Extension of finite volume evolution Galerkin scheme for low Froude number flows based on asymptotic expansion

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Outline

1. Introduction
   - Shallow water equations
   - Bicharacteristic evolution operators
   - FVEG scheme

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2. FVEG Scheme for Low Froude Numbers
   - Advection of a vortex
   - Asymptotic analysis
3. A Flux-splitting Method
   - A two-grid algorithm

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3. A Flux-splitting Method
   - A two-grid algorithm
4. Numerical Case Studies

Outline

The shallow water system

- System of balance laws

$$\mathbf{u}_t + \mathbf{f}_1(\mathbf{u})_x + \mathbf{f}_2(\mathbf{u})_y = \mathbf{s}(\mathbf{u}, x, y).$$

- Conserved variables and fluxes

$$\mathbf{u} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \; \mathbf{f}_1(\mathbf{u}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix}, \; \mathbf{f}_2(\mathbf{u}) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix}.$$

- Source terms

$$\mathbf{s}(\mathbf{u}, x, y) = -gh \begin{pmatrix} 0 \\ b_x \\ b_y \end{pmatrix}.$$

Linearised system

- Linearised system in primitive variables

$$\mathbf{v}_t + A_1(\tilde{\mathbf{v}})\mathbf{v}_x + A_2(\tilde{\mathbf{v}})\mathbf{v}_y = \mathbf{t}(\tilde{\mathbf{v}}).$$

- Matrix pencil $A := \cos\theta A_1 + \sin\theta A_2$.
- Eigenvalues and eigenvectors

$$\lambda_{1,3} = \cos\theta\tilde{u} + \sin\theta\tilde{v} \mp \sqrt{g\tilde{h}},\ \ \lambda_2 = \cos\theta\tilde{u} + \sin\theta\tilde{v}$$

$$\mathbf{l}_{1,3} = \frac{1}{2}\left(\mp 1, \frac{\tilde{c}}{g}\cos\theta, \frac{\tilde{c}}{g}\sin\theta\right),\ \ \mathbf{l}_2 = (0, \sin\theta, -\cos\theta)$$

$$\mathbf{r}_{1,3} = \begin{pmatrix} \mp 1 \\ \frac{g}{\tilde{c}}\cos\theta \\ \frac{g}{\tilde{c}}\sin\theta \end{pmatrix},\ \ \mathbf{r}_2 = \begin{pmatrix} 0 \\ \sin\theta \\ -\cos\theta \end{pmatrix}$$

Bicharacteristics

- Bicharacteristic curves of the linearised system (Courant, Hilbert)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tilde{u} \mp \tilde{c}\cos\theta, \;\; \frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{v} \mp \tilde{c}\sin\theta, \;\; \frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tilde{u}, \;\; \frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{v}, \;\; \frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.$$

- Bicharacteristic cone

`diagrams/cone.pdf`

Transport equations

Transport equations along the bicharacteristics

$$-\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{\tilde{c}}{g}\left(\cos\theta\frac{\mathrm{d}u}{\mathrm{d}t} + \sin\theta\frac{\mathrm{d}v}{\mathrm{d}t}\right) + \frac{\tilde{c}^2}{g}\left(\sin\theta\frac{\partial u}{\partial\lambda} - \cos\theta\frac{\partial v}{\partial\lambda}\right) = -\tilde{c}\frac{\partial b}{\partial\mu},$$

$$\cos\theta\frac{\mathrm{d}u}{\mathrm{d}t} + \sin\theta\frac{\mathrm{d}v}{\mathrm{d}t} - g\frac{\partial(h+b)}{\partial\lambda} = 0,$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{\tilde{c}}{g}\left(\cos\theta\frac{\mathrm{d}u}{\mathrm{d}t} + \sin\theta\frac{\mathrm{d}v}{\mathrm{d}t}\right) - \frac{\tilde{c}^2}{g}\left(\sin\theta\frac{\partial u}{\partial\lambda} - \cos\theta\frac{\partial v}{\partial\lambda}\right) = -\tilde{c}\frac{\partial b}{\partial\mu}.$$

Tangential and normal derivatives $\partial/\partial\lambda$ and $\partial/\partial\mu$

$$\frac{\partial}{\partial\lambda} := -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial\mu} := \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}.$$

Remark

- *The derivatives* $\mathrm{d}/\mathrm{d}t$ *are the only ones involving* ∂/∂_t.
- *The tangential derivatives can be expressed in terms of* $\partial/\partial\theta$.

