Adaptive mesh refinement techniques for well-balanced schemes for shallow water flows

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NumHyp2011, Roscoff, September 2011

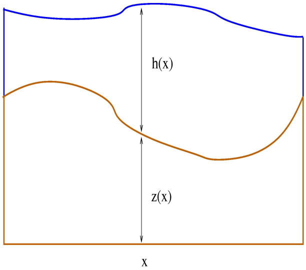
Outline

1. Shock capturing schemes for Shallow water flows
2. Adaptive Mesh Refinement
   - Adaptive schemes
   - Grid hierarchy
3. Well-balanced Adaptive techniques
   - Well-balanced schemes
   - Well-balanced AMR
   - Homogeneous discretization for SWE
   - Well-balanced interpolation
4. Numerical results
   - Numerical results
5. Conclusions

Shallow water flow

- Shallow water equations (SWE) are obtained from incompressible Navier-Stokes equations by depth-averaging and neglecting some terms:

$$h_t + \mathrm{div}(hv) = 0$$
$$(hv)_t + \mathrm{div}(hv \otimes v + \frac{gh^2}{2} I_2) = -gh\nabla z$$



$h \equiv$ water depth,
$v = (v^x, v^y) \equiv$ depth-averaged velocity,
$g \equiv$ gravity acceleration,
$z \equiv$ bottom elevation.

- To simplify, we do the exposition in 1D:

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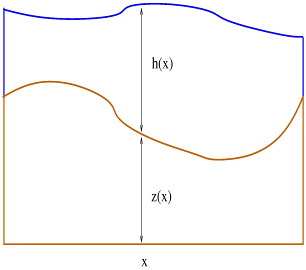

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Shock capturing schemes

- Use notation:

$$u = \begin{bmatrix} h \\ hv \end{bmatrix}, f(u) = \begin{bmatrix} hv \\ hv^2 + \frac{gh^2}{2} \end{bmatrix}, s(x,u) = \begin{bmatrix} 0 \\ -ghz_x \end{bmatrix}$$

so that SWE system can be written as: $\boxed{u_t + f(u)_x = s(x,u)}$.

- Nonlinear hyperbolic system $\Rightarrow$ solutions can develop discontinuities $\Rightarrow$ use **shock capturing** schemes:

$$u_i^{n+1} = u_i^n - \Delta t\left(\frac{\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n}{\Delta x} - s_i^n\right),$$

where $s_i^n(u(x,t)) \approx s(x_i, u(x_i,t_n))$ and the numerical fluxes $\hat{f}_{i+1/2} = \hat{f}(u_{i-s}, \dots, u_{i+s+1})$ verify

$$\left[\frac{\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n}{\Delta x}\right](u(x,t)) \approx f(u)_x(x_i,t_n), \quad x_i = i\Delta x, t_n = n\Delta t$$

and appropriate stability conditions (through **upwinding** and adding numerical viscosity to comply with entropy conditions).

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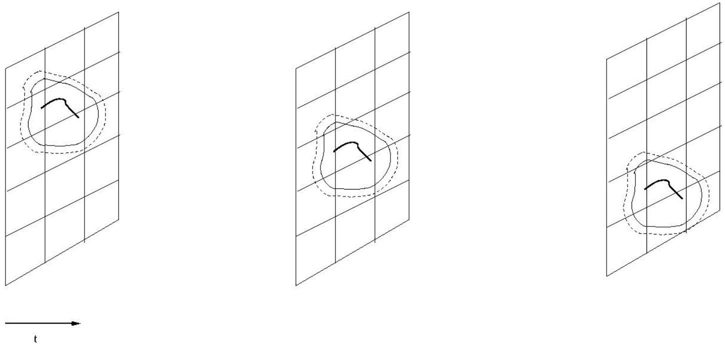
Adaptive schemes

