Well-balanced schemes for linear models of Boltzmann equation: a legacy of Chandrasekhar, Case, Cercignani and Siewert

Laurent Gosse

IAC-CNR (Bari and then Rome)

Plan of the talk

1. Introduction
   - What need for well-balanced ?
   - Localization process with BV theory
2. Cercignani's decomposition of linear Boltzmann equation
   - Decoupling shear effects/heat transfer
   - Shear flows : Ψ_3, Ψ_4 and Ψ_5
   - Heat transfer : $\Psi := (\Psi_1 \ \Psi_2)^T$
   - Well-balanced Godunov schemes
3. Numerical results
   - Radiative transfer with discontinuous opacity (R. Sentis)
   - Couette flow with $\neq$ accomodation coefficients
   - Heat transfer with different Knudsen's (C.W. Siewert)
   - Linear 1-D chemotaxis (Othmer-Hillen, Natalini)
   - 1-D n^+nn^+ MOSFET simulation (I.Gamba, A.Jungel)
   - Gravitational Vlasov-Poisson (Dolbeault, Bouchut)

Well-Balanced schemes : a quick story

A.-Y. LeRoux, may 1995, during the Ph.D. of Alain Zelmanse :

> *If you simulate a glass of water with a small stone inside, it makes any numerical scheme unstable.*

The best discretization of $\partial_t u = Au + Bu$ with A, B of different nature (*e.g.* $Au = -\partial_x f(u)$, $B(u) = g(x, u)$ or differential and Fredholm operators) is probably not a linear combination of already existing ones for $\partial_t u = Au$ and $\partial_t u = Bu$ alone.

Main ingredients :

- introduce the auxiliary equation for $\Delta x > 0$,

$$\partial_t u + \partial_x f(u) = \Delta x \sum_j g(x, u)\delta(x - (j - \frac{1}{2})\Delta x)$$

- define correctly the non-conservative products
- set up the Godunov scheme for NC homogeneous eqn
- homogeneous Godunov has no viscosity at steady-state

Several remaining open problems

- justify the "localization step" for general 1-D hyperbolic systems of balance laws (the limit $\Delta x \to 0$ has been studied with wavefront-tracking techniques in [Amadori, L.G., Guerra, ARMA 2002]) and prove the definition of NC products.
- understand the "nonlinear resonance" phenomenon when $det(f_u)$ vanishes : intricate Riemann problems (cf. Liu, Isaacson-Temple, Vasseur, Goatin, Le Floch, [A.G.G. JDE'04]). Is it really useful numerically ?
- extend to multi-D, to high order ...
- extend to more general equations (the topic of today).

The "localization step" : results with BV theory

- 1-D scalar balance law : $\partial_t u + \partial_x f(u) = k(x)g(u)$, $f' > 0$,
- 2-velocity kinetic model : $\partial_t f^{\pm} \pm \partial_x f^{\pm} = \mp(f^+ - f^-)$.

Write $k(x) = \partial_x a^1(x)$, define a sequence $a^\varepsilon(x) \to a(x) \in BV(\mathbb{R})$ which induces a sequence u^ε of usual Kruzkov solutions. Following early computations, [L.G., MCOM 2002], the 2×2 system

$$\partial_t u + \partial_x f(u) - g(u)\partial_x a = 0, \ \partial_t a = 0,$$

is Temple class. Its Riemann invariants are a, $w(u,a) = \phi^{-1}(\phi(u) - a)$, $\phi' = f'/g$. BV norm decays with time and 1-to-1 if $f' > 0$. Thus u^ε is uniformly BV-bounded and compact in L^1_{loc}. Hence $g(u^\varepsilon)\partial_x a^\varepsilon$ becomes a well-defined non-conservative product in the sense of LeFloch-Tzavaras. NC product uses steady-state eqns $\partial_x f(u) = k(x)g(u)$.

Linear models of Boltzmann equation

For simplicity, let's start with BGK model ($\rightarrow$ Gross-Jackson) :

$$\partial_t f + \xi \partial_x f = \nu(\mathcal{M}(f) - f) := \mathcal{J} f, \qquad \xi := \mathbf{v}_1,$$

If $f(t,x,\mathbf{v}) = M(\mathbf{v})[1 + h(t,x,\mathbf{v})]$, the eqn can be linearized : $\partial_t h + \xi \partial_x h = -\nu(Id - \mathcal{P})h$, with $\mathcal{P}$ the orthogonal projection onto the vector space spanned by 5 collisional invariants. $\mathcal{P}h(t,x,\mathbf{v}) =$

$$\int_{\mathbb{R}^3} M(\mathbf{v}') \left[1 + 2\mathbf{v}.\mathbf{v}' + \frac{2}{3}\left(|\mathbf{v}|^2 - \frac{3}{2}\right)\left(|\mathbf{v}'|^2 - \frac{3}{2}\right)\right] h(t,x,\mathbf{v}') d\mathbf{v}'.$$

For $\Delta x > 0$ the space-step of the computational grid,

$$\partial_t h + \xi \partial_x h = \nu \Delta x \sum_{j \in \mathbb{Z}} (\mathcal{P} - Id) h \ \delta\left(x - (j - \frac{1}{2})\Delta x \right).$$

Define the 5-components vector :

$$\Phi(\mathbf{v}) = \frac{1}{\pi^{\frac{3}{4}}} \begin{pmatrix} 1 & \sqrt{\frac{2}{3}}(|\mathbf{v}|^2 - \frac{3}{2}) & \sqrt{2}\mathbf{v}_2 & \sqrt{2}\mathbf{v}_3 & \sqrt{2}\mathbf{v}_1 \end{pmatrix}^T .$$

