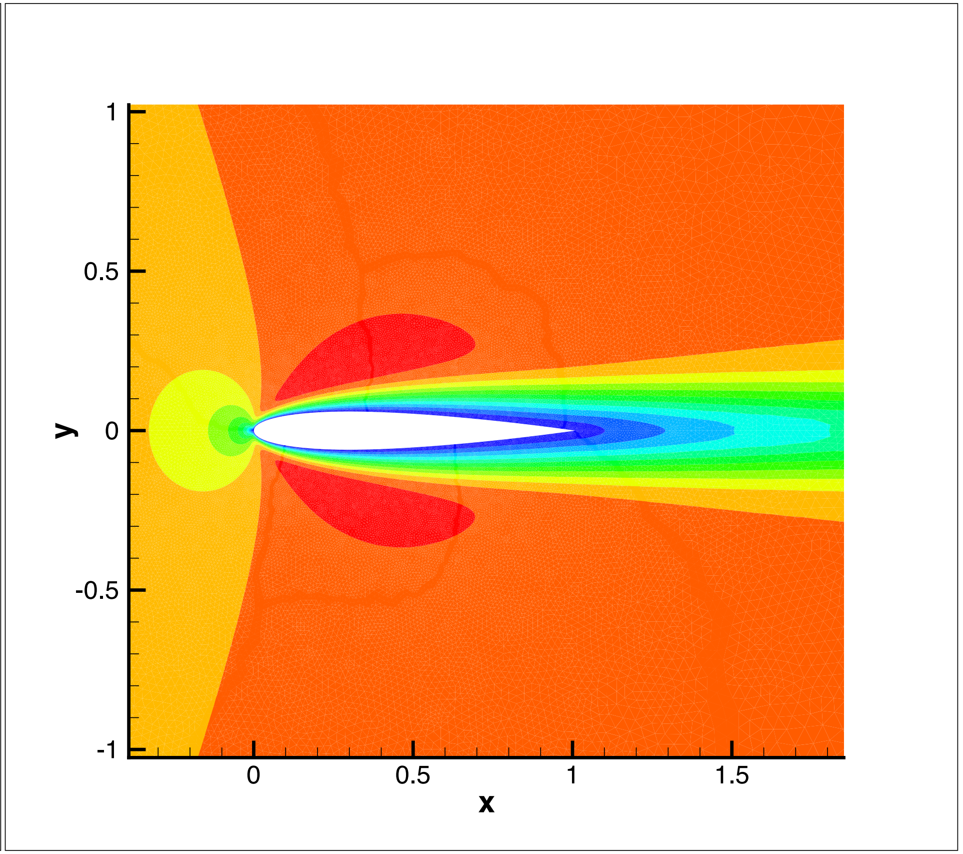
Navier-Stokes

NACA0012, $M = 0.5$, $Re = 500$

2nd Order

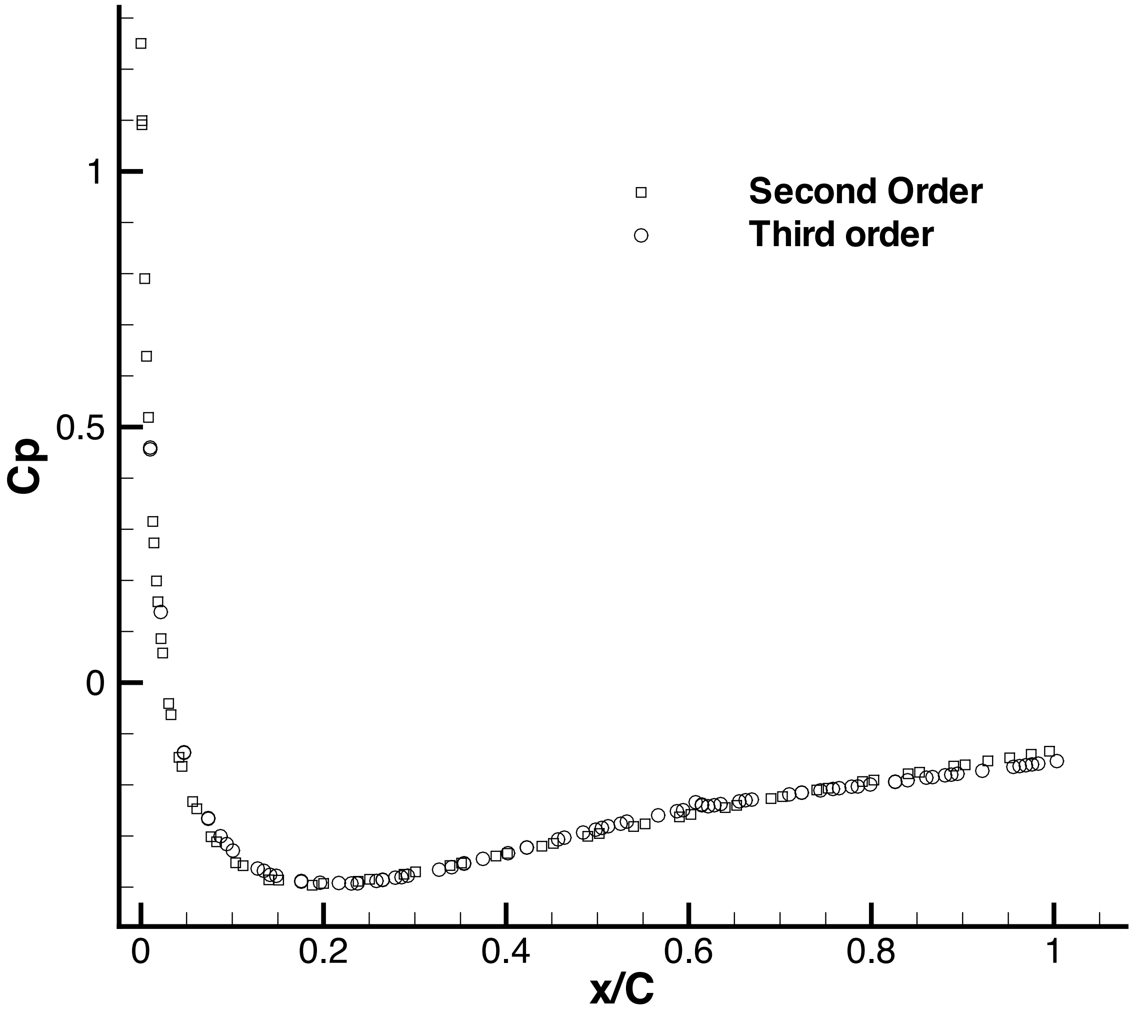


3rd Order



NAVIER-STOKES

NACA0012, $M = 0.5$, $Re = 500$



Overview

Conclusions and perspectives

Conclusions

- Convergent higher order non-oscillatory *RD* schemes, steady, unsteady.
- General procedure: hybrid conformal meshes
- Efficient method for solving the non linear system (not shown, uses Petsc)
- Viscous terms, in progress
- Easily parallelisable (3D + viscous, Scotch partitionning)
- Possibility to handle discontinuous elements, other approximations, in the same framework.
- Other physics: MHD, Shallow water, multispecies (combustion), multiphase in progress. Relativistic compressible fluid dynamics (J. Rossmanith, Wisconsin U)

Conclusions and perspectives

Perspectives

- Better time dependant (order $>$ 2+other elements) ▸ Comment
- More complex physical models: multiphase, ...
- Efficient discretizations (fewer DoF and op.s w.r.t. DG): to be checked.
- For systems less matrix algebra than with upwind schemes

Unsteady, explicit version. LxF

- No simple time-space splitting can work
- However, explicit possible (2nd order) for now+triangles

$\dfrac{\partial u}{\partial t} + \text{DIV } f(u) = 0$

Remark

- Evaluation of the total residual: $\phi_T = \int_{\partial T} f_h \cdot \hat{n} dl = \int_T \text{div } f_h dx$, div f_h constant if Lagrange interpolation.
- Rewrite $\beta_i^T \phi_T = \int_T (\varphi_i + \gamma_i^T) \text{div } f_h dx \qquad$ with $\gamma_i^T = \beta_i^T - 1/3$

Steady $\longrightarrow$ Unsteady

- Choose a RK type scheme, for example $u^n \to u^1 \to u^{n+1} = u^2$

$$0 = \frac{\delta u^1}{\Delta t} + \text{ div } f(u^n) := r^1 \qquad 0 = \frac{\delta u^2}{\Delta t} + \frac{1}{2}\Big(\text{div } f(u^n) + \text{ div } f(u^1) \Big) := r^2$$

