Coupling of Hyperbolic PDEs : Thin versus Thick Coupling Interfaces

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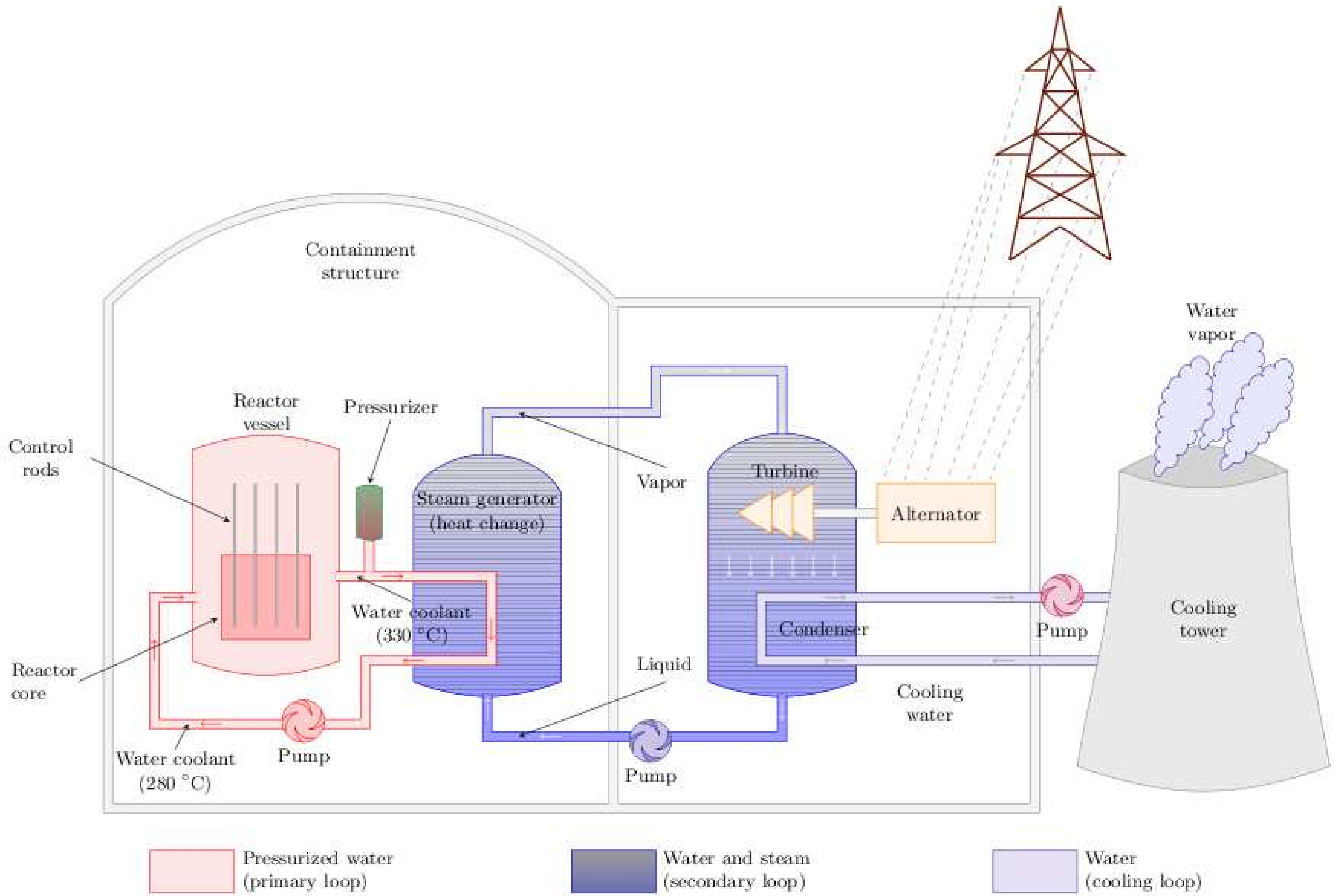
Foreword : A Physical Picture of the Industrial Question

A large operating system made up of various sub-components
modeled by flow problems with distinct physical scales



Foreword : Coupling distinct flow solvers

A vividly rising question in many distinct industrial settings

A second life for existing in-house softwares with enhanced capabilities

- A universal closure law is too expensive to describe the whole operating system : the device is decomposed into subcomponents, each being simulated by a specific software.
- *Target* : transient coupling of the existing softwares to improve the performance and reliability in simulating the whole operating system and not simply subcomponents of it.

The real industrial question is not only to know how to couple existing softwares but to lower the manpower cost needed to implement and validate the mathematical solution

Outline

- ▷ The industrial question : coupling of existing simulation softwares
- ▷ Phrasing mathematically the problem : coupling PDEs within a nested hierarchy of relaxation models
- ▷ From an ideal to the real world : Discontinuity in the modeling
- ▷ Coupling via infinitely thin interfaces : Preserve as far as possible the existing softwares
- ▷ The resonance phenomena : Failure of uniqueness
- ▷ Coupling via regularized (thick) interfaces : Restore uniqueness

Special emphasis put on
Well-Balanced numerical issues :

- ▷ **Hyperbolic equations with singular (measure-valued) source term**
- ▷ **Hyperbolic equations with smooth source term**

Main Collaborators

- **Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie - Paris 6**
  Christophe Chalons, Filipa Caetano, FC, Edwige Godlewski,
  Frédéric Lagoutière, Philippe Le Floch, Pierre-Arnaud Raviart, Nicolas Seguin
- **French Atomic Agency (CEA Saclay)**
  Annalisa Ambroso, Benjamin Boutin, Thomas Galiié, Samuel Kokh, Jacques Segré

- In close interaction with
  French Electricity Company (EDF R&D)
  Jean-Marc Hérard, Laetitia Girault, Olivier Hurisse, Khaled Saleh

Phrasing Mathematically the Problem

Let be given a multi-scale flow problem (typically a multiphase flow problem) over a physical domain, where scales are well separated in given sub-regions whose boundaries may vary with time, or simply appear or disappear

- At your hand : a complete hierarchy of nested (hyperbolic) PDE models formally arranged according to the typical scale their are supposed to capture
- *Target* : Determine on the fly (*during the computation*) the PDE model that fits the best in terms of computational effort in a given sub-region

Phrasing Mathematically the Problem. *Con't*

- *Target* : Determine on the fly the PDE model that fits the best in a given sub-region
  - Propose a mathematical coupling theory of hyperbolic PDEs with distinct phase space dimension, with different physical space dimension.
  - Explore coupling theory in the regime of a singular coupling interface (*i.e.* infinitely thin) and the one of a regularized interface (shake-hand coupling region).
  - Propose a mathematical *a posteriori* modeling error analysis (coupled with "a standard" *a posteriori* discretization error analysis) to adapt in time the location of the boundaries and the meshing of the sub-regions (with or without overlapping).

Coupling of numerical codes in an ideal world

Two PDEs models with distinct size separated by a coupling zone $|x| < \eta$:

CODE 1	CODE 2
$\partial_t \mathbf{W} + \partial_x \mathbf{F}(\mathbf{W}) = 0$ $x < -\eta$	$\partial_t \mathbf{w} + \partial_x \mathbf{f}^{eq}(\mathbf{w}) = 0$ $x > \eta$

$x = 0$

$\begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{g}(\mathbf{w}, \mathbf{v}) = 0, \\ \partial_t \mathbf{v} + \partial_x \mathbf{h}(\mathbf{w}, \mathbf{v}) = \frac{1}{\tau}(\mathbf{v}^{eq}(\mathbf{w}) - \mathbf{v}) \end{cases}$	$\partial_t \mathbf{w} + \partial_x \mathbf{g}(\mathbf{w}, \mathbf{v}^{eq}(\mathbf{w})) = 0.$
$\tau(\mathbf{W}) = \mathcal{O}(1)$	$\tau(\mathbf{W}) = \mathcal{O}(\epsilon) << 1$

From ideal to real worlds

- We do not deal in practice with the expected equilibrium model in the limit $\epsilon \to 0$, namely with closure flux function $\mathbf{f}^{eq}(\mathbf{w})$.
- We thus deal with a distinct closure flux function in $\{x > 0, t > 0\}$, say $\mathbf{f}_+(\mathbf{w})$

behaviour close to equilibrium	Avalaible equilibrium model
$\partial_t \mathbf{w} + \partial_x \mathbf{f}^{eq}(\mathbf{w}) = 0$	$\partial_t \mathbf{w} + \partial_x \mathbf{f}_+(\mathbf{w}) = 0$
$x < 0$	$x > 0$

$x = 0$

- The flux functions are discontinuous at the coupling interface $\{x = 0\}$ (*i.e.* at the exit of possible relaxation boundary layers).