- For $N = 1/\Delta$ and d dimensions, computational cost of scheme is $\mathcal{O}(N^{d+1})$, storage is $\mathcal{O}(N^d)$, huge to get small errors.
- Numerical errors are not uniformly distributed:
  - larger errors at discontinuities
  - smaller errors at smooth regions
- An **Adaptive Scheme**, with a smaller Δ where higher errors occur, would be necessary for $d \geq 2$ and high precision needs.
- Many approaches [Cohen et al., 2003, Müller and Stiriba, 2007] $\cdots$, we briefly review the (**S**tructured) **A**daptive **M**esh **R**efinement algorithm, proposed by [Berger and Oliger, 1984] and extended by many authors (Colella, Quirk, $\cdots$) to FV schemes.

AMR algorithm



- Time evolution for some grid size $\Delta \equiv \Delta x$ and Δt.

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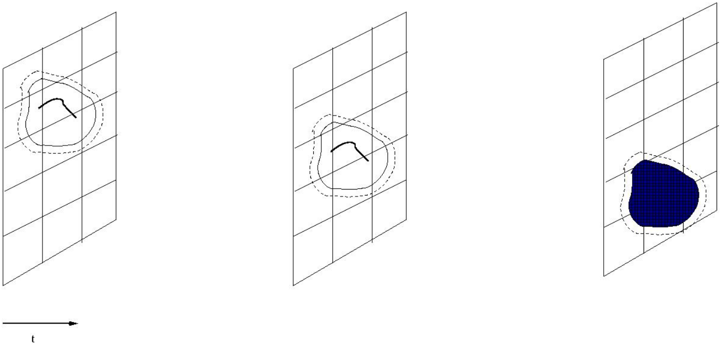

- Want to zoom at **R**egion **O**f **I**nterest, say by using $\Delta/2$.

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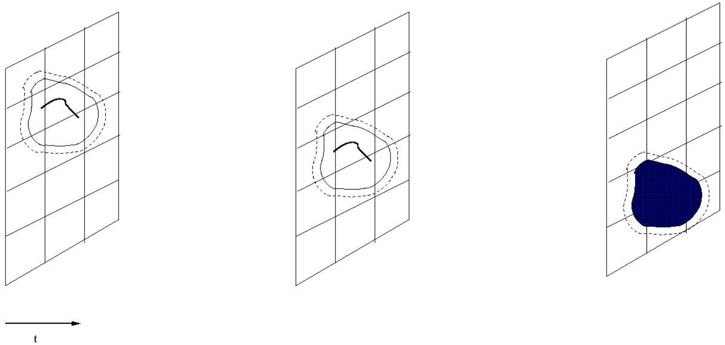

- A: use **interpolation** (zoom), but this causes large errors near shocks.
- B: discard results with Δ, start over with $\Delta/2$.
- C: track region of interest through time evolution.

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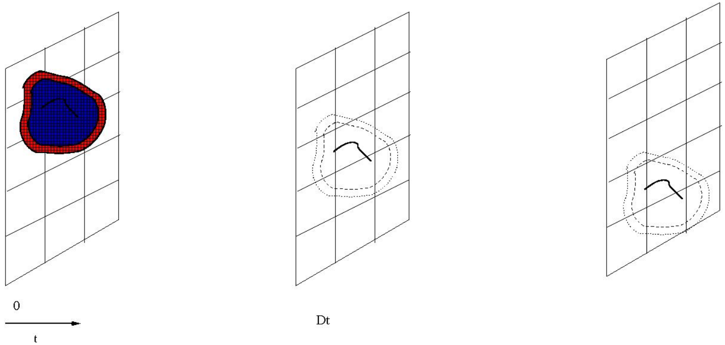

- Before going to B plan, notice that solution on $\Omega \times [0, \Delta t]$ (hopefully) depends on solution at **D**omain of **D**ependence $\widetilde{\Omega} \times \{0\}$ (**by hyperbolicity**).
- Can compute solution at $\Omega \times \{\frac{\Delta t}{2}\}$ (assuming $\Delta/2$ at ROI, same CFL)