A direct computation shows that : (^{T} denotes transposition)

$$\mathcal{P}h(t,x,\mathbf{v}) = \int_{\mathbb{R}^3} M(\mathbf{v}') \left[\Phi(\mathbf{v})^T \Phi(\mathbf{v}')\right] h(t,x,\mathbf{v}') d\mathbf{v}'.$$

Cercignani introduces the (normalized) functions :

$$\mathbf{G}(\mathbf{v}_2,\mathbf{v}_3) = \begin{pmatrix} g_1(\mathbf{v}_2,\mathbf{v}_3) \\ g_2(\mathbf{v}_2,\mathbf{v}_3) \\ g_3(\mathbf{v}_2,\mathbf{v}_3) \\ g_4(\mathbf{v}_2,\mathbf{v}_3) \end{pmatrix} = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 1 \\ (\mathbf{v}_2^2 + \mathbf{v}_3^2 - 1) \\ \sqrt{2}\mathbf{v}_2 \\ \sqrt{2}\mathbf{v}_3 \end{pmatrix},$$

which are orthogonal in the following sense,

$$\int_{\mathbb{R}^2} g_i(\mathbf{v}_2,\mathbf{v}_3) g_j(\mathbf{v}_2,\mathbf{v}_3) \exp(-\mathbf{v}_2^2 - \mathbf{v}_3^2) d\mathbf{v}_2 d\mathbf{v}_3 = \delta_{i,j}.$$

With the notation $\mathbf{v}_1 = \xi$, $h(t,x,\mathbf{v})$ can be expanded :

$$h(t,x,\mathbf{v}) = \sum_{i=1}^{4} \Psi_i(t,x,\xi) g_i(\mathbf{v}_2,\mathbf{v}_3) + \Psi_5(t,x,\mathbf{v}),$$

where the components Ψ_i are "unusual moments" which read,

$$\Psi_i(t,x,\xi) = \int_{\mathbb{R}^2} h(t,x,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) g_i(\mathbf{v}_2,\mathbf{v}_3) \exp(-\mathbf{v}_2^2 - \mathbf{v}_3^2) d\mathbf{v}_2 d\mathbf{v}_3.$$

$$\partial_t \Psi + \xi \partial_x \Psi + \Psi = \int_{\mathbb{R}} [\mathbf{P}(\xi)\mathbf{P}(\xi')^T + \mathbf{Q}(\xi)\mathbf{Q}(\xi')^T] \Psi(\xi') \frac{\exp(-|\xi'|^2)}{\sqrt{\pi}} d\xi,$$

$$\mathbf{P}(\xi) = \begin{pmatrix} \sqrt{\frac{2}{3}}(\xi^2 - \frac{1}{2}) & 1 & 0 & 0 \\ \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{Q}(\xi) = \sqrt{2}\xi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Microscopic (easy) eqn : $\partial_t \Psi_5 + \xi \partial_x \Psi_5 + \Psi_5 = 0$. Shear flows realize slns of "reduced viscosity equation" for $\psi := \Psi_3, \Psi_4$:

$$\partial_t \psi + \xi \partial_x \psi = \int_{\mathbb{R}} \psi(t, x, \xi') \frac{\exp(-|\xi'|^2)}{\sqrt{\pi}} d\xi' - \psi,$$

Well-balanced stems on solving the "forward-backward problem" :

$$\xi \partial_x \psi = \int_{\mathbb{R}} \psi(x, \xi') \frac{\exp(-|\xi'|^2)}{\sqrt{\pi}} d\xi' - \psi, \qquad x \in (0, \Delta x),$$

with inflow boundary conditions. According to Chandrasekhar, $\psi(x, \xi) = \exp(-x/\nu)\varphi_\nu(\xi)$ [translation invariance, Sturm-Liouville] which leads to a third kind linear integral eqn (cf. [Bart, Warnock, SIMA'73]) which admits "elementary slns" (Case, Zweifel, Cercignani, Siewert ...) and a superposition principle :

$$\psi(x, \xi) = \alpha + \beta(x - \xi) + \int_{\mathbb{R}} A(\nu) \exp(-x/\nu)\varphi_\nu(\xi) d\nu.$$

Forward-backward boundary-value problem for $\Psi(x,\xi) \in \mathbb{R}^2$:

$$\xi \partial_x \Psi + \Psi = \int_{\mathbb{R}} [\mathbf{P}(\xi)\mathbf{P}(\xi')^T + 2\xi\xi' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] \Psi(\xi') \frac{\exp(-|\xi'|^2)}{\sqrt{\pi}} d\xi',$$

Flux $J = \int_{\mathbb{R}} \xi' \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \Psi(x,\xi') \frac{\exp(-|\xi'|^2)}{\sqrt{\pi}} d\xi'$ doesn't depend on x.
Same technique of "elementary slns" yields the explicit sln :

$$\Psi(x,\xi) = \sum_{i=1}^{2} \mathbf{P}(\xi)[\alpha_i + \beta_i(x-\xi)]\vec{e}_i + \int_{\mathbb{R}} A_i(\nu)\Phi_i(\nu,\xi)\exp(-x/\nu)d\nu.$$