The thin interface : a double IBVP formalism

$\partial_t \mathbf{w} + \partial_x \mathbf{f}_-(\mathbf{w}) = 0$	Model coupling	$\partial_t \mathbf{w} + \partial_x \mathbf{f}_+(\mathbf{w}) = 0$
$IBVP \quad x < 0$	BC at $x = 0$	$IBVP \quad x > 0$

Express boundary conditions to link $\mathbf{w}(t, 0^-)$ with $\mathbf{w}(t, 0^+)$

General rule : **infinitely many distinct** pairs of boundary conditions may be prescribed :

- They model some expected continuity properties at $x = 0$ for the solution $\mathbf{w}$ or for some nonlinear transform of it
- Various conservation properties may be privileged but in general these dictate in turn the resulting continuity properties.
- The different solutions of the resulting coupled problems stay close (numerical evidences) provided that $\mathbf{f}_+$ does not depart too much from $\mathbf{f}_-$

Coupling modeling via thin interfaces

Possibly expected continuity properties at $x = 0$ for the solution of the coupled problem

- Flux continuity, namely a conservative coupling $\mathbf{f}_-(\mathbf{w}(t, 0^-)) = \mathbf{f}_+(\mathbf{w}(t, 0^+))$
- Unknown continuity, a non conservative coupling $\mathbf{w}(t, 0^-) = \mathbf{w}(t, 0^+)$
- Continuity of others nonlinear transformations $\gamma_\pm$ (invertible !) $\gamma_-^{-1}(\mathbf{w}(t, 0^-)) = \gamma_+^{-1}(\mathbf{w}(t, 0^+))$
- Possible blending ensuring specific conservation properties with other continuity properties

Example : The 3×3 *Euler equations with two distinct pressure laws*
$\mathbf{w} = (\rho,\ \rho u,\ \rho E) \to \mathbf{u} = (\rho, \rho u, p_\pm)$

- Some additional information from the physics must be added to promote a particular set of continuity properties.
- Small differences in the closure $|\mathbf{f}_+ - \mathbf{f}_-|$ result in small variations in the coupled solutions

Coupling modeling via thin interfaces

- A useful change of variable

$$\mathbf{u}(x,t) = \begin{cases} \gamma_-^{-1}(\mathbf{w})(x,t), & x < 0, \quad t > 0, \\ \gamma_+^{-1}(\mathbf{w})(x,t), & x > 0, \quad t > 0. \end{cases}$$

$\partial_t \gamma_-(\mathbf{u}) + \partial_x \mathbf{f}_-(\gamma_-(\mathbf{u})) = 0$	$\partial_t \gamma_+(\mathbf{u}) + \partial_x \mathbf{f}_+(\gamma_+(\mathbf{u})) = 0$

$$\mathbf{u}(t,0^-) = \mathbf{u}(t,0^+)$$

Choosing $\gamma_\pm$ invertible does preserve the time arrow.

- Propose guidelines to promote a particular set of transmission conditions

Partial Guidelines for promoting the transmission conditions

Needed additional information for promoting a given set of transmission conditions

- ▷ The definition of thermal and mechanical equilibria are dictated (*i.e.* the mathematical definition of constant in time and space solution)

  Two distinct states $\mathbf{w}_-, \mathbf{w}_+$ with the continuity property $\mathbf{u} = \gamma_-(\mathbf{w}_-) = \gamma_+(\mathbf{w}_+)$

  examples : constant (density, velocity, pressure) versus constant (velocity , pressure, temperature)

- ▷ The transient behaviour of the coupling interface : Numerical investigation, Mathematical analysis of the long time-behaviour of the coupled solutions :

  A common corner stone : the study of the Coupled Riemann Problem

$$\begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{f}_-(\mathbf{w}) = 0, \quad x < 0, \\ \mathbf{w}(0,x) = \mathbf{w}_L, \\ \mathbf{w}(t,0^-) = \gamma_-(\mathbf{u}(t,0^+)) \end{cases} \qquad \begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{f}_+(\mathbf{w}) = 0, \quad x > 0, \\ \mathbf{w}(0,x) = \mathbf{w}_R, \\ \mathbf{w}(t,0^+) = \gamma_+(\mathbf{u}(t,0^-)) \end{cases}$$

where in a strong sense $\mathbf{u}(t,0^-)'' = ''\mathbf{u}(t,0^+), \quad t > 0.$

A canonical example : Euler equations with distinct pressure laws

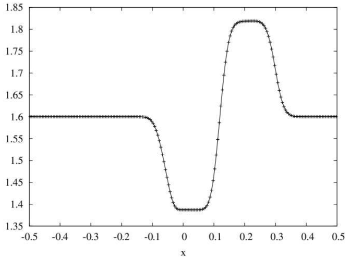
CODE 1	CODE 2
$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p_-) = 0 \\ \partial_t \rho E + \partial_x (u(\rho E + p_-)) = 0 \end{cases}$	$\begin{cases} \partial_t \rho + \partial_x \rho u = 0 \\ \partial_t \rho u + \partial_x (\rho u^2 + p_+) = 0 \\ \partial_t \rho E + \partial_x (u(\rho E + p_+)) = 0 \end{cases}$
$x < 0$	$x > 0$

$x = 0$

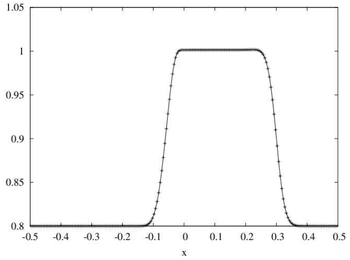
The two-pressure law are distinct : $p_-(.) \not\equiv p_+(.)$

For illustration purposes, p_- strongly departs from p_+

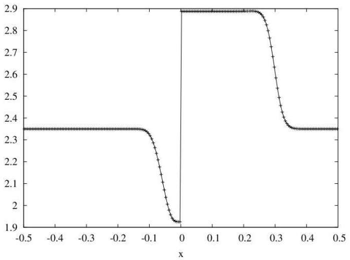
A First Coupling Strategy



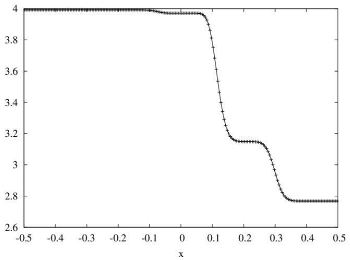
(a) Densité



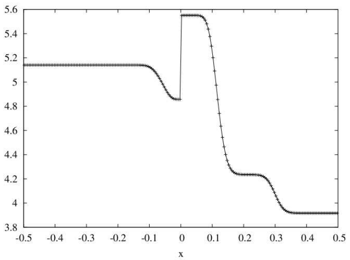
(b) Vitesse



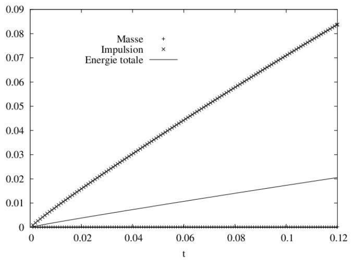
(c) Pression



(d) Énergie



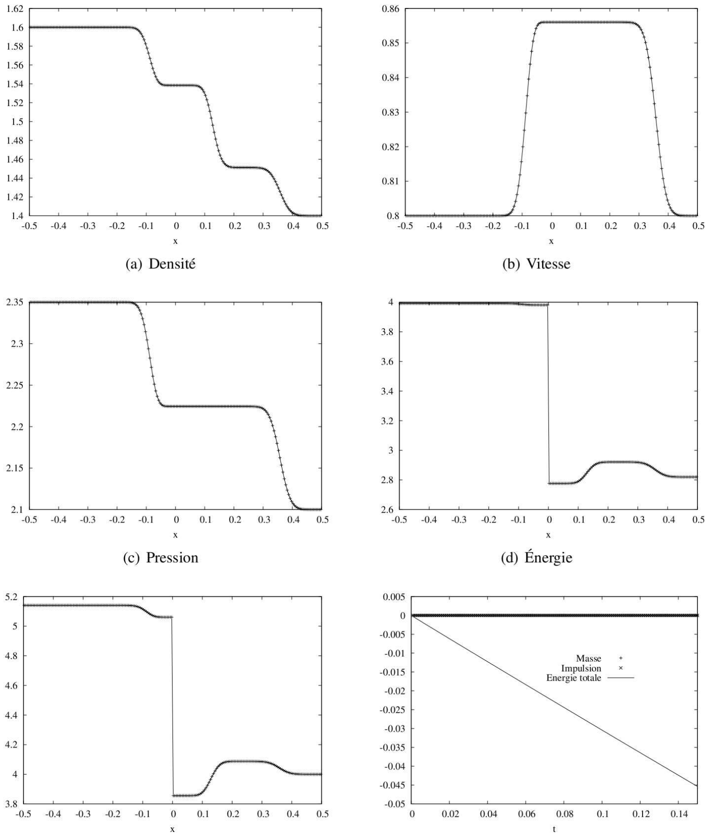
(e) Enthalpie



(f) Pertes de conservation relatives

$$(\rho\ ,\rho u,\ \rho E)(0^-,t) = (\rho\ ,\rho u,\ \rho E)(0^+,t)$$

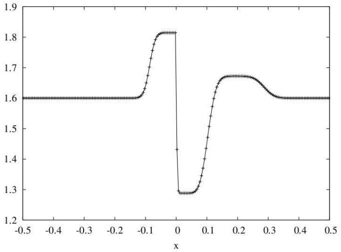
A Distinct Coupling Strategy



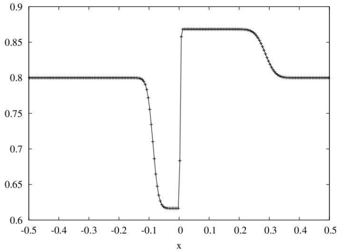
(a) Densité (b) Vitesse
(c) Pression (d) Énergie
(e) Enthalpie (f) Pertes de conservation relatives