Exact evolution operators (Lukacova, Noelle, Kraft)

$$
\begin{aligned}
h(P) = {}& \frac{1}{2\pi}\int_0^{2\pi}\left(h(Q) - \frac{\tilde{c}}{g}u(Q)\cos\theta - \frac{\tilde{c}}{g}v(Q)\sin\theta\right)\mathrm{d}\theta \\
& - \frac{1}{2\pi}\int_{t^n}^{t^{n+1}}\frac{1}{t^{n+1}-\tilde{t}}\int_0^{2\pi}\frac{\tilde{c}}{g}\left(u(\tilde{Q})\cos\theta + v(\tilde{Q})\sin\theta\right)\mathrm{d}\theta\mathrm{d}\tilde{t} \\
& + \frac{\tilde{c}}{2\pi}\int_{t^n}^{t^{n+1}}\int_0^{2\pi} b_\mu(\tilde{Q})\mathrm{d}\theta\mathrm{d}\tilde{t},
\end{aligned}
$$

$$
\begin{aligned}
u(P) = {}& \frac{1}{2\pi}\int_0^{2\pi}\left(-\frac{g}{\tilde{c}}h(Q)\cos\theta + u(Q)\cos^2\theta + v(Q)\sin\theta\cos\theta\right)\mathrm{d}\theta \\
& + \frac{1}{2}u(Q_0) - \frac{g}{2}\int_{t^n}^{t^{n+1}}\left(h_x(\tilde{Q}_0) + b_x(\tilde{Q}_0)\right)\mathrm{d}\tilde{t} \\
& + \frac{1}{2\pi}\int_{t^n}^{t^{n+1}}\frac{1}{t^{n+1}-\tilde{t}}\int_0^{2\pi}\frac{\tilde{c}}{g}\left(u(\tilde{Q})\cos 2\theta + v(\tilde{Q})\sin 2\theta\right)\mathrm{d}\theta\mathrm{d}\tilde{t} \\
& - \frac{g}{2\pi}\int_{t^n}^{t^{n+1}}\int_0^{2\pi} b_\mu(\tilde{Q})\cos\theta\mathrm{d}\theta\mathrm{d}\tilde{t}.
\end{aligned}
$$

Need for a better linearisation

Linearisation about a constant state

1. We have linearised the shallow water system about a constant state and the resulting linearised bicharacteristics are always straight lines.
2. In many situations, e.g. at low Froude numbers, the gravitational waves travel much faster than the advection waves.
3. The fast gravity waves quickly run across the shore and leave the domain, while the nonlinear advection waves slow down and come to halt as they approach the shore.
4. In such cases, the nonlinear bicharacteristic cone can have a very anisotropic structure compared to the linearised one.

Linearisation contd..

Linearisation about a space dependent state

1. Similar situations arise when the flow behaviour is complex, such as wave run-up over beaches, interaction with solid structures like wave barriers.
2. The spatial inhomogeneity in the wave speed causes the diffraction of the bicharacteristics and they no longer remain as straight lines.
3. Hence, the idea of linearisation about a constant state may not be satisfactory, and we require a better linearisation, e.g. about a space-dependent state.

Bicharacteristics

- Linearise the shallow water system about a state $(u, v, c) = (\tilde{u}, \tilde{v}, \tilde{c})(x, y)$.
- Evolution of the bicharacteristics are governed by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tilde{u} \mp \tilde{c}\cos\theta,\ \frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{v} \mp \tilde{c}\sin\theta,\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\cos\theta\frac{\partial\tilde{u}}{\partial\lambda} - \sin\theta\frac{\partial\tilde{v}}{\partial\lambda} \pm \frac{\partial\tilde{c}}{\partial\lambda}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tilde{u},\ \frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{v},\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = -\cos\theta\frac{\partial\tilde{u}}{\partial\lambda} - \sin\theta\frac{\partial\tilde{v}}{\partial\lambda}$$

Remark

- *Spatial variation of* $(\tilde{u}, \tilde{v}, \tilde{c})$ *is taken into account.*
- *Normal to the wavefront* $(\cos\theta, \sin\theta)$ *changes its orientation with time.*
- *In other words, the bicharacteristics diffract.*