We have stationary slns and so we can build WB schemes !
Gaussian quadrature for strictly hyperbolic *DO* eqns :

$$\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_N) \in (0,1)^N, \ \boldsymbol{\omega} = (\omega_1, ..., \omega_N) \in \mathbb{R}^+.$$

NC Riemann solver and WB Godunov scheme

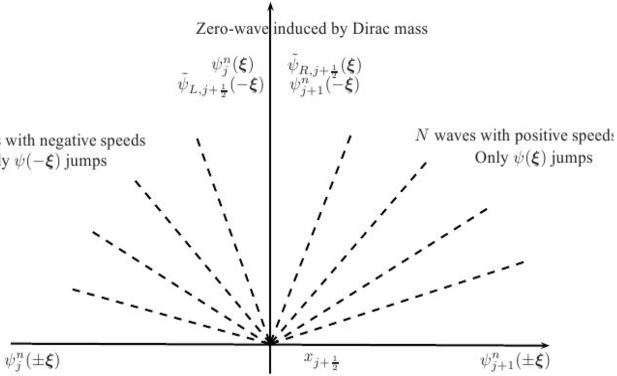


FIGURE 3. Self-similar solution of non-conservative Riemann problem.

Consistent Godunov scheme for shear or heat flows :

$$\begin{cases} \psi_j^{n+1}(\xi_k) = \psi_j^n(\xi_k) - \xi_k \dfrac{\Delta t}{\Delta x}\left(\psi_j^n(\xi_k) - \tilde{\psi}_{R,j-\frac{1}{2}}(\xi_k)\right), \\ \psi_j^{n+1}(-\xi_k) = \psi_j^n(-\xi_k) + \xi_k \dfrac{\Delta t}{\Delta x}\left(\tilde{\psi}_{L,j+\frac{1}{2}}(-\xi_k) - \psi_j^n(-\xi_k)\right), \end{cases}$$

with for all j in the computational domain and $N \in \mathbb{N}$:

$$\begin{pmatrix} \tilde{\psi}_{R,j+\frac{1}{2}}(\boldsymbol{\xi}) \\ \tilde{\psi}_{L,j+\frac{1}{2}}(-\boldsymbol{\xi}) \end{pmatrix} = \tilde{M}\ M^{-1} \begin{pmatrix} \psi_j^n(\boldsymbol{\xi}) \\ \psi_{j+1}^n(-\boldsymbol{\xi}) \end{pmatrix}.$$

M and $\tilde{M}$ are matrices of passage between the base of elementary slns and the usual space (x, ξ). (pre-processing step)
We are exact in x, only ξ has been discretized.

At each interface $j + \frac{1}{2}$, solve a linear problem :

$$M \begin{pmatrix} \mathbf{A} \\ \alpha \\ \mathbf{B} \\ \beta \end{pmatrix} = \begin{pmatrix} \psi_j^n(\boldsymbol{\xi}) \\ \psi_{j+1}^n(-\boldsymbol{\xi}) \end{pmatrix} \in \mathbb{R}_+^{2N},$$

where the $2N \times 2N$ matrices M, $\tilde{M}$ read for $\boldsymbol{\nu} := \{\nu_1, \nu_2, ..., \nu_{N-1}\}$,

$$M = \begin{pmatrix} \left(1 - \boldsymbol{\xi} \otimes \boldsymbol{\nu}^{-1}\right)^{-1} & 1_{\mathbb{R}^N} & \left(1 + \boldsymbol{\xi} \otimes \boldsymbol{\nu}^{-1}\right)^{-1} \exp(-\frac{\Delta x}{\boldsymbol{\nu}}) & -\boldsymbol{\xi} \\ \left(1 + \boldsymbol{\xi} \otimes \boldsymbol{\nu}^{-1}\right)^{-1} \exp(-\frac{\Delta x}{\boldsymbol{\nu}}) & 1_{\mathbb{R}^N} & \left(1 - \boldsymbol{\xi} \otimes \boldsymbol{\nu}^{-1}\right)^{-1} & \Delta x + \boldsymbol{\xi} \end{pmatrix}.$$

$$\tilde{M} = \begin{pmatrix} \left(1 - \boldsymbol{\xi} \otimes \boldsymbol{\nu}^{-1}\right)^{-1} \exp(-\frac{\Delta x}{\boldsymbol{\nu}}) & 1_{\mathbb{R}^N} & \left(1 + \boldsymbol{\xi} \otimes \boldsymbol{\nu}^{-1}\right)^{-1} & \Delta x - \boldsymbol{\xi} \\ \left(1 + \boldsymbol{\xi} \otimes \boldsymbol{\nu}^{-1}\right)^{-1} & 1_{\mathbb{R}^N} & \left(1 - \boldsymbol{\xi} \otimes \boldsymbol{\nu}^{-1}\right)^{-1} \exp(-\frac{\Delta x}{\boldsymbol{\nu}}) & \boldsymbol{\xi} \end{pmatrix},$$

the interface values are given by :

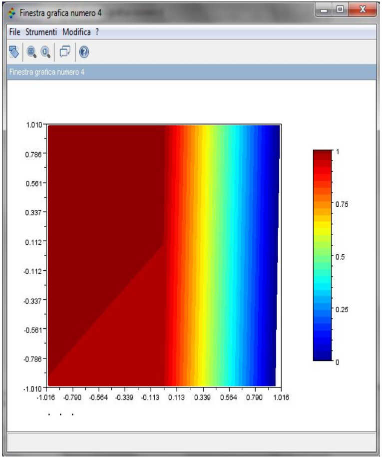
$$\begin{pmatrix} \tilde{\psi}_{R,j+\frac{1}{2}}(\boldsymbol{\xi}) \\ \tilde{\psi}_{L,j+\frac{1}{2}}(-\boldsymbol{\xi}) \end{pmatrix} = \tilde{M}\ M^{-1} \begin{pmatrix} \psi_j^n(\boldsymbol{\xi}) \\ \psi_{j+1}^n(-\boldsymbol{\xi}) \end{pmatrix}.$$