$$(\rho\ ,\ u,\ p_-)(0^-,t) = (\rho\ ,u,\ p_+)(0^+,t)$$

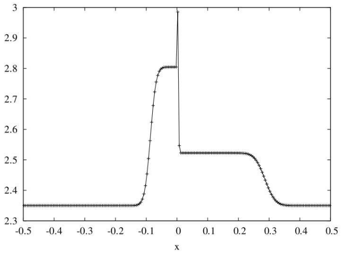
Fully conservative coupling for the Euler equations via relaxation



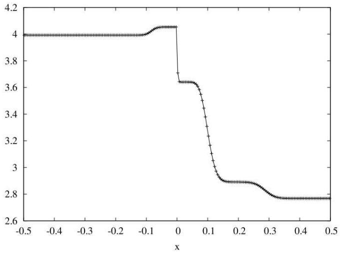
(a) Densité



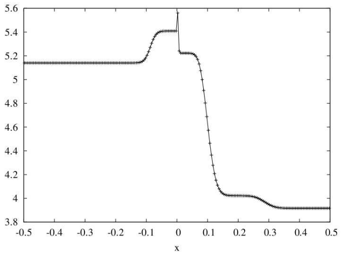
(b) Vitesse



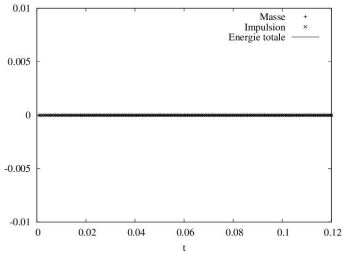
(c) Pression



(d) Énergie



(e) Enthalpie



(f) Pertes de conservation relatives

Closing the Gap with Well-Balanced Numerical Issues

$$\begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{f}_-(\mathbf{w}) = 0, \quad x < 0, \\ \mathbf{w}(0,x) = \mathbf{w}_0(x), \\ \mathbf{w}(t,0^-) = \gamma_-(\mathbf{u}(t,0^+)) \end{cases} \qquad \begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{f}_+(\mathbf{w}) = 0, \quad x > 0, \\ \mathbf{w}(0,x) = \mathbf{w}_0(x), \\ \mathbf{w}(t,0^+) = \gamma_+(\mathbf{u}(t,0^-)) \end{cases}$$

where in a strong sense $\mathbf{u}(t,0^-)'' = ''\mathbf{u}(t,0^+), \quad t > 0.$

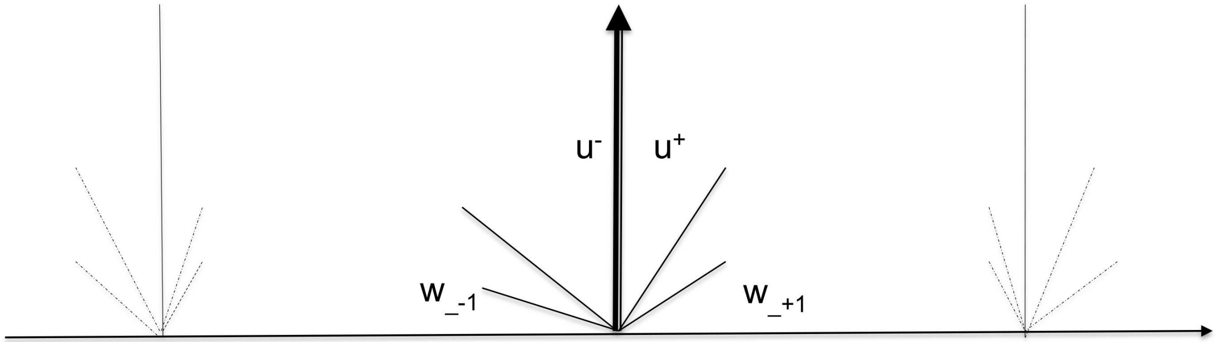
which can be readily rephrased as

$$\begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w},x) = \mathcal{M}(\mathbf{w})\, \delta_{x=0}, \quad x \in \mathbb{R}, \\ \mathbf{w}(0,x) = \mathbf{w}_0(x) \end{cases} \qquad \mathbf{f}(\mathbf{w},x) = \begin{cases} \mathbf{f}_-(\mathbf{w}), \ x < 0, \\ \mathbf{f}_+(\mathbf{w}), \ x > 0, \end{cases}$$

and where the Dirac mass $\mathcal{M}(\mathbf{w})$ is such that $\mathbf{u}(t,0^-)'' = ''\mathbf{u}(t,0^+), \quad t > 0.$

Well-Balanced numerical issues
for hyperbolic equations
with measure-valued source terms

The Exact Riemann Solver in the Thin Coupling Setting



x < 0
Riemann fan for $\mathbf{f_-}$

x = 0
Coupled Riemann fan

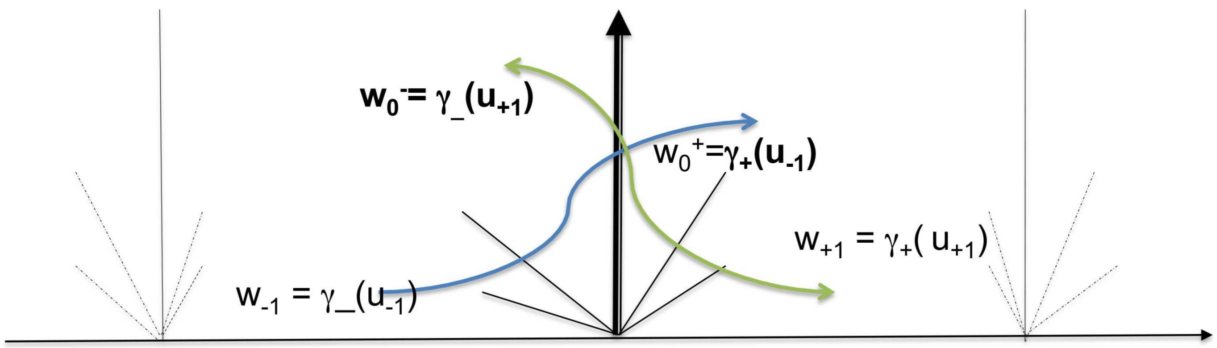
x > 0
Riemann fan for $\mathbf{f_+}$

Use two fluxes at the coupling interface x=0 :
left $f_-(\gamma_-(u_-))$ and **right** $f_+(\gamma_+(u_+))$

with $f_+(\gamma_+(u_+)) - f_-(\gamma_-(u_-)) = M(w)$
accounting for the u-transmission

A canonical well-balanced method but barely practicable

An Approximate Well-Balanced Numerical Solver



$x < 0$
Numerical flux $\mathbf{g_-}$ for $\mathbf{f_-}$

$x = 0$
Two Approximate
Riemann fans

$x > 0$
Numerical flux $\mathbf{g_+}$ for $\mathbf{f_+}$

Use the two numerical solvers at x=0
with reconstructed left and right data :

left $g_-(w_{-1}, \gamma_-(u_{+1}))$
right $g_+(\gamma_+(u_{--1}), w_{+1})$

Connexions with :
L. Gosse
S. Osher and co-workers

Towards an Augmented PDEs Formulation

Model the coupling problem over the entire real line :

$$\begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}, x) = \mathcal{M}(\mathbf{w})\, \delta_{x=0}, & x \in \mathbb{R}, \\ \mathbf{w}(0, x) = \mathbf{w}_0(x) \end{cases} \qquad \mathbf{f}(\mathbf{w}, x) = \begin{cases} \mathbf{f}_-(\mathbf{w}), & x < 0, \\ \mathbf{f}_+(\mathbf{w}), & x > 0, \end{cases}$$

and where the Dirac mass $\mathcal{M}(\mathbf{w})$ is such that $\quad \mathbf{u}(t, 0^-)'' = ''\mathbf{u}(t, 0^+), \quad t > 0.$