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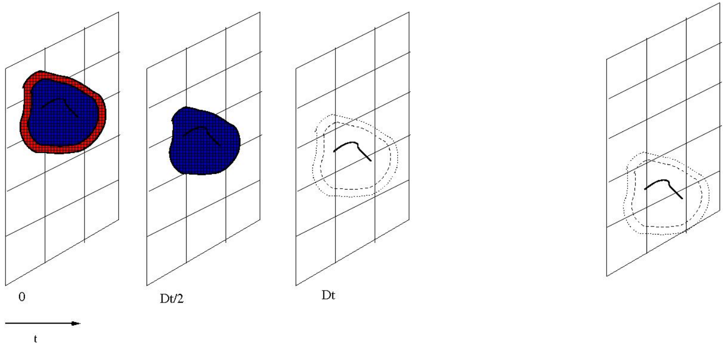

- How can new DD of region of interest be computed?
- **Zooming by** (x, t)**-interpolation**, OK at (supposedly smooth) surrounding band (**coarse** $\rightarrow$ **fine interpolation**)

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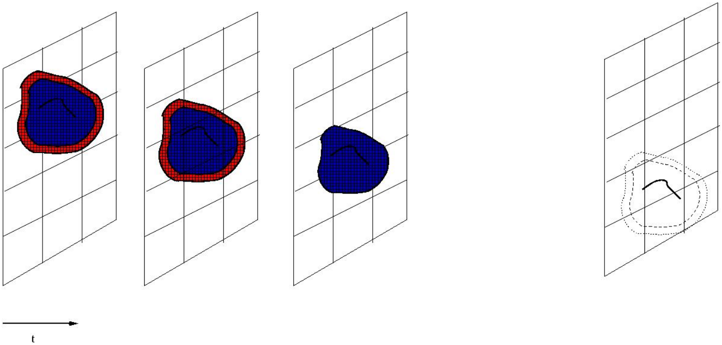

- Recursion $\Rightarrow$ need **nested Grid Hierarchy** (for interpolation), indexed by level l from $l = 0$ (**coarsest**) to $l = L$ (**finest**).
- Must synchronize data through GH at same (x, t) (**fine $\rightarrow$ coarse project.**)
- More (shorter) time steps at finer resolutions (**local time stepping**).

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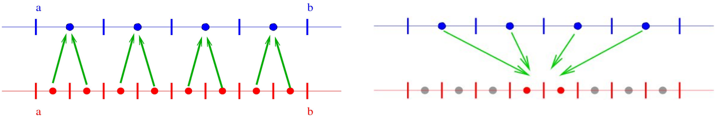
- **Based on cell averages:** Points in the grid hierarchy (show 1D, 2D obtained by cartesian product): $x_i^l = (i + \frac{1}{2})\Delta_0/2^l$, $i = 0, \ldots, N_0 2^l - 1$ (**cell centers**).
- Since $\frac{1}{2}(x_{2i}^{l+1} + x_{2i+1}^{l+1}) = x_i^l$, project solution by averaging

$$\text{Proj}_{l+1 \to l}(u^{l+1})_i = \frac{1}{2}(u_{2i}^{l+1} + u_{2i+1}^{l+1}), \quad i = 0, \ldots, N_0 2^l - 1.$$



fine → coarse coarse → fine

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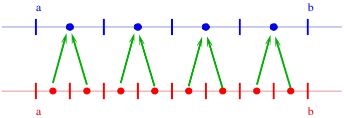

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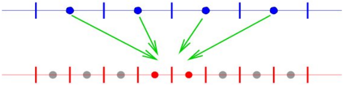

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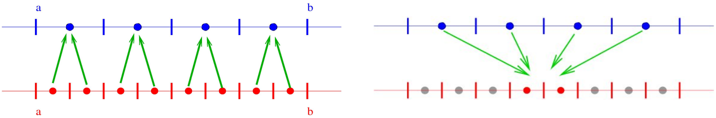