Numerical results with several models

IBVP for linear kinetic equation :

$$\partial_t f + v\partial_x f = \kappa(x)\left(\frac{1}{2}\int_{-1}^{1} f(t,x,v')dv' - f\right),$$

with variable (discontinuous) opacity κ and inflow boundary conditions. $\kappa(x) = 1 + 49\chi(x > 0)$ with $x \in]-1,1[$. We want the flow $\int_{-1}^{1} v' f(t,x,v')dv'$ to be constant at numerical steady-state ! Positive inflow in the transparent region ($x < 0$, $\kappa = 1$) and zero re-entry in the opaque zone ($x > 0$, $\kappa = 50$). [JQSRT'11]
The WB Godunov scheme can be converted into an AP scheme for the diffusion approximation in the parabolic scaling (exactly like in [G-Toscani, 2002 & 2004]).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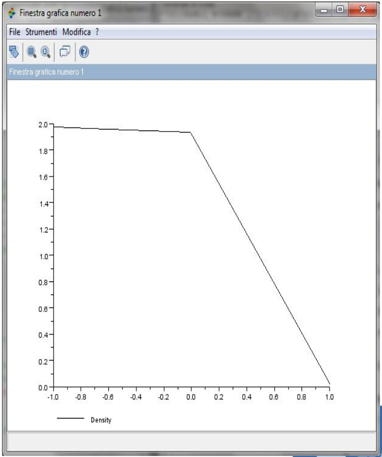
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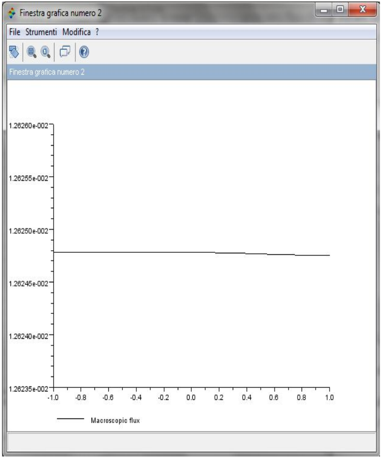
Density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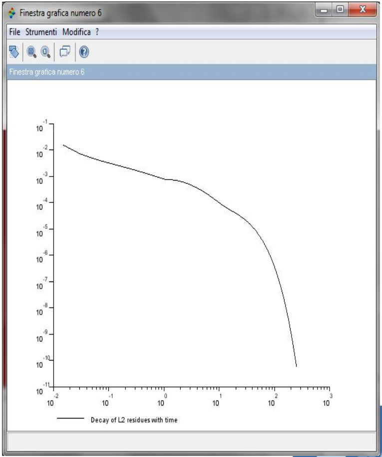
Macroscopic flux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Decay of L2 residues with time

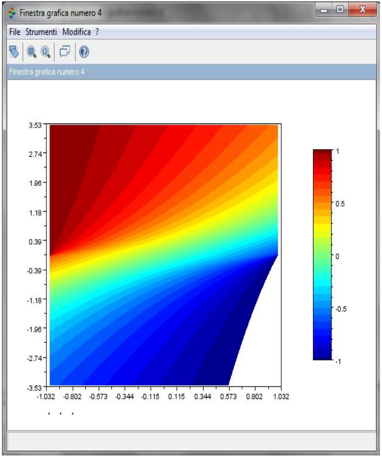
IBVP for linear kinetic equation :

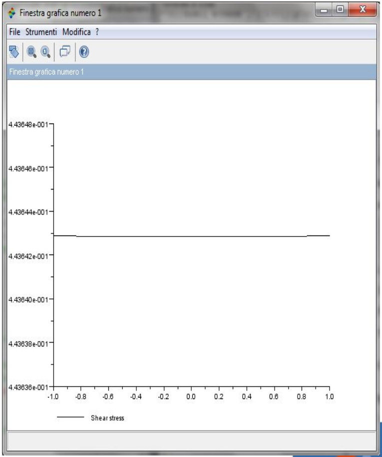
$$\partial_t \psi + v \partial_x \psi = \int_R \psi(t, x, v') \frac{\exp(-|v'|^2)}{\sqrt{\pi}} dv' - \psi,$$

with inflow boundary conditions corresponding to infinite plates moving at $\pm u_{wall}$ and $\alpha \in [0, 1]$ is the accomodation coefficient :

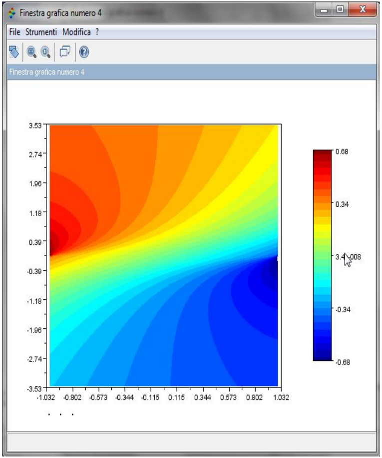
$$\psi(t, x = \pm 1, \mp |v|) = \mp \alpha u_{wall} + (1 - \alpha) \psi(t, x = \pm 1, \pm |v|).$$