Anisotropic cone

Example of an anisotropic cone (Arun, Kraft, Lukacova, Prasad)

diagrams/oblique_cone.pdf

Figure: Anisotropic cone

Generalised evolution operators

$$
\begin{aligned}
h(P) = \frac{1}{2\pi}\int_0^{2\pi}\left(h - \frac{\tilde{c}}{g}u\cos\theta - \frac{\tilde{c}}{g}v\sin\theta\right)(Q)\mathrm{d}\omega \\
- \frac{1}{2\pi g}\int_0^{2\pi}\int_{t^n}^{t^{n+1}}\left\{\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{c}\cos\theta)u + \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{c}\sin\theta)v\right\}(\tilde{Q})\mathrm{d}\tilde{t}\mathrm{d}\omega \\
+ \frac{1}{2\pi g}\int_0^{2\pi}\int_{t^n}^{t^{n+1}}\tilde{c}^2\left(\sin\theta\frac{\partial u}{\partial\lambda} - \cos\theta\frac{\partial v}{\partial\lambda}\right)(\tilde{Q})\mathrm{d}\tilde{t}\mathrm{d}\omega \\
+ \frac{1}{2\pi}\int_0^{2\pi}\int_{t^n}^{t^{n+1}}\tilde{c}b_\mu(\tilde{Q})\mathrm{d}\tilde{t}\mathrm{d}\omega,
\end{aligned}
$$

Remark

This evolution operator is the generalisation of the ones obtained by Lukacova and collaborators.

Generalised evolution operators contd...

$$
\begin{aligned}
u(P) = {} & \frac{g}{2\pi\tilde{c}(P)} \int_0^{2\pi} \cos\omega \left(-h + \frac{\tilde{c}}{g} u\cos\theta + \frac{\tilde{c}}{g} v\sin\theta \right)(Q)\mathrm{d}\omega \\
& + \frac{1}{2\pi} \int_0^{2\pi} \sin\omega \left(u\sin\theta - v\cos\theta \right)(Q_0)\mathrm{d}\omega \\
& + \frac{1}{2\pi\tilde{c}(P)} \int_0^{2\pi} \cos\omega \int_{t^n}^{t^{n+1}} \left\{ \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{c}\cos\theta)u + \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{c}\sin\theta)v \right\}(\tilde{Q})\mathrm{d}\tilde{t}\mathrm{d}\omega \\
& + \frac{1}{2\pi} \int_0^{2\pi} \sin\omega \int_{t^n}^{t^{n+1}} \left\{ \frac{\mathrm{d}}{\mathrm{d}t}(\sin\theta)u - \frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta)v \right\}(\tilde{Q}_0)\mathrm{d}\tilde{t}\mathrm{d}\omega \\
& + \frac{g}{2\pi} \int_0^{2\pi} \sin\omega \int_{t^n}^{t^{n+1}} \frac{\partial(h+b)}{\partial\lambda}(\tilde{Q})\mathrm{d}\tilde{t}\mathrm{d}\omega \\
& - \frac{1}{2\pi\tilde{c}(P)} \int_0^{2\pi} \cos\omega \int_{t^n}^{t^{n+1}} \tilde{c}^2 \left(\sin\theta\frac{\partial u}{\partial\lambda} - \cos\theta\frac{\partial v}{\partial\lambda} \right)(\tilde{Q})\mathrm{d}\tilde{t}\mathrm{d}\omega \\
& - \frac{g}{2\pi\tilde{c}(P)} \int_0^{2\pi} \cos\omega \int_{t^n}^{t^{n+1}} \tilde{c} b_\mu(\tilde{Q})\mathrm{d}\tilde{t}\mathrm{d}\omega.
\end{aligned}
$$

Remarks

- *Exact evolution operators are implicit.*
- *They contain the time integrals of the unknowns and their derivatives.*
- *Integrals along the Mach cone are too complex and to be simplified.*
- *We approximate the time integrals at $t = t_n$ to get an explicit relation.*
- *Unknown derivatives can be eliminated by integration by parts along the Mach cone.*
- *Leads to the so-called approximate evolution operators.*