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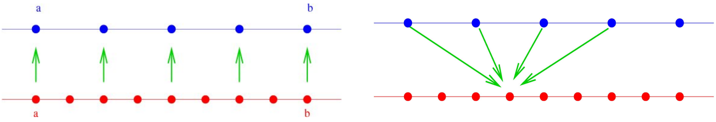

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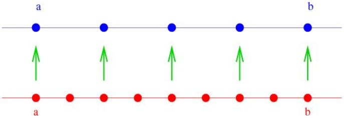

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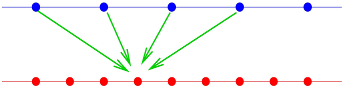

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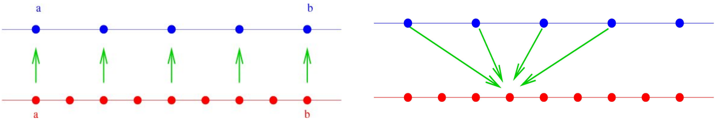

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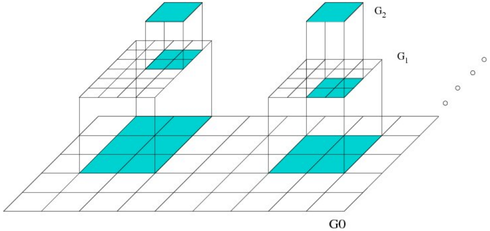

(surrounding band not shown)

- AMR algorithm $\equiv$ **"time evolution" of grid functions** $(u_0^{t_0}, G_0^{t_0}), \dots, (u_L^{t_L}, G_L^{t_L})$ with data $u_l^{t_l} = (u_{l,i}^{t_l} / i \in G_l^{t_l})$ attached to grid points indexed by subsets $G_l^{t_l}$ and associated to times $t_0 \geq t_1 \geq \cdots \geq t_L$ (coarser levels evolve "faster" to provide interpolation data to finer levels)

$$u_{l,i}^{t_l} \approx \begin{cases} u(x_{l,i}, t_l) & \text{point values} \\ \fint_{x_{l,i-\frac{1}{2}}}^{x_{l,i+\frac{1}{2}}} u(x, t_l)dx & \text{cell averages} \end{cases}$$

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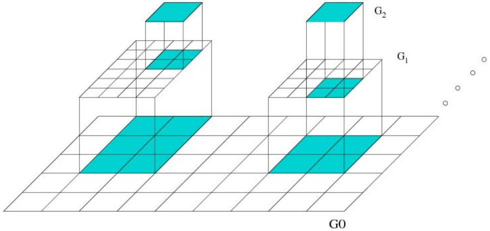

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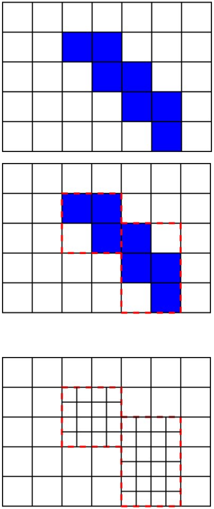
Grid adaption

- Index sets $G_l^{t_l}$ have to evolve in time to track ROI.
- Coarse cells are **marked**, including surrounding band (not shown here), by some **criterion**.
- Marked coarse cells are then **grouped into rectangular patches**, with the **goal** of having (relatively) few large patches for efficiency.
- Coarse cells in rectangular patches are finally **refined**.