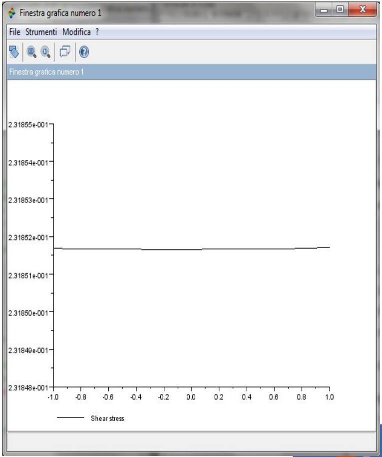
We want the shear stress $\int_{-1}^{1} v' \psi(t, x, v') dv'$ to be constant at numerical steady-state ! (cf. papers by L. Barichello & C.W. Siewert + collaborators)





$\alpha = 1$: no specular reflection.





$\alpha = \frac{1}{2}$: half specular reflection.

2×2 kinetic system for density and temperature fluctuations :

$$\partial_t \Psi + \xi \partial_x \Psi + \Psi = \int_{\mathbb{R}} [\mathbf{P}(\xi)\mathbf{P}(\xi')^T + \mathbf{Q}(\xi)\mathbf{Q}(\xi')^T] \Psi(\xi') \frac{\exp(-|\xi'|^2)}{\sqrt{\pi}} d\xi,$$

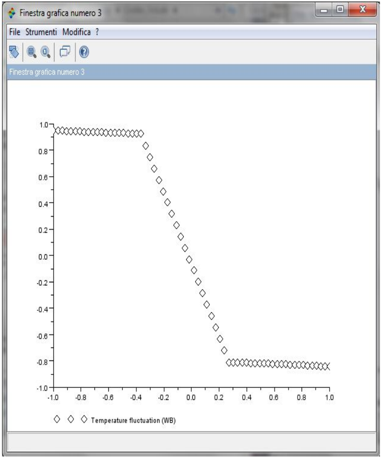
$$\mathbf{P}(\xi) = \begin{pmatrix} \sqrt{\frac{2}{3}}(\xi^2 - \frac{1}{2}) & 1 \\ \sqrt{\frac{2}{3}} & 0 \end{pmatrix}, \quad \mathbf{Q}(\xi) = \sqrt{2}\xi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We want the macroscopic mass flow and the normalized heat flow to be constant at numerical steady-state with $\neq$ acc. coeffts α_i.

$$\Psi(t, \pm 1, \mp|\xi|) = (1 - \alpha_i)\Psi(t, \pm 1, \pm|\xi|) \mp \alpha_i \delta_i \sqrt{\pi} \begin{pmatrix} \xi^2 + \beta_i \\ 1 \end{pmatrix}.$$

Zero macroscopic flux through the infinite walls [L.G.'11].
Variable Knudsen number in the domain :

$$\varepsilon = 1 \text{ for } x \in \left(-1, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right), \quad \varepsilon = 10^{-2} \text{ elsewhere.}$$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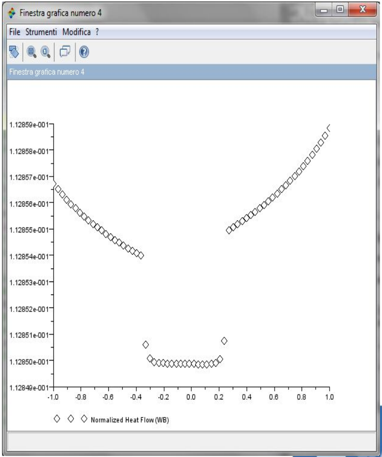
Temperature fluctuation (WB)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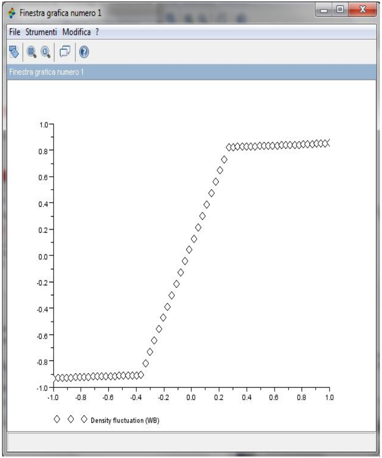
Normalized Heat Flow (WB)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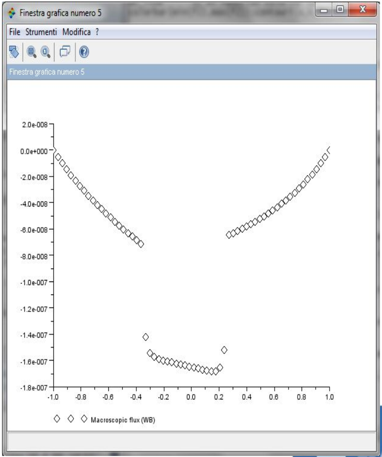
Density fluctuation (WB)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-1.6e-007
-1.8e-007
-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
Macroscopic flux (WB)

Weakly nonlinear system with "biased" velocity redistribution :