Example: approximate evolution operators for the wave equation system

$$\phi(P) = \frac{1}{2\pi} \int_0^{2\pi} \left(\phi(Q_1) - 2u(Q_1)\cos\theta - 2v(Q_1)\sin\theta\right) \mathrm{d}\theta,$$

$$\begin{aligned} u(P) = \frac{1}{2}u(Q_2) + \frac{1}{2\pi}\int_0^{2\pi} &\left(-\phi(Q_1)\cos\theta + u(Q_1)(3\cos^2\theta - 1)\right. \\ &\left. + 3v(Q_1)\sin\theta\cos\theta\right) \mathrm{d}\theta. \end{aligned}$$

Remarks

- Approximate evolution operators do not depend on the derivatives of (h, u, v).
- If one uses a Taylor expansion for prediction, the space derivatives are inevitable.
- Hence, evolution operators are less dependent on the reconstruction of piecewise constant data than the Taylor expansion.
- Integrations along the Mach cone correctly takes into account of the domain of dependence.
- This results in a robust algorithm, particularly for multidimensional problems.

Theorem (Arun, Noelle)

Suppose that the reconstructions satisfy for all x, y

$$h^n(x, y) + b(x, y) = \textit{const},\ u^n(x, y) = v^n(x, y) = 0,$$

then the approximate evolution operators satisfy the well-balanced property:

$$\hat{h} + \hat{b} = \textit{const},\ \hat{u} = \hat{v} = 0.$$

Finite volume update

- Cartesian grid with cell averages, need to compute fluxes over cell interfaces.
- Choose Gauss quadrature points on the cell interface $\mathcal{E}$.
- Construct local Mach cones at the quadrature nodes.
- Use the evolution operator to predict the solution at half time step $t^{n+\frac{1}{2}} := t^n + \frac{\Delta t}{2}$.
- Compute the interface flux using

$$\frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} \mathbf{f}\left(\mathbf{u}^{n+\frac{1}{2}}\right) \mathrm{d}s = \sum_k \alpha_k \mathbf{f}\left(\mathbf{u}_k^{n+\frac{1}{2}}\right).$$

Discretisation of source terms

- In each cell, we need to discretise $-ghb_x$ and $-ghb_y$.
- We use same quadrature as used in the flux evaluation.
- In a cell $C_{i,j}$ we take

$$\mathbf{s}_{i,j}^{n+\frac{1}{2}} = -g \sum_k \alpha_k \begin{pmatrix} 0 \\ \frac{1}{2}\left(\hat{h}_k^r + \hat{h}_k^l\right)\left(\hat{b}_k^r - \hat{b}_k^l\right) \\ \frac{1}{2}\left(\hat{h}_k^t + \hat{h}_k^b\right)\left(\hat{b}_k^t - \hat{b}_k^b\right) \end{pmatrix}$$

Scheme

Final scheme

$$\mathbf{u}_{i,j}^{n+1} = \mathbf{u}_{i,j}^{n} - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+\frac{1}{2},j}^{n+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2},j}^{n+\frac{1}{2}}\right) - \frac{\Delta t}{\Delta y}\left(\mathcal{F}_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} - \mathcal{F}_{i,j-\frac{1}{2}}^{n+\frac{1}{2}}\right) + \mathbf{s}_{i,j}^{n+\frac{1}{2}}.$$

- Multidimensional generalisation of Godunov's idea to reconstruct, evolve and average.
- Genuine multidimensional flow features are captured very well.
- Very accurate compared to dimension split finite volume schemes.

Theorem (Lukacova, Noelle, Kraft; Arun, Noelle)

If the predicted values $(\hat{h}, \hat{u}, \hat{v})$ *at the interfaces satisfy*

$$\hat{h} + \hat{b} = \mathit{const}, \;\; \hat{u} = \hat{v} = 0,$$

then the above FVEG scheme preserves the steady lake at rest, i.e.

$$h + b \equiv \mathit{const}, \;\; u \equiv v \equiv 0.$$

Advection of a vortex

- To test the performance of FVEG scheme for low Froude numbers.
- Consider a linearised problem

$$
\begin{aligned}
h_t + \tilde{u} h_x + \tilde{v} h_y + \tilde{c}(u_x + v_y) &= 0,\\
u_t + \tilde{u} u_x + \tilde{v} u_y + \tilde{c} h_x &= 0,\\
v_t + \tilde{u} v_x + \tilde{v} v_y + \tilde{c} h_y &= 0.
\end{aligned}
$$

- Froude number

$$
Fr := \frac{\sqrt{\tilde{u}^2 + \tilde{v}^2}}{\tilde{c}}.
$$

- Initial velocity: vortex

$$
u(x,y,0) = -\Gamma y e^{\left(\frac{1-r^2}{2}\right)},\ v(x,y,0) = \Gamma x e^{\left(\frac{1-r^2}{2}\right)}.
$$

- Vorticity advects with the flow: $\omega(x,y,t) = \omega(x - \tilde{u}t, y - \tilde{v}t)$.