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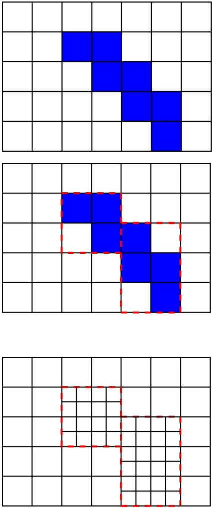

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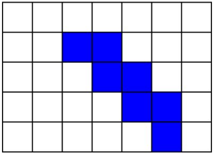

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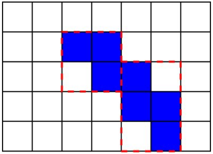

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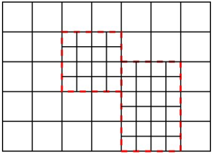

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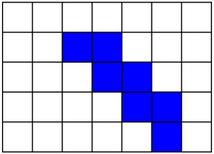

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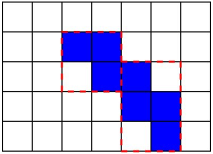

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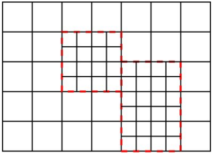

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- Crucial part of algorithm: decide which cells should be refined so as salient flow features are contained in properly refined patches.
- Cells are marked by thresholding based on:
- Large **local truncation errors** [Berger and Oliger, 1984], $\cdots$:
  - Not easy to implement.
- Large **gradients** [Quirk, 1996] $\cdots$
  - Easy, but thresholding is difficult to control (e.g. in rarefactions)
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- Cells are marked by thresholding based on:
- Large **local truncation errors** [Berger and Oliger, 1984], $\cdots$:
  - Not easy to implement.
- Large **gradients** [Quirk, 1996] $\cdots$
  - Easy, but thresholding is difficult to control (e.g. in rarefactions)
- Large **interpolation errors** (related to wavelet coefficient thresholding [Cohen et al., 2003], refine cells that cannot be accurately predicted)
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   - Grid hierarchy
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   - Well-balanced AMR
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Well-balanced schemes

- The convergence of the scheme is usually proved (when possible) through its consistence and stability (this being the harder part).
- When converging to a steady state or dealing with quasi-stationary solutions, the requirement of **preserving steady states** is plausible.
- When the scheme

$$u_i^{n+1} = u_i^n - \Delta t \Big(\frac{\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n}{\Delta x} - s_i^n \Big)$$

does so, that is:

$$f(u(x))_x = s(x, u(x)) \Longrightarrow \left[\frac{\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n}{\Delta x} - s_i^n \right] (u(x)) = 0$$

then the scheme is termed **well-balanced** [Greenberg and Leroux, 1996].
- Special steady state for SWE, **water at rest** ($h + z$ =constant, $v = 0$).
- If a scheme preserves this steady state solution, then the scheme is said to verify the **C-property** [Bermudez and Vazquez, 1994].

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  [Berger-Calhoun-Helzel-LeVeque, 2009, George, 2011].
- Goal: obtain AMR code that preserves steady states (at least water at rest).
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Homogeneous discretization

- We build on [Gascón and Corberán, 2001, Caselles-Donat-Haro, 2009, Donat and Martínez-Gavara, 2011]: PDE can be rewritten in “homogeneous” form:

$$u_t + f(u)_x = s(x,u) \Leftrightarrow u_t + g[u]_x = 0$$

where the **functional** g (dependent on f and s) acts on $u = u(x,t)$ as:

$$g[u](x,t) = f(u(x,t)) - \int_0^x s(r, u(r,t))\, dr$$

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- [Donat and Martínez-Gavara, 2011] propose a **Lax-Wendroff**-type finite differences discretization for $u_t + g[u]_x = 0$, which is hybridized with a first order monotone scheme through **flux-limiting** techniques.
- The scheme applied to exact solution $u(x,t)$ is:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(\overbrace{A_i^n \Delta g_{i-\frac{1}{2}}^n + B_i^n \Delta g_{i+\frac{1}{2}}^n}^{G_{i+\frac{1}{2}}^n - G_{i-\frac{1}{2}}^n})$$

where $G_{i+\frac{1}{2}}$ are numerical fluxes for $g[u]$ and:

$$g_i^n = g[u](x_i, t_n) = f(u(x_i,t_n)) - \int_0^{x_i} s(r, u(r,t_n))dr$$

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C-property preserving interpolation: cell-averages