More convenient to deal with the u-transmission property :

$$\begin{cases} \mathcal{A}_0(\mathbf{u}, x)\partial_t \mathbf{u} + \mathcal{A}_1(\mathbf{u}, x)\partial_x \mathbf{u} = 0, & x \in \mathbb{R}, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) \end{cases}$$

with relevant consistency conditions on $\mathcal{A}_0(., x)$ and $\mathcal{A}_1(., x)$ for $\pm x > 0$

Discontinuity in the mappings $x \to \mathcal{A}_{0,1}(., x)$ at $x = 0$
Use instead of x a discontinuous color function,
say v, for an augmented PDE model

Coupling via an augmented PDE Model

Rewrite the coupled problem in the transmitted variable $\mathbf{u}$ plus a color function v

$\partial_t \gamma_-(\mathbf{u}) + \partial_x \mathbf{f}_-(\gamma_-(\mathbf{u})) = 0,$ $v(x) = -1$	$\partial_t \gamma_+(\mathbf{u}) + \partial_x \mathbf{f}_+(\gamma_+(\mathbf{u})) = 0,$ $v(x) = +1$

Intermediate values

$$\mathbf{u}(0^-, t) = \mathbf{u}(0^+, t), \quad v \in (-1, 1)$$

may be seen as modeling a transition from a system to the other

Set the coupling problem over the entire real axis $\mathbb{R}$

$$\begin{cases} \mathcal{A}_0(\mathbf{u}, v)\partial_t \mathbf{u} + \mathcal{A}_1(\mathbf{u}, v)\partial_x \mathbf{u} = 0, \quad t > 0, \ x \in \mathbb{R} \\ \partial_t v = 0. \end{cases}$$

with the consistency property

$$\mathcal{A}_0(\mathbf{u}, \pm 1) = D\gamma_\pm(\mathbf{u}), \quad \mathcal{A}_1(\mathbf{u}, \pm 1) = \nabla f_\pm(\gamma_-(\mathbf{u}))D\gamma_\pm(\mathbf{u})$$

Coupling via an augmented PDE Model

$\partial_t \gamma_-(\mathbf{u}) + \partial_x \mathbf{f}_-(\gamma_-(\mathbf{u})) = 0,$ $v(x) = -1$	$\partial_t \gamma_+(\mathbf{u}) + \partial_x \mathbf{f}_+(\gamma_+(\mathbf{u})) = 0,$ $v(x) = +1$

$$\mathbf{u}(0^-, t) = \mathbf{u}(0^+, t), \quad v \in (-1, 1)$$

$$\begin{cases} \mathcal{A}_0(\mathbf{u}, v)\partial_t \mathbf{u} + \mathcal{A}_1(\mathbf{u}, v)\partial_x \mathbf{u} = 0, \quad t > 0, \ x \in \mathbb{R} \\ \partial_t v = 0. \end{cases}$$

where for instance (see additional conditions hereafter)

$$\mathcal{A}_0(\mathbf{u}, v) = \frac{(1-v)}{2} D\gamma_-(\mathbf{u}) + \frac{(1+v)}{2} D\gamma_+(\mathbf{u}),$$

$$\mathcal{A}_1(\mathbf{u}, v) = \frac{(1-v)}{2} \nabla f_-(\gamma_-(\mathbf{u})) D\gamma_-(\mathbf{u}) + \frac{(1+v)}{2} \nabla f_+(\gamma_+(\mathbf{u})) D\gamma_+(\mathbf{u}).$$

In what sense the expected transmission conditions in $\mathbf{u}$ are satisfied?

Coupling via an augmented PDE model with thin interfaces

$$\begin{cases} \mathcal{A}_0(\mathbf{u}, v)\partial_t \mathbf{u} + \mathcal{A}_1(\mathbf{u}, v)\partial_x \mathbf{u} = 0, \quad t > 0,\ x \in \mathbb{R} \\ \partial_t v = 0. \end{cases}$$

where by assumption

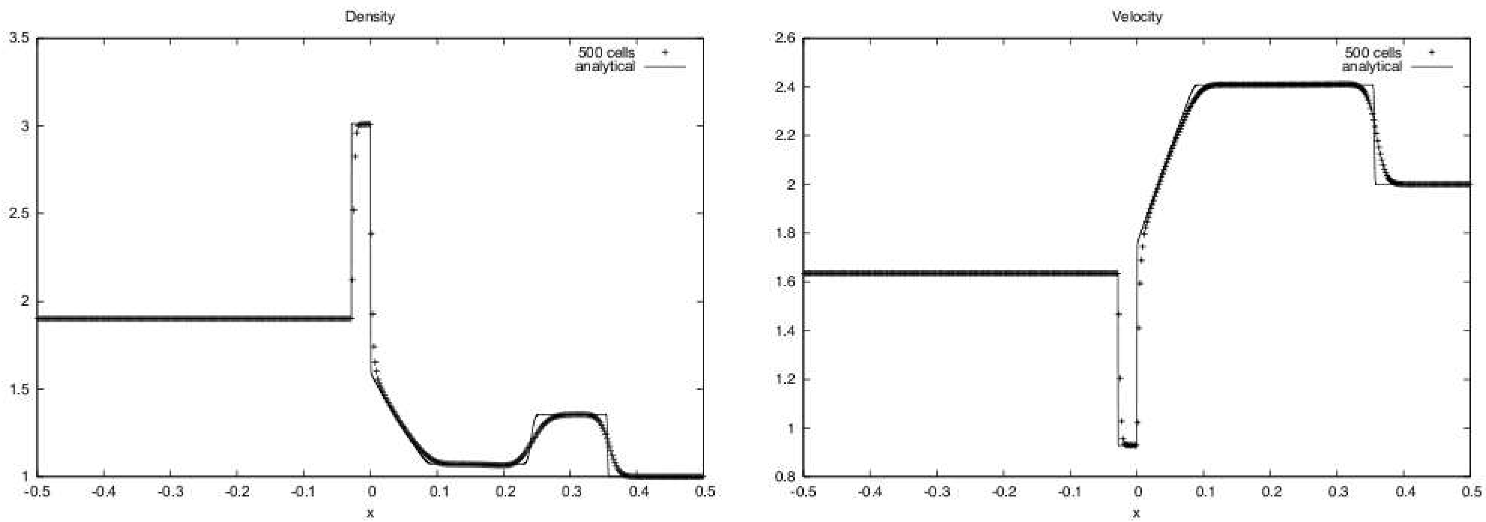
$\mathcal{A}_0(\mathbf{u}, v)$ is invertible, $\mathcal{A}_0^{-1}(\mathbf{u}, v) \times \mathcal{A}_1(\mathbf{u}, v)$ is $\mathbb{R}$ diagonalizable

- As soon as $Det(\mathcal{A}_1(\mathbf{u}, v)) \neq 0$, the augmented PDE model is $\mathbb{R}$ diagonalizable
- Otherwise, the basis of right eigenvector is in general locally lost (the so-called resonant phenomena)
- v is associated with a standing wave, whose Riemann invariants satisfy $\mathcal{A}_1(\mathbf{u}, v)D\mathbf{u} = 0$

$$Det(\mathcal{A}_1(\mathbf{u}, v)) \neq 0 \longrightarrow D\mathbf{u} = 0, \quad i.e. \quad \mathbf{u}(0^-, t) = \mathbf{u}(0^+, t).$$

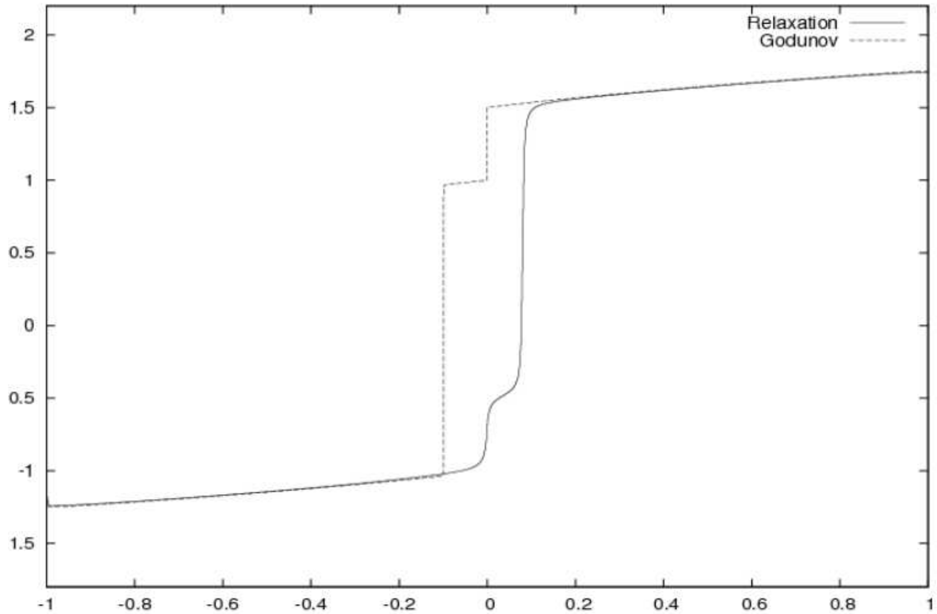
The expected transmission conditions $\mathbf{u}(0^+, t) = \mathbf{u}(0^-, t)$ are restored away from resonance