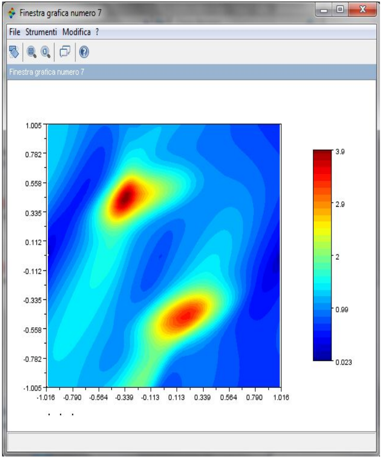
$$\partial_t f + v\partial_x f + f = \frac{1}{2}\int_{-1}^{1} \left[1 + 2v.(a(\partial_x S)v')\right] f(t,x,v')dv' - f,$$

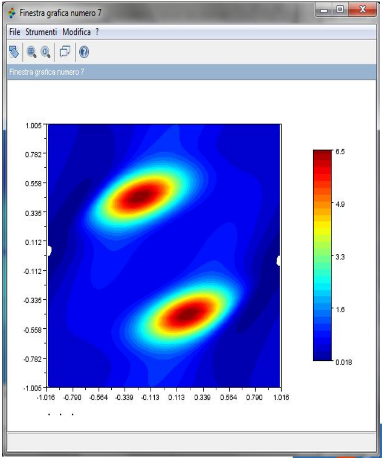
specular BC, and $S(t,x)$ chemo-attractant concentration sln of

$$\partial_t S - D\partial_{xx} S = \alpha\rho - \beta S \qquad + \text{Neumann BC.}$$

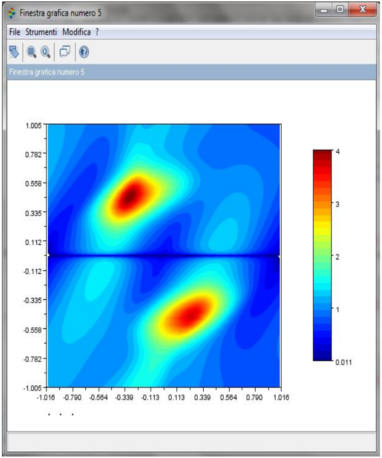
Steady-states are constant in x [OH] : we want to see the differences due to $\partial_x S$ compared to radiative transfer eqn. 2×2 model studied in detail by extending [G.Toscani '04]. Especially, $v \in \{-1, 1\}$ implies that : any numerical stationary-state with non-zero flux is not Maxwellian :

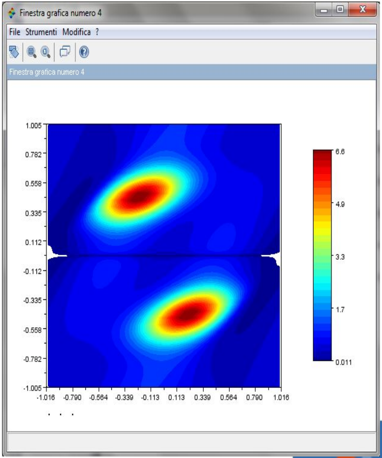
$$f^+ = f^- \Rightarrow J = f^+ - f^- = 0, \quad \text{[Natalini, Ribot, G.]}$$





Numerical solution with time-splitting. (viscous !)





Numerical solution with kinetic well-balanced.

Weakly nonlinear system (Vlasov-Poisson + linear collisions) :

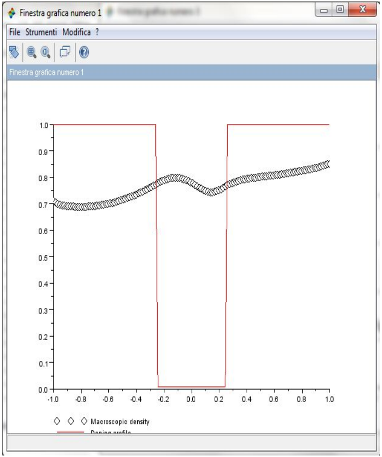
$$\partial_t f + v\partial_x f + E(t,x)\partial_v f = \frac{1}{\varepsilon}\left(\frac{\exp(-|v|^2)}{\sqrt{\pi}}\int_{\mathbb{R}} f(t,x,v')dv' - f\right),$$

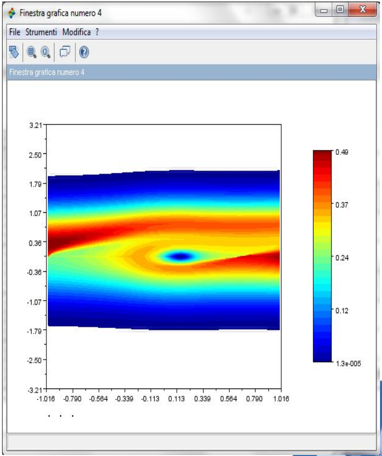
with the electric field $E = -\partial_x \Phi$ and

$$-\lambda^2 \partial_{xx}\Phi(t,x) = \int_{\mathbb{R}} f(t,x,v)dv - D(x),$$

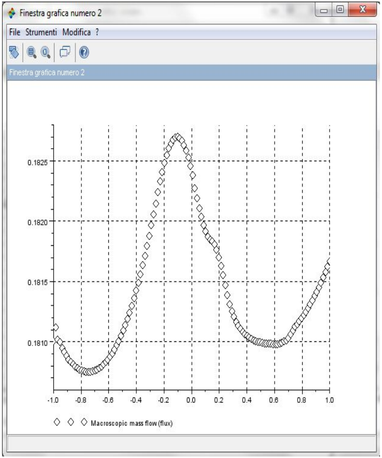
D the discontinuous doping profile and the relaxation time passes from $\varepsilon \simeq 0.01/0.0001$ in doped areas to $\varepsilon \simeq 1/\infty$ in the channel (few collisions or ballistic). $\lambda =$ scaled Debye length.
We want $\int_{\mathbb{R}} vf(t,x,v)dv$ at least continuous in x, or even constant at numerical steady-state in order to compute the current-voltage relation without (too much ?) averaging.