- In cell-based grid hierarchy, projection is given by $h_{i+\frac{1}{2}} = \frac{1}{2}(h_i + h_{i+1})$, where indexes indicate the point the data is attached to.
- If $h_i = h(x_i)$ correspond to a water at rest solution, does $h_{i+\frac{1}{2}} = \frac{1}{2}(h_i + h_{i+1})$ correspond to point values (at $x_{i+\frac{1}{2}}$) of the solution?
- If it were so, from $h(x) = \eta - z(x)$ we get

$$h_{i+\frac{1}{2}} = h(x_{i+\frac{1}{2}}) = \eta - z(x_{i+\frac{1}{2}}),$$

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$$h_{i+\frac{1}{2}} = \frac{1}{2}\Big(h(x_i) + h(x_{i+1})\Big) = \eta - \frac{1}{2}\Big(z(x_i) + z(x_{i+1})\Big)w$$

so z should verify

$$\frac{z(x_i) + z(x_{i+1})}{2} = z\Big(\frac{x_i + x_{i+1}}{2}\Big), \forall i,$$

which does not hold for general $z \Rightarrow$ Projection not OK for point values
- Projection OK if h_i are cell-averages of stationary solution, but then underlying scheme should preserve them (OK for well-balanced schemes as in Carlos Parés' course, not OK for our scheme).

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- For point value grid hierarchy, the projection from level $l+1$ to level l is given by copying values with even indexes, corresponding to the same point-values, so this projection is automatically well-balanced.
- **Well-balanced** interpolation (related to hydrostatic reconstruction [Audusse-Bouchut-Bristeau-Klein-Perthame, 2004], appears in Carlos Pare's course and Professor Valiani's talk): if we only want to preserve water at rest solutions, given interpolator $I((w_i);x)$ (i.e., $I((w_i);x_j)=w_j$), and

$$V(x,\begin{bmatrix}h\\q\end{bmatrix})=\begin{bmatrix}h+z(x)\\q\end{bmatrix},\quad V(x,\cdot)^{-1}\begin{bmatrix}\eta\\q\end{bmatrix}=\begin{bmatrix}\eta-z(x)\\q\end{bmatrix}$$

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- For point value grid hierarchy, the projection from level $l+1$ to level l is given by copying values with even indexes, corresponding to the same point-values, so this projection is automatically well-balanced.
- **Well-balanced** interpolation (related to hydrostatic reconstruction [Audusse-Bouchut-Bristeau-Klein-Perthame, 2004], appears in Carlos Pare's course and Professor Valiani's talk): if we only want to preserve water at rest solutions, given interpolator $I((w_i);x)$ (i.e., $I((w_i);x_j)=w_j$), and

$$V(x, \begin{bmatrix} h \\ q \end{bmatrix}) = \begin{bmatrix} h+z(x) \\ q \end{bmatrix}, \quad V(x,\cdot)^{-1} \begin{bmatrix} \eta \\ q \end{bmatrix} = \begin{bmatrix} \eta - z(x) \\ q \end{bmatrix}$$

  then we can define an interpolator by

$$\tilde{I}((u_i);x) = V(x,\cdot)^{-1}(I((V_i);x)), \quad V_i = V(x_i, u_i)$$

  (i.e.,interpolate total heights, then subtract bottom height).
- I preserves constants $\Rightarrow \tilde{I}$ preserves water at rest.
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General well-balanced interpolation

- If we can re-write $f(u)_x = s(x,u)$ as $V(x,u)_x = 0$, then $u(x)$ is solution of PDE $\Leftrightarrow V(x,u(x))$ is constant at regions of smoothness + jump conditions .
- $V(x,u) \equiv$ **equilibrium variables**, which are for SWE:

$$V(x, \begin{bmatrix} h \\ hv \end{bmatrix}) = \begin{bmatrix} \frac{v^2}{2} + g(h+z(x)) \\ hv \end{bmatrix}$$