Numerical results

figures/vorticity_ex.pdf figures/vorticity_m1p0.pdf

figures/vorticity_m0p1.pdf figures/vorticity_m0p01.pdf

Figure: Advection of a vortex: $\tilde{u} = 1, \tilde{v} = 0$. Froude numbers are $Fr = 1, 0.1, 0.01$.

Numerical results contd.

figures/vorticity_sect.pdf

Figure: Cross section of the vortex along the x-axis at $t = 3$.

Developments

Numerical simulation of low Mach number flows.

- Early development: simulation of incompressible flows by Chorin.
- Rigorous convergence results: compressible isentropic flow $\rightarrow$ incompressible flow by Klainerman and Majda, Schochet, for combustion by Majda and Sethian.
- Difficulties associate with low Froude number:
  1. Stiffness: disparity in wavespeeds of gravity and advection waves, poses severe restriction on the time-step due to CFL condition.
  2. Cancellation: water height variable has to accommodate $\mathcal{O}(1)$ constant height and physically valid $\mathcal{O}(\varepsilon^2)$ fluctuations, leading to round-off errors.
  3. Accuracy: numerical viscosity depends on the Froude number, can cause truncation error to grow as $\varepsilon \rightarrow 0$.
- Stiffness issue: Preconditioning approach by Turkel, characteristic time-stepping by vanLeer, Roe and others.

Developments contd.

- Combination of compressible flow solution and projection methods based on asymptotic expansion by Klein, Munz and coworkers.
- Multiple pressure variable approach:

$$p = p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)}.$$

- Numerical analysis of Roe-type schemes on Cartesian grids by Guillard and Viozat.
- Analysis of Godunov-type schemes by Dellacherie, Omnes and Rieper.
- Two-grid algorithm due to Le Maître and coworkers.
- Low Mach number scheme of Jin and coworkers.

Non-dimensional system

Non-dimensional shallow water system: (Le Maître et al.)

$$\eta_t + \nabla \cdot ((\eta + h)\mathbf{u}) = 0,$$
$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\varepsilon^2} \nabla \eta = 0.$$

- $z = \eta(x, y, t)$: free surface elevation,
- $z = b(x, y)$: bottom topography,
- $h = \eta - b$: water height
- $\mathbf{u}$: velocity vector.

Reference values:

- $L_{ref}, t_{ref}, u_{ref}$.
- For gravity waves $c_{ref} = \sqrt{g h_{ref}}$.
- Froude number $\varepsilon := u_{ref}/c_{ref}$.
- Interested in the regime $\varepsilon \ll 1$.

Asymptotic analysis

- Aim: to identify behaviour of the system as $\varepsilon \to 0$.
- Asymptotic ansatz

$$f(x,t;\varepsilon) = f^{(0)}(x,\xi,t) + \varepsilon f^{(1)}(x,\xi,t) + \varepsilon^2 f^{(2)}(x,\xi,t) + \cdots ,$$

  $\xi = \varepsilon x$.
- Multiple space scales and single time scale.
- Use the ansatz for all the unknowns.
- Balancing the powers of ε gives the asymptotic equations.

Summary of asymptotic analysis (Klein, Le Maître et al.)