Resonance phenomenon : an illustration in the Euler setting



Expected ransmission conditions (ρ, u, p) for the coupling of Euler equations with distinct pressure laws $p_-(\mathbf{u})$ et $p_+(\mathbf{u})$
(T. Galié, with courtesy)

Resonance in the setting of non-convex scalar conservation laws



(B. Boutin, with courtesy)

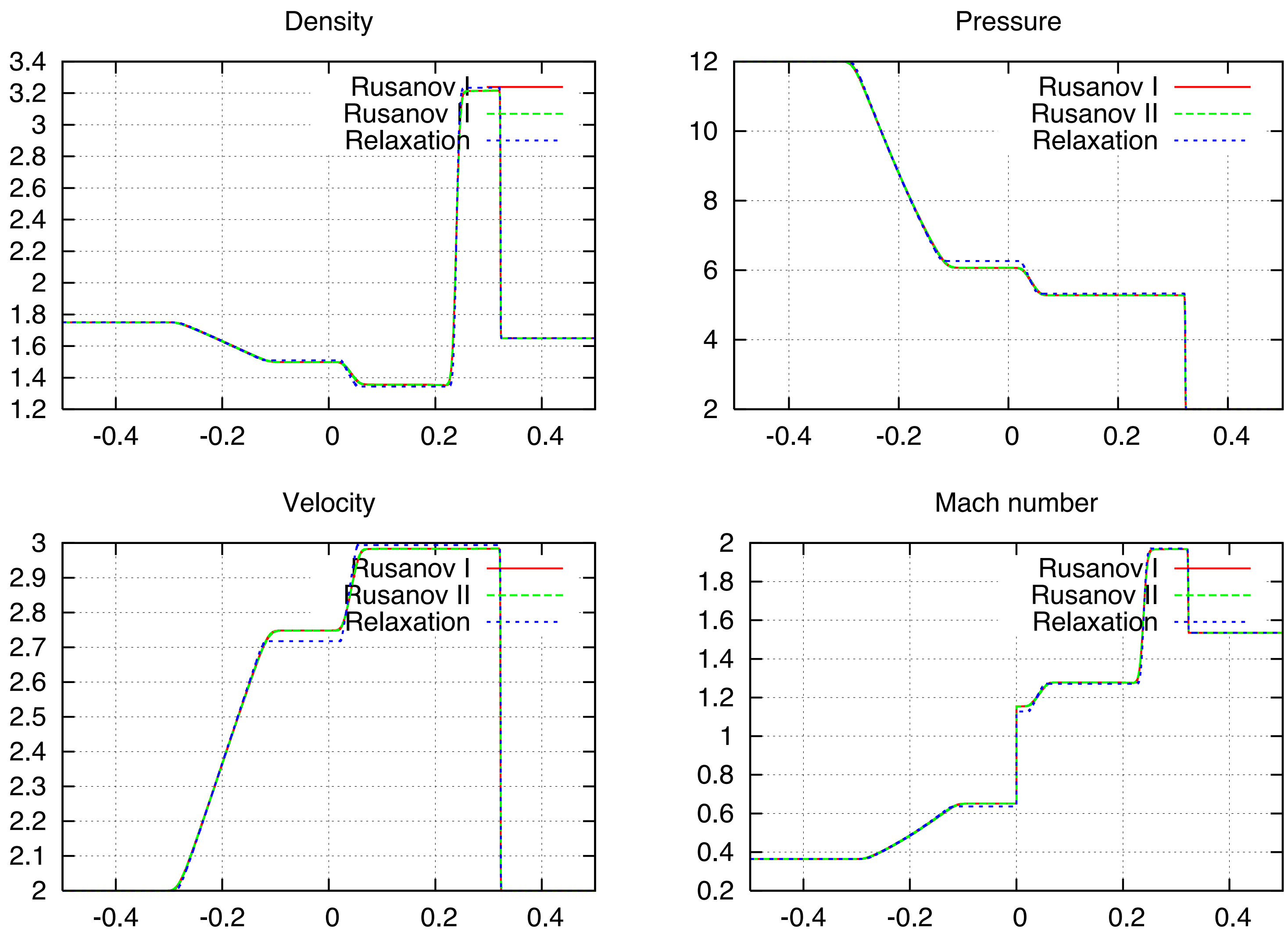
- ▷ Multiple solutions
- ▷ Strong sensitiveness of the discrete solutions with respect to the numerical solver
- ▷ Virtually identical behaviour as the one already observed in the classical resonant 2×2 nonlinear framework (Isaacson-Temple)

The system setting

Similar results are in order

- Multiple solutions generally arise when some (nonlinear) eigenvalue of $\nabla \mathbf{f}_-$ or $\nabla \mathbf{f}_+$ locally vanishes
- In such a case, we will speak of a resonant coupling interface, or for short of resonance.
- The resonance phenomenon we speak about is identical to the classical one taking place in weakly nonlinear hyperbolic equations in non-conservation form
  - Euler equations in varying duct, or in porous media, or shallow water equations with varying bathymetry, *etc*. Multiple Riemann solutions may indeed be built (see P. LeFloch and co-workers)
  - Their numerical capture turns very sensitive to the numerical solvers (see N. Andrianov, N.Seguin).

Multiple solutions with PDEs models of Industrial Interest



Analyzing the Coupled Riemann Problem

Consider the Dafermos *ansatz* for analyzing the time asymptotic behavior of the solution of the Cauchy problem with viscous perturbation :

$$\begin{cases} \mathcal{A}_0(\mathbf{u}^\epsilon, v^\epsilon)\partial_t \mathbf{u}^\epsilon + \mathcal{A}_1(\mathbf{u}^\epsilon, v^\epsilon)\partial_x \mathbf{u}^\epsilon = \epsilon \, t \, \partial_x(\mathcal{B}(\mathbf{u}^\epsilon, v^\epsilon)\partial_x \mathbf{u}^\epsilon), \quad t > 0, \; x \in \mathbb{R} \\ \partial_t v^\epsilon = \epsilon^2 \, t \, \partial_{xx} v^\epsilon. \end{cases}$$

$\hookrightarrow$ Self-similar solutions : $\xi = x/t$

$$\begin{cases} \Big(-\xi \mathcal{A}_0(\mathbf{u}^\epsilon, v^\epsilon) + \mathcal{A}_1(\mathbf{u}^\epsilon, v^\epsilon)\Big) d_\xi \mathbf{u}^\epsilon = \epsilon \, d_\xi(\mathcal{B}(\mathbf{u}^\epsilon, v^\epsilon) d_\xi \mathbf{u}^\epsilon), \quad \xi \in \mathbb{R} \\ -\xi d_\xi v^\epsilon = \epsilon^2 \, d_{\xi\xi} v^\epsilon, \\ (\mathbf{u}^\epsilon, v^\epsilon)(-\infty) = (\mathbf{u}_L, -1) \quad \text{et} \quad (\mathbf{u}^\epsilon, v^\epsilon)(+\infty) = (\mathbf{u}_R, +1) \end{cases}$$