- If $V(x,\cdot)$ is bijective onto some relevant range then we can define an interpolator that preserves equilibrium variables by:

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Outline

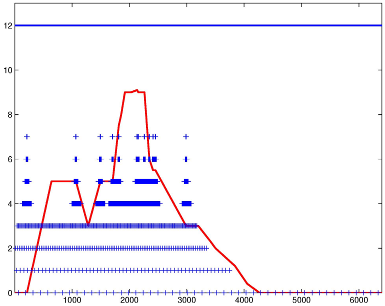
1. Shock capturing schemes for Shallow water flows
2. Adaptive Mesh Refinement
   - Adaptive schemes
   - Grid hierarchy
3. Well-balanced Adaptive techniques
   - Well-balanced schemes
   - Well-balanced AMR
   - Homogeneous discretization for SWE
   - Well-balanced interpolation
4. Numerical results
   - Numerical results
5. Conclusions

Tests setup

- Based on code developed by A. Baeza for cell-based AMR.
- We use point-value-based grid hierarchy, with well-balanced interpolation based on linear interpolation.
- Refinement criterion: mark cells to refine when interpolation error exceeds some relative error **rtol** with respect to the maximal interpolation error at each level.

Test for stationary 1D solutions

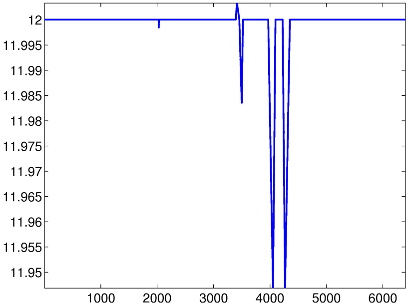
- Water at rest solution of total height=12, bottom topography below. Solution at $T = 200$.
- Have used rtol=10^{-1}, $N_0 = 50$, and eight levels ($L = 7$, $N_7 = 6400$) to obtain:



  with a CPU speedup ≈ 11.5.
- Scheme gives approximated solution such that $||h + z - 12||_\infty = 1.06 \cdot 10^{-14}$ and $||v||_\infty = 3.36 \cdot 10^{-14} \Rightarrow$ **C-property OK to double precision**.

Test for stationary 1D solutions

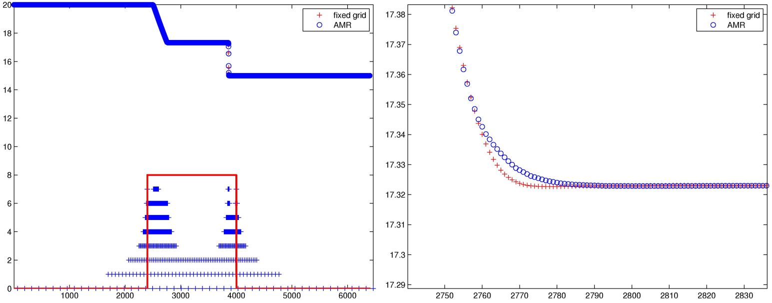
- Same setup, but without well balanced interpolation:



- Scheme gives approximated solution such that $||h+z-12||_\infty = 5.31 \cdot 10^{-2}$ and $||v||_\infty = 2.16 \cdot 10^{-14} \Rightarrow$ **loss of exact C-property.**

Test for non stationary 1D solutions

- Dam break problem with square bump bottom topography.
- Solution at $T = 15$. Have used rtol=10^{-3}, $N_0 = 50$, and eight levels ($L = 7$, $N_7 = 6400$) to obtain:

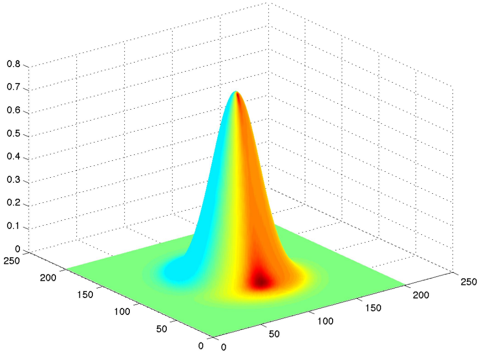


with CPU speedup ≈ 14.04.