- Leading order elevation $\eta^{(0)}$ is constant in space, i.e. $\eta^{(0)} = \eta^{(0)}(t)$.
- It can change due to mass flux from boundary:

$$\eta_t^{(0)} = -\frac{1}{|A|} \int_{\partial A} \left(h + \eta^{(0)}\right) \mathbf{u}^{(0)} \cdot \mathbf{n} \mathrm{d}\sigma.$$

- Leads to a divergence constraint on the leading order momentum

$$\nabla_x \cdot \left(h + \eta^{(0)}\right) = -\eta_t^{(0)}.$$

- First order term $\eta^{(1)}$ does not admit small-scale variations, i.e. $\eta^{(1)} = \eta^{(1)}(\xi, t)$.
- It can be interpreted as the amplitude of a gravity wave.
- Large scale components are filtered out by the averaging operator

$$\overline{f}(\xi, t) := \frac{1}{|B(0, \frac{1}{\varepsilon})|} \int_{B(0, \frac{1}{\varepsilon})} f(x, \xi, t) \mathrm{d}x.$$

.

Summary contd.

- Long scale equations.

$$\eta^{(1)} + \nabla_\xi \cdot \overline{\left(h+\eta^{(0)}\right)\mathbf{u}^{(0)}} = 0,$$
$$\left(\overline{\left(h+\eta^{(0)}\right)\mathbf{u}^{(0)}}\right)_t + \overline{\left(h+\eta^{(0)}\right)\nabla_x \eta^{(2)}} = -\overline{\left(h+\eta^{(0)}\right)\nabla_\xi \eta^{(1)}}.$$

Remark

- *Gravity wavespeed becomes infinite as* $\varepsilon \to 0$.
- *Large scale derivatives vanish and* $\eta^{(1)}$ *becomes constant in space and time.*
- *Final zero Froude number limit equations cannot contain gravity waves.*
- *In accordance with Klainerman and Majda.*

Zero Froude number equations

$$\nabla \cdot (\eta + h)\mathbf{u} = -\frac{\mathrm{d}\eta}{\mathrm{d}t},$$
$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\eta^{(2)} = 0.$$

Decomposition

- Conservation form

$$
\begin{aligned}
H_t + \nabla \cdot H\mathbf{u} &= 0, \\
(H\mathbf{u})_t + \nabla \cdot \left(H\mathbf{u} \otimes \mathbf{u} + \frac{H^2}{2\varepsilon^2} \right) &= \frac{H \nabla h}{\varepsilon^2},
\end{aligned}
$$

$H := (h + \eta)$.

- Splitting into two subsystems:
- 'Fast' subsystem (linear part)

$$
\begin{aligned}
H_t + \nabla \cdot H\mathbf{u} &= 0, \\
(H\mathbf{u})_t + \nabla \left(\frac{\eta h}{\varepsilon^2} \right) &= \frac{\eta \nabla h}{\varepsilon^2}.
\end{aligned}
$$

- 'Slow' subsystem (nonlinear part)

$$
\begin{aligned}
H_t &= 0, \\
(H\mathbf{u})_t + \nabla \cdot \left(H\mathbf{u} \otimes \mathbf{u} + \frac{\eta^2}{2\varepsilon^2} \right) &= 0.
\end{aligned}
$$

Eigenvalues

- Full system

$$\lambda_{1,3} = u\cos\theta + v\sin\theta \mp \frac{\sqrt{h}}{\varepsilon},$$
$$\lambda_2 = u\cos\theta + v\sin\theta.$$

- Fast subsystem

$$\lambda_1 = -\frac{\sqrt{h}}{\varepsilon},\ \lambda_2 = 0,\ \lambda_3 = \frac{\sqrt{h}}{\varepsilon}.$$

- Slow subsystem

$$\lambda_1 = 2u\cos\theta + v\sin\theta,\ \lambda_2 = 0,\ \lambda_3 = u\cos\theta + 2v\sin\theta.$$

- *Eigenvalues are* $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$ *for fast subsystem.*
- *They are* $\mathcal{O}\left(1\right)$ *for slow subsystem.*
- *Both the subsystems are hyperbolic.*

Flux splitting method

- Full system in the quasi one-dimensional case

$$W_t + F_1(W)_x = 0.$$

- Split fluxes

$$F_1(W) = \tilde{F}_1(W) + \hat{F}_1(W),$$

- $\tilde{F}_1$: fast subsystem, $\hat{F}_1$: slow subsystem.
- Finite volume discretisation

$$W_j^{n+1} = W_j^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}}\right).$$

Strategy:

$$\mathcal{F}_{j+\frac{1}{2}} = \tilde{\mathcal{F}}_{j+\frac{1}{2}} + \hat{\mathcal{F}}_{j+\frac{1}{2}}$$

Contd.