Recover and analyze the coupled Riemann solutions in the limit $\epsilon \to 0$

Existence of Riemann solutions for the Augmented Model

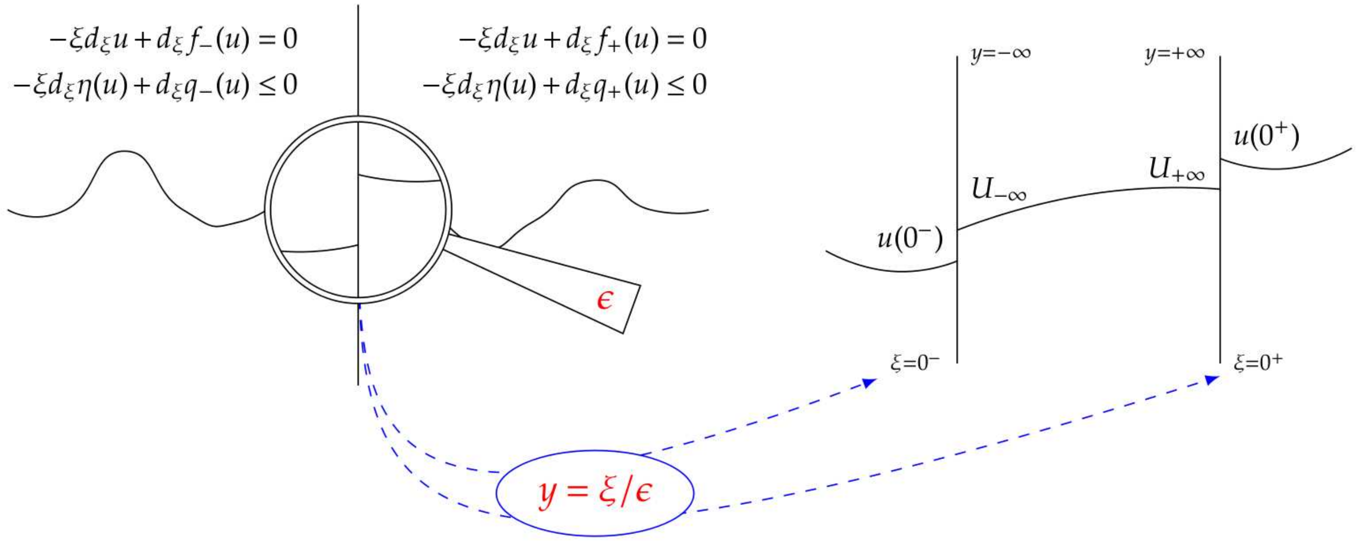
Theorem (B. Boutin, FC, P. LeFloch) *For sufficiently close states* $\mathbf{u}_L$ *et* $\mathbf{u}_R$ (*with possible resonance*) *and under general assumption on the coupling matrices* $\mathcal{A}_0, \mathcal{A}_1$ *and the viscous tensor* $\mathcal{B}$, *then there exists a solution* $\mathbf{u}^\epsilon$ any given $\epsilon > 0$ *and extracted subsequences* $\{\mathbf{u}^\epsilon\}_{\epsilon>0}$ *which simply converge to a limit* $\mathbf{u}$ *with bounded variation*. *In each half space*, $\mathbf{u}$ *is a self-similar (entropy) weak solution of*

$$\partial_t \gamma_-(\mathbf{u}) + \partial_x \mathbf{f}_-(\gamma_-(\mathbf{u})) = 0,\ x < 0, \quad \partial_t \gamma_+(\mathbf{u}) + \partial_x \mathbf{f}_+(\gamma_+(\mathbf{u})) = 0,\ x > 0.$$

What about the properties of the limit self-similar functions at the coupling interface, in particular when resonant ?

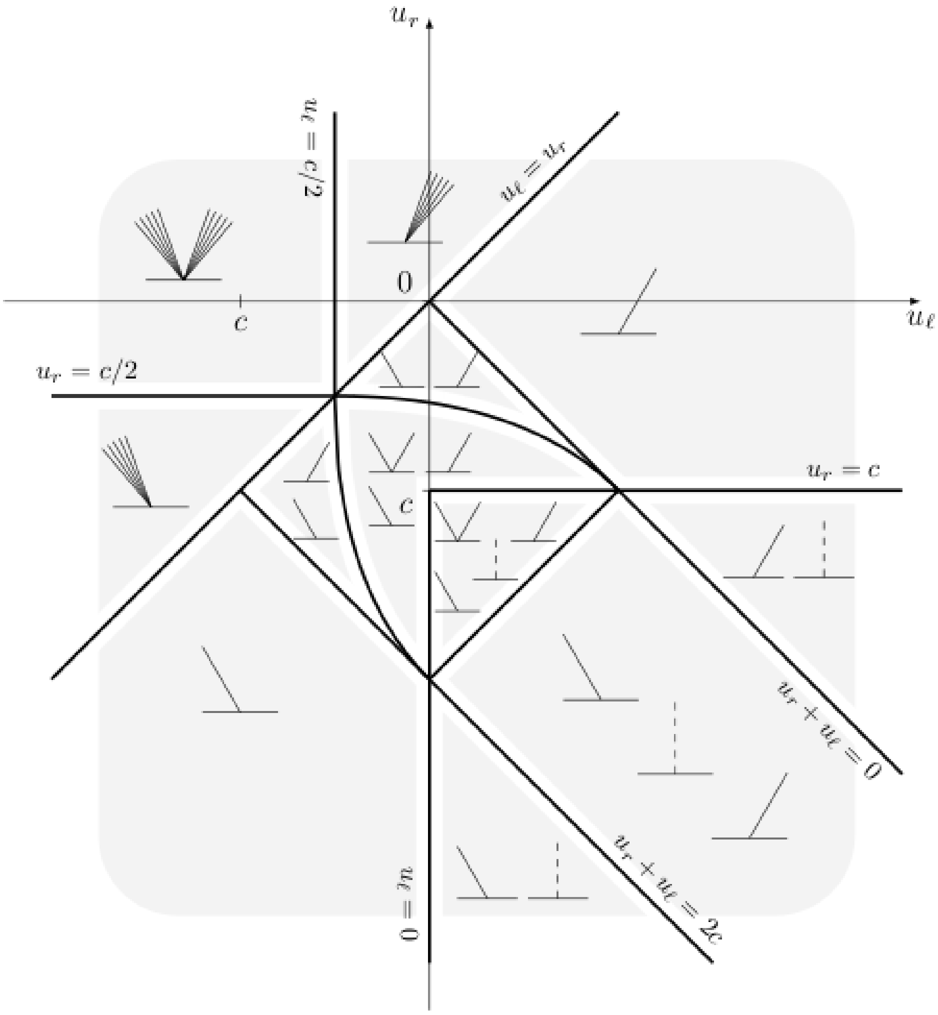
Characterizing the internal structure of the resonant coupling interfac

Blow up of the limit solution at the coupling interface : $\mathcal{U}^\epsilon(y) = \mathbf{u}^\epsilon(\epsilon\xi)$



▷ Allow a complete characterization of the limit solutions in the convex scalar setting (B. Boutin, FC, P. Le Floch, E. Godlewski)

Multiple limit self-similar solutions are recovered in the case of resonant interfaces



u_r
u_ℓ
0
c
$u_\ell = c/2$
$u_\ell = u_r$
$u_r = c/2$
$u_r = c$
c
$u_r + u_\ell = 0$
$u_r + u_\ell = 2c$
$u_\ell = 0$

About the failure of uniqueness in the Riemann solutions

The Riemann problem governs the time asymptotic behaviour of the solutions of the Cauchy problem
(with viscous pertubation)

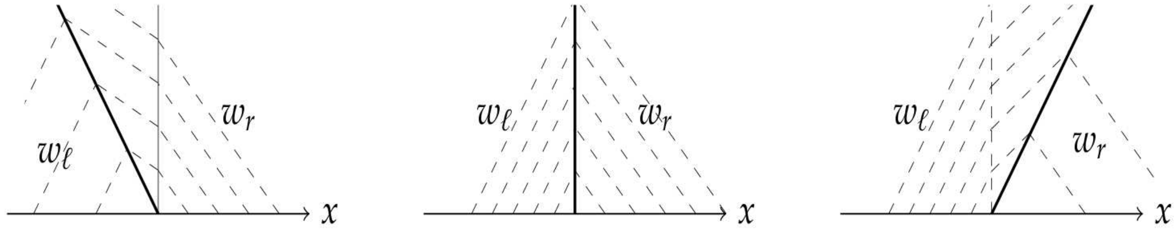
- Investigating the solutions of the Cauchy problem with initial data $\mathbf{u}_0$ kept self-similar but with a regularized color function

$$\begin{cases} \mathcal{A}_0(\mathbf{u}, v)\partial_t \mathbf{u} + \mathcal{A}_1(\mathbf{u}, v)\partial_x \mathbf{u} = 0, \quad t > 0,\ x \in \mathbb{R} \\ \partial_t v = 0, \\ (\mathbf{u}(0,x), v(0,x)) = (\mathbf{u}_0(x), v_0^\eta(x)), \quad v_0^\eta(x) = \rho^\eta(x) * v_0(x), \quad \eta > 0. \end{cases}$$

- Numerical investigation of the sensitiveness of the solutions of the "regularized" Cauchy problem with

$$v_0^{\nu,\theta}(x) = \frac{(Erf(x/\eta + \zeta) + 1)}{2}, \quad \eta \text{ thickening }, \quad \zeta \text{ translation}$$