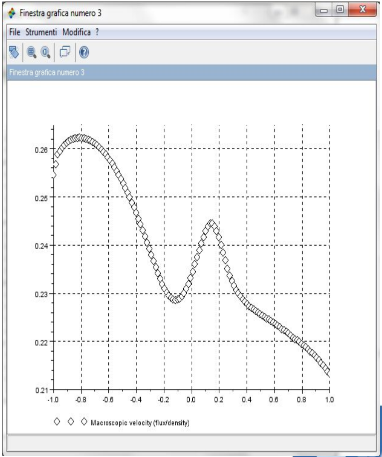
Big value of ε : Vlasov-type behavior

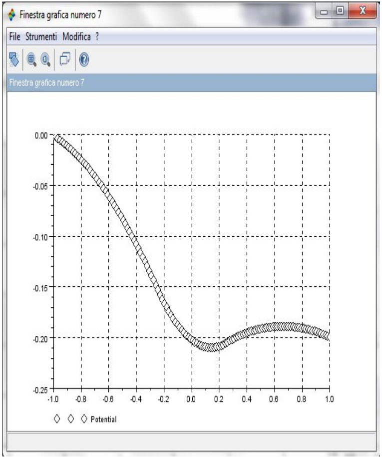


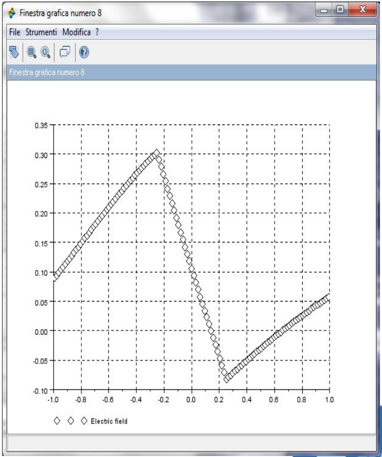


Maxwellian boundary conditions ($\alpha = 1$, no spec reflexion).

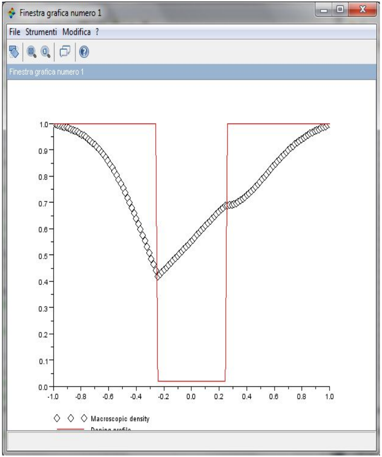

Finestra grafica numero 2
File Strumenti Modifica ?
0.1825
0.1820
0.1815
0.1810
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Macroscopic mass flow (flux)

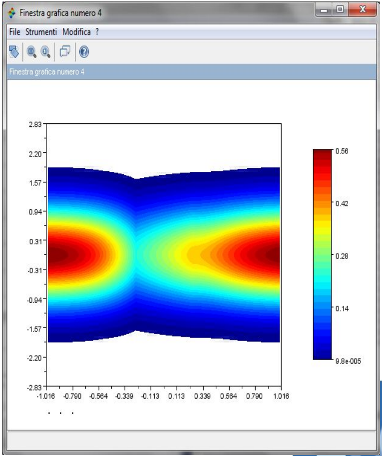

Finestra grafica numero 3
File Strumenti Modifica ?
0.26
0.25
0.24
0.23
0.22
0.21
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Macroscopic velocity (flux/density)


Finestra grafica numero 7
File Strumenti Modifica ?
Finestra grafica numero 7
0.00
-0.05
-0.10
-0.15
-0.20
-0.25
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Potential

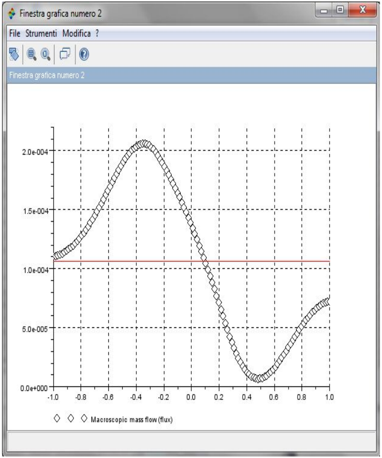

Finestra grafica numero 8
File Strumenti Modifica ?
Finestra grafica numero 8
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
-0.05
-0.10
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Electric field

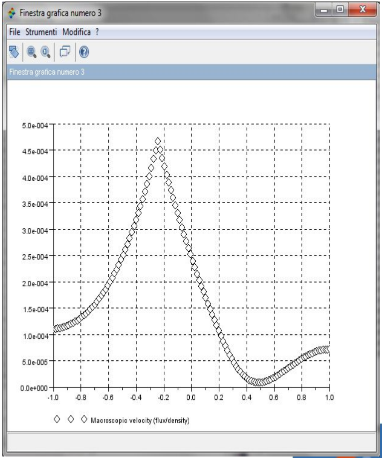
$\varepsilon = 0.0002$ & 1 (channel) : strong collisions