- *Since slow system has* $\mathcal{O}(1)$ *eigenvalues, we compute the numerical fluxes on a fine grid with mesh size* $\Delta x = \mathcal{O}(1)$.
- *Since the eigenvalues of the fast system are* $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$, *numerical fluxes for the fast system are computed on a coarse grid with mesh size* $\Delta\xi = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$.
- *Interpolate the coarse fluxes onto the fine grid.*

- *Time-step for slow system*

$$\Delta t_f \frac{\hat{\lambda}_{max}}{\Delta x} = CFL.$$

- *Time-step for fast system*

$$\Delta t_c \frac{\tilde{\lambda}_{\max}}{\Delta \xi} = CFL.$$

- *Both* Δt_f *and* Δt_c *are* $\mathcal{O}(1)$.
- *Choose*

$$\Delta t = \min(\Delta t_f, \Delta t_c).$$

A test problem (Le Maître et al.)

- One dimensional channel of length $L_{ref} = 3600Km$ with flat bottom.
- Reference depth $h_{ref} = 1Km$.
- Periodic boundary conditions.
- Initial values

$$u(x,0) = v(x,0) = 0,\ \eta(x,0) = ae^{-\left(\frac{\left(\frac{x}{L_{ref}} - \frac{1}{2}\right)^2}{0.005}\right)},$$

  $a = 0.5m$
- Linearised solution consists of two waves going to the left and right with speed $c = \sqrt{gh} \sim 100m/s$.
- Linearised solution

$$\eta(x,t) = \frac{1}{2}\left\{\eta(x-ct) + \eta(x+ct)\right\}$$
$$u(x,t) = \frac{g}{2c}\left\{\eta(x-ct) - \eta(x+ct)\right\}$$

- Reference velocity $u_{ref} = ga/c = 0.05m/s$.
- Froude number $\varepsilon = u_{ref}/c_{ref} = 5 \times 10^{-4}$.

setup contd.

- Use 360 fine mesh points, i.e. $\Delta x = 10Km$.
- Coarse mesh size $\Delta \xi = r\Delta x = 30Km$, i.e. $r = 3$.
- First order Lax-Friedrichs numerical fluxes.
- Euler time-stepping.
- $CFL = 0.95$
- Linear interpolation of coarse flux-differences onto the fine grid.
- After each time step, solution is smoothed using

$$W_j \to \frac{1}{2}\left(W_{j-1} + W_{j+1}\right)$$

Remark

Without smoothing, the solution contains high frequency oscillations!

Results

figures/elevation_t0t4.pdf

Figure: Surface elevation computed using the split scheme. Final time $t = 4Hrs$

Results

- Effect of grid coarsening.

figures/elevation_r3.pdf

figures/elevation_r6.pdf

Figure: Effect of grid coarsening

Results contd.

figures/elevation_r12.pdf

figures/elevation_r24.pdf

Figure: Effect of grid coarsening

Conclusions

FVEG scheme for shallow water equations

- Generalised evolution operators are derived by considering a space dependent linearisation state.
- Evolution operators are approximated using appropriate numerical quadratures.
- Approximate evolution operators satisfy the conditions for well-balancing.
- Well-balanced genuinely multidimensional FVEG scheme is derived using the approximate evolution operators.

Conclusions contd..

Flux-splitting scheme for low Froude number

- Conservative hyperbolic splitting is introduced.
- Guided by the asymptotic considerations, the shallow water system is split into two subsystems.
- Wave velocities of the 'fast' subsystem are $\mathcal{O}(\frac{1}{\varepsilon})$ and that of 'slow' system are $\mathcal{O}(1)$.
- Two-grid algorithm based on fine-coarse grid is proposed.
- Numerical fluxes are evaluated on different grids.
- Fluxes are assembled to get the flux on a single grid.
- Two-grid algorithm overcomes the stiffness due to CFL restriction.

Future work

- Two-grid framework has to be incorporated to the FVEG scheme.
- The method has be validated against benchmark problems in multidimensions.
- Include nonzero bottom topography and study its influence on the propagation of long waves.
- Well-balancing has to be introduced in the new framework.
- Efficient filtering has to be introduced to remove the high frequency oscillations.
- Better flux interpolation strategies.
- Extension to high order accuracy.

Thank You for Your Kind Attention!