Multiple Discontinuous Self-Similar Solutions



(a) A left-shock (b) A steady shock (c) A right-shock

Figure 4.10: Possible solutions for the thick coupling

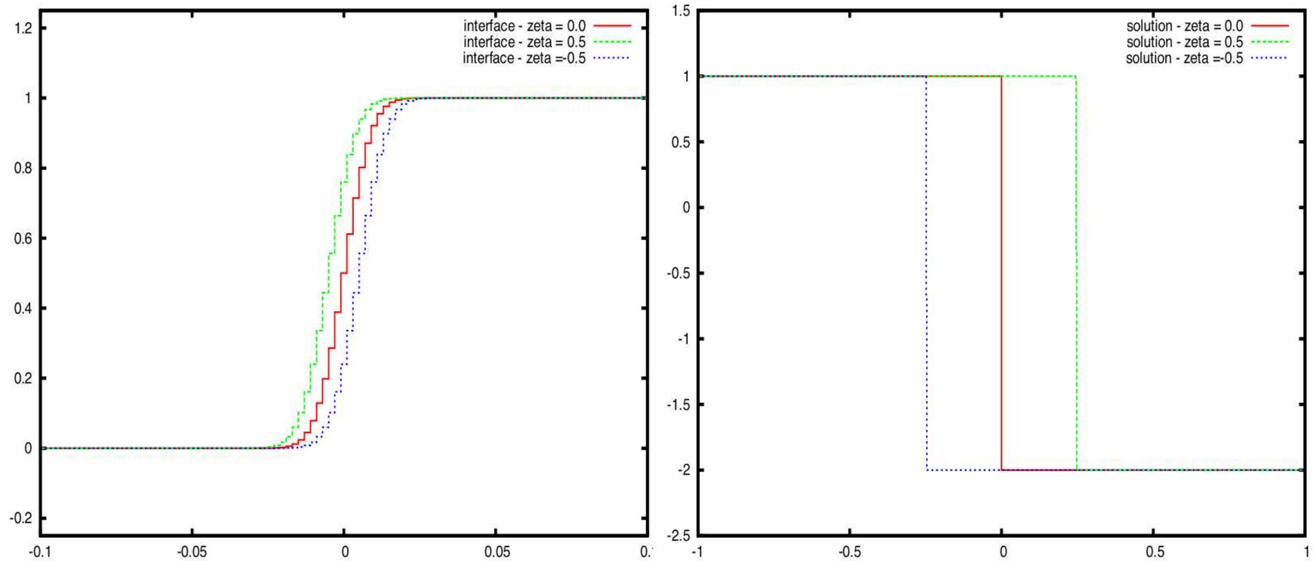


Figure 4.11: Three different interfaces (left) and corresponding solutions (right) - N = 1000

Coupling of Hyperbolic Equations via Thick Interfaces

- Use regularized color function $v_0^\eta(x) = \rho^\eta(x) * v_0(x)$

  Additional informations must be provided in order to promote a given regularized profile (*hint* : keep in mind that one of the two PDE models contains more physics and could be thus privileged)

- Understood as a coupling technic with a shake-hand coupling zone

$$\mathcal{A}_0(\mathbf{u}, v_0^\eta(x))\partial_t \mathbf{u} + \mathcal{A}_1(\mathbf{u}, v_0^\eta(x))\partial_x \mathbf{u} = 0,$$

$\partial_t \gamma_-(\mathbf{u}) + \partial_x \mathbf{F}(\gamma_-(\mathbf{u})) = 0$	Coupling zone	$\partial_t \gamma_+(\mathbf{u}) + \partial_x \mathbf{F}(\gamma_+(\mathbf{u}))$
$x < -\eta$	$-\eta < x < +\eta$	$x > +\eta$

Why can we expect uniqueness ?

Coupling of Hyperbolic Equations via Thick Interfaces

$$\mathcal{A}_0(\mathbf{u}, v_0^\eta(x))\partial_t \mathbf{u} + \mathcal{A}_1(\mathbf{u}, v_0^\eta(x))\partial_x \mathbf{u} = 0,$$

- The coupled PDE model can be equivalently rewritten defining the new unknown

$$\mathbf{w}(x,t) = \frac{(1 - v_0^\eta(x))}{2}\gamma_-(\mathbf{u}(x,t)) + \frac{(1 + v_0^\eta(x))}{2}\gamma_+(\mathbf{u}(x,t))$$

System of conservation laws with smooth spatial inhomogeneities and a smooth source term

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}, v_0^\eta(x)) = \mathbf{l}(\mathbf{w}, v_0^\eta(x))\frac{d}{dx}v_0^\eta(x)$$

Well balanced numerical issues
for hyperbolic PDEs with smooth source terms

Coupling of Hyperbolic Equations via Thick Interfaces

Usual notion of entropy weak solutions

$$\begin{cases} \partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}, v_0^\eta(x)) = l(\mathbf{w}, v_0^\eta(x)) \dfrac{d}{dx} v_0^\eta(x), \quad \mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x), \\ \partial_t \mathcal{U}(\mathbf{w}) + \partial_x \mathcal{F}(\mathbf{w}, v_0^\eta(x)) = \mathcal{L}(\mathbf{w}, v_0^\eta(x)) \dfrac{d}{dx} v_0^\eta(x), \end{cases}$$

for any convex entropy pair $(\mathcal{U}(\mathbf{w}),\ \mathcal{F}(\mathbf{w}))$.

$\hookrightarrow$ Uniqueness Kruzkov's Theorem in the setting of SCL with Lipschitz-continuous inhomogeneities

Let be given $w_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $v_0^\eta \in W^{2,\infty}(\mathbb{R})$,

then there exists a unique entropy weak solution

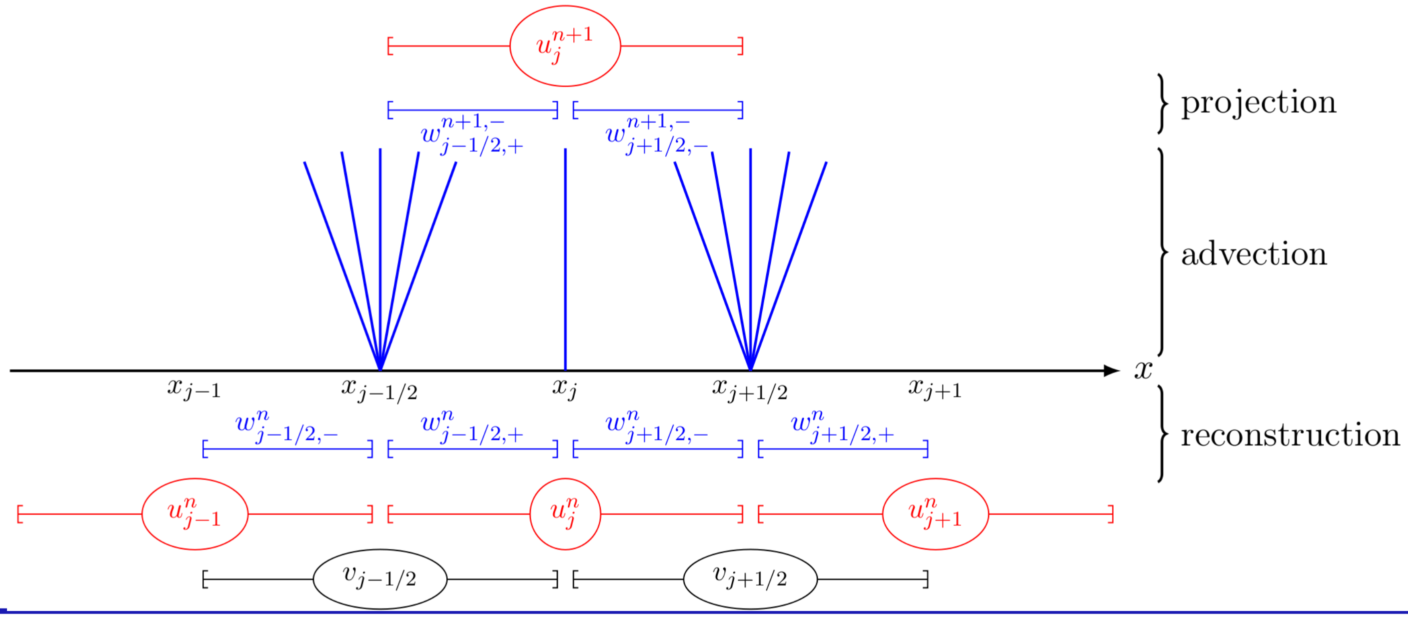
$w \in L^\infty(\mathbb{R}_t, L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ to the Coupled Cauchy problem with thick interfaces.