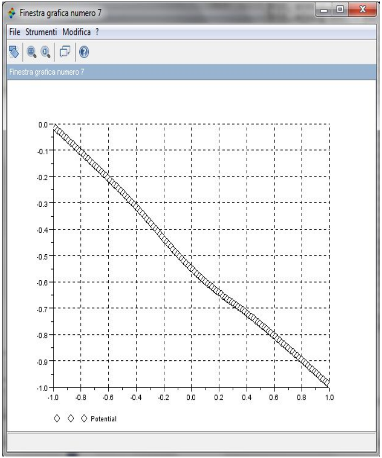


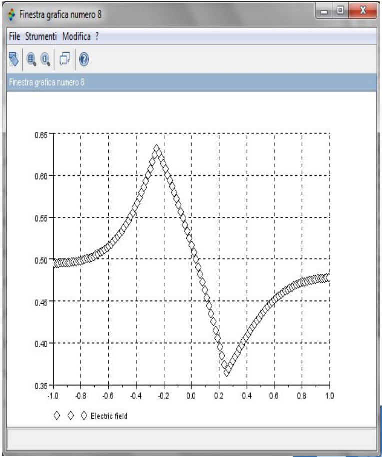
"Neighbor" boundary conditions ($\alpha = 1$, no spec reflexion).





Macroscopic flow (current) continuous but not constant.


Finestra grafica numero 7
File Strumenti Modifica ?
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
Potential


Finestra grafica numero 8
File Strumenti Modifica ?
0.65
0.60
0.55
0.50
0.45
0.40
0.35
-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
Electric field

Approximate elementary solutions for Vlasov-collisions model :

$$v\partial_x f + E\partial_v f = M(v)\int f(v')dv' - f \ + \ B.C.$$

Observe that $\tilde{M}(x,v) = \exp(-v^2 + 2Ex)$ is an exact solution (both sides vanish). One seeks f in the form : $f(x,v) = \tilde{M}(x,v)g(x,v)$,

$$v\partial_x(g\tilde{M}) + E\partial_v(g\tilde{M}) = \tilde{M}\left[v\partial_x g + \underbrace{E\partial_v g}_{=0!}\right] = M(v)\int \tilde{M}'g'dv' - \tilde{M}g.$$

Simplify by $\tilde{M}$ and observe that : $\frac{M(v)}{\tilde{M}(x,v)}\tilde{M}(x,v') = M(v')$. Thus $v\partial_x g + g = \int M(v')g(x,v')dv'$ and this admits usual elementary solutions by writing $g(x,v) = \exp(-x/\nu)\varphi_\nu(v)$ and then ...

Note : $g(x,v)\exp(2Ex)$ "nearly" satisfies Vlasov eqn.

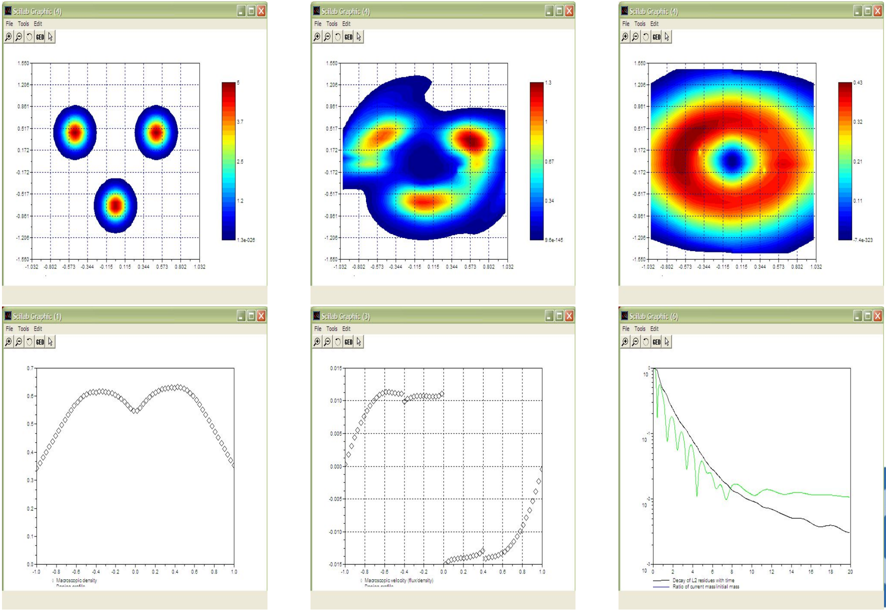
Weakly nonlinear system (Vlasov-Poisson) :

$$\partial_t f + v\partial_x f + E(t,x)\partial_v f = 0,$$

with the attractive gravity field $E = -\partial_x \Phi$ and

$$\partial_{xx}\Phi(t,x) = \int_{\mathbb{R}} f(t,x,v)dv.$$

We want to see circular motion even if macroscopic quantities display strong discontinuities by using a variant of the Perthame-Simeoni scheme [PS Calcolo'01] which consists in rewriting it with interface values computed with matrix products (matrtix depends on interpolation in v).


Scilab Graphic (4)
Scilab Graphic (4)
Scilab Graphic (4)
Scilab Graphic (1)
Macroscopic density
Scilab Graphic (3)
Macroscopic velocity (flux/density)
Scilab Graphic (6)
Decay of L2 residues with time
Ratio of current mass/initial mass