- Evaluation of residuals

$$\int_T \varphi_i \, r^j dx + \int_T \gamma_i \left(\widetilde{\frac{\delta u^j}{\Delta t}} + \text{DIV } f \right) dx = \int_T \varphi_i \left(\frac{\delta u^j}{\Delta t} - \widetilde{\frac{\delta u^j}{\Delta t}} \right) + \beta_i \int_T \left(\widetilde{\frac{\delta u^j}{\Delta t}} + \text{DIV } f \right) dx$$

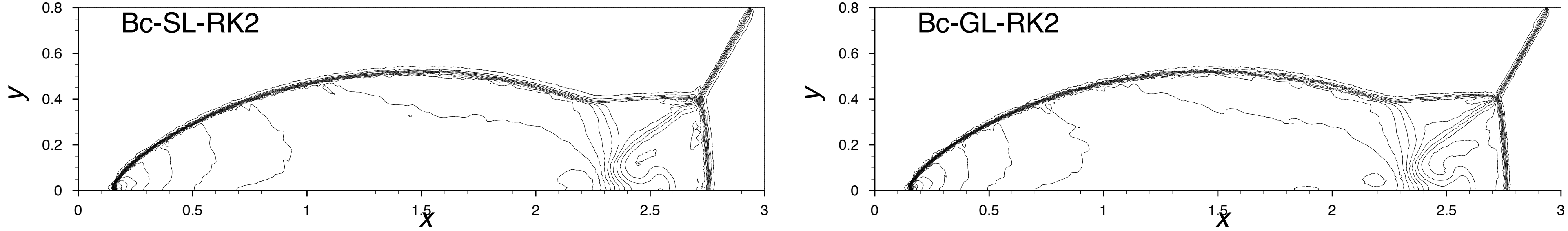
UNSTEADY, ONE EXAMPLE OF SECOND ORDER SCHEME

After mass lumping,

$$|S_i|\frac{u_i^1 - u_i^n}{\Delta t} = -\sum_{T|i\in T} \beta_i^T \phi(t^n)$$

$$|S_i|\frac{u_i^{n+1} - u_i^1}{\Delta t} = -\sum_{T|i\in T} \beta_i^T \left(\int_T \frac{u^1 - u^n}{\Delta t} + \frac{1}{2}\Big(\text{div } f(u^n) + \text{ div } f(u^1)\Big) dx \right)$$

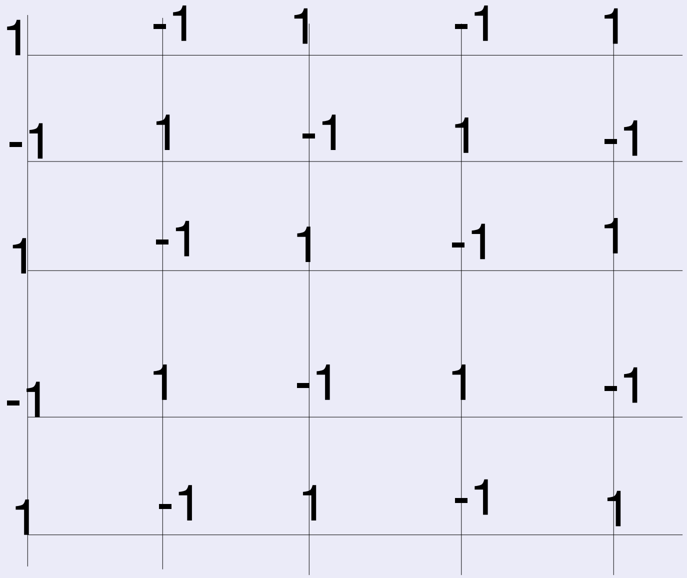
S_i : area of dual cell.



Back

Back

SOLVE $\dfrac{\partial u}{\partial x} = 0$ ON $[0.1]^2$



Initialisation 1

Initialisation 2

- In both cases, $\phi^K = 0$: these are steady solutions.
- Cure :

$$\phi_i^{H,K} = \beta_i^K \phi^K \longrightarrow \beta_i^K \phi^K \qquad + h_K \int_K \big(\nabla f_u(u^h) \cdot \nabla v^h\big) \mathcal{T} \big(\nabla f_u(u^h) \cdot \nabla u^h\big) dx$$