Well-Balanced Finite Volumes Scheme : Principle

$$\partial_t \mathbf{w} + \partial_x \mathbf{f}(\mathbf{w}, v) = \mathbf{l}(\mathbf{w}, v)\frac{d}{dx}v, \quad \text{versus} \quad \mathcal{A}_0(\mathbf{u}, v)\partial_t \mathbf{u} + \mathcal{A}_1(\mathbf{u}, v)\partial_x \mathbf{u} = 0.$$

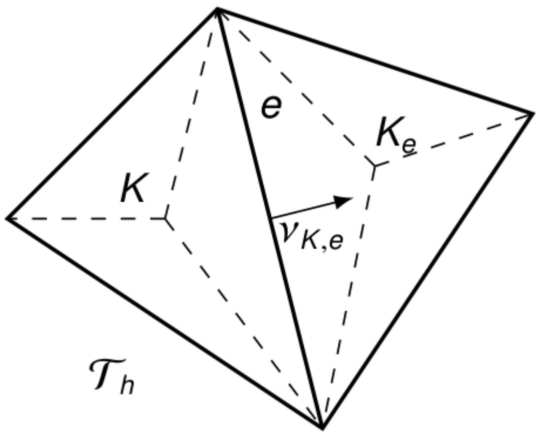
Approximate in a consistant manner the PDE for w and preserve the u equilibrium (locally constant u)

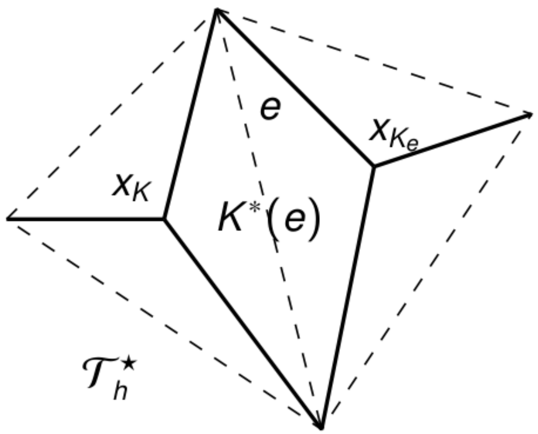
▷ non-colocalized approximation for u and v with reconstruction à la Bouchut-Perthame :



Multi-dimensional Finite Volume Scheme

- Augmented PDE formalism turns very flexible : easy extension to coupling problems with several space dimensions, general partition of the physicial domain in distinct hyperbolic equations with possible covering
- Non-colocalized approximation via a dual mesh approach





Multi-dimensional Finite Volume Scheme

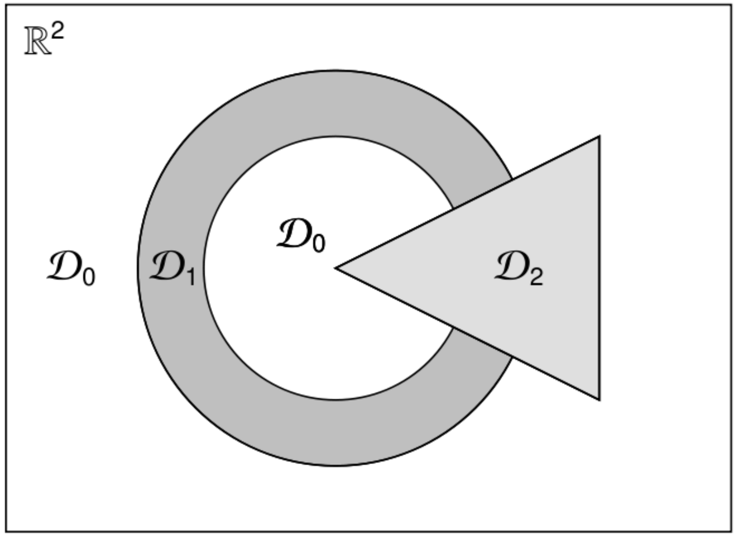
Theorem (B. Boutin, F.C., P. LeFloch) *Given* $w_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *and* $v_0 \in W^{2,\infty}(\mathbb{R}^d)$,

then under some classical CFL restriction, the family of approximate solutions $\{w_h\}_{h>0}$ *converges to the unique Kruzkov's solution*

$w \in L^\infty(\mathbb{R}_t, L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ *of the Coupled Cauchy problem with thick interfaces.*

A 2D coupling problem with recovering

Configuration géométrique :



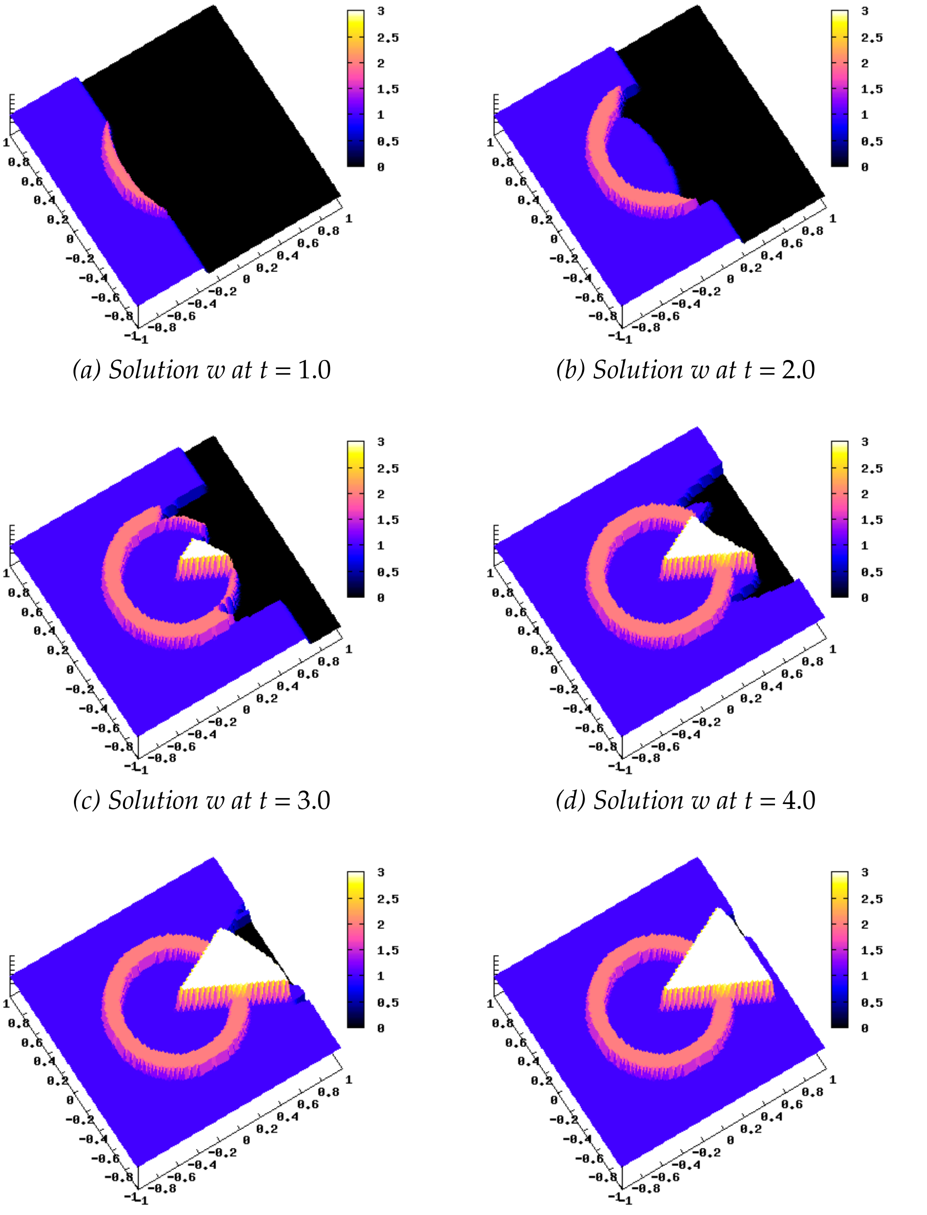
$$\partial_t w + \partial_x f_i(w) = 0, \quad x \in \mathcal{D}_i,\ i = 0, 1, 2.$$

$$f_0(w) = w^2/2\begin{pmatrix}1\\0\end{pmatrix}, \qquad \theta_0(w) = w,$$

$$f_1(w) = w^2/2\begin{pmatrix}0.5\\0\end{pmatrix}, \qquad \theta_1(w) = w/2,$$

$$f_2(w) = w^2/2\begin{pmatrix}0\\1\end{pmatrix}, \qquad \theta_2(w) = w/3.$$

$$\theta_i(w(x^i, t)) = \theta_j(w(x^j, t)), \qquad i \neq j.$$



(a) Solution w at $t = 1.0$

(b) Solution w at $t = 2.0$

(c) Solution w at $t = 3.0$

(d) Solution w at $t = 4.0$

(e) Solution w at $t = 5.0$

(f) Solution w at $t = 6.0$

Figure 5.6: Three domains - evolution of the solution w.

Conclusions

- You can develop several coupling mathematical coupling theories with singular interfaces, which always require you to add extra (physical) information to uniquely define the coupling strategy
  - A one based on a purely geometric modeling of the coupling interface
  - Another based on a purely PDE modeling of the coupling interface
- In both settings, you do have several existence results : a complete existence theory in the scalar case, and general existence results in the case of systems under the usual flatness assumption on the Cauchy data.
- You get multiple solutions in both setting. Those are stable (observable numerically speaking) and multiplicity comes from a nonlinear resonance phenomena at the singular interface
- You can recover uniqueness provided you regularize the coupling interface when adding further (physical) information to model the coupling within the hand-shake region
- There is a real interest in developing *a posteriori* modeling error analysis for moving in time the coupling interfaces (thin or thick)

Thank you for your attention !