- Scheme gives approximated solution such that $||h_{AMR} - h_{fixed}||_1 = 1.44 \cdot 10^{-4}$, $||v_{AMR} - v_{fixed}||_1 = 1.47 \cdot 10^{-4}$

Test for stationary 2D solutions

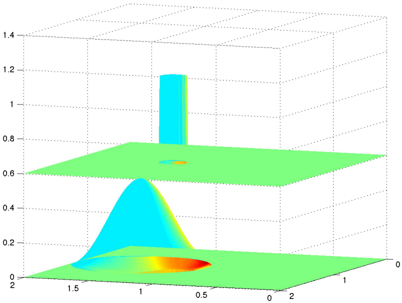
- ([LeVeque, 1998]) Water at rest, total height$= 1$ and bottom:



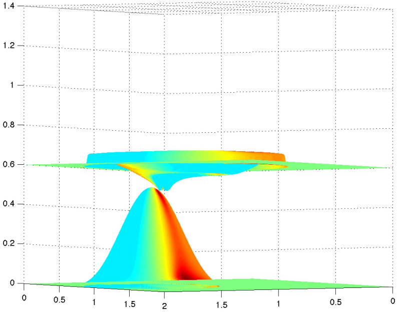
- Have used rtol=10^{-1}, $N_0 = 25$, and 4 levels ($L = 3$, $N_3 = 200$), $T = 0.1$ to obtain: $||h + z - 1||_\infty = 1.11 \cdot 10^{-15}$, $||v^x||_\infty = 3.52 \cdot 10^{-15}$, $||v^y||_\infty = 3.88 \cdot 10^{-15} \Rightarrow$ **C-property OK to double precision.**
- CPU speedup=3.96

Test for non stationary 2D solutions

- Circular dam break problem ([Castro-Fernández-Nieto-Ferreiro-García-Rodríguez-Parés, 2009]). Have used rtol=10^{-1}, $N_0 = 100$, and 5 levels ($L = 4$, $N_4 = 1600$), $T = 0.25$



$T = 0$



$T = 0.25$

- CPU speedup=5.22
- $\|h_{AMR} - h_{fixed}\|_1 = 8.33 \cdot 10^{-4}$, $\|v^x_{AMR} - v^x_{fixed}\|_1 = 1.5 \cdot 10^{-3}$, $\|v^y_{AMR} - v^y_{fixed}\|_1 = 1.4 \cdot 10^{-3}$, difference of mass$\approx 7 \cdot 10^{-4}$.

Test for non stationary 2D solutions

water and grids

In grid hierarchy, ligther color means finer resolution.

Conclusions and future research

Conclusions

- We have presented a technique for obtaining well-balanced point-value-based adaptive mesh refinement schemes for shallow water equations.
- We have seen some of the difficulties for getting well-balanced adaptive mesh refinement schemes for SWE based on cell-averages.
- We have tested the scheme with Donat&Martinez-Gavara homogenized SWE solver and we have obtained an adaptive scheme with the exact C-property.

Future research

- We are working on its parallelization and extension to deal with dry zones.
- Possibility of getting an adaptive scheme that preserves more stationary solutions if underlying scheme does so.
- Comparison of present code with AMR without well-balanced interpolation
- Comparison of present code with AMR with cell-average-based AMR.

Audusse-Bouchut-Bristeau-Klein-Perthame (2004).
A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows.
SIAM J. Sci. Comp, 25:2050–2065.

Baeza, A. and Mulet, P. (2006).
Adaptive mesh refinement techniques for high-order shock capturing schemes for multi-dimensional hydrodynamic simulations.
Internat. J. Numer. Methods Fluids, 52(4):455–471.

Berger, M. J. and Oliger, J. (1984).
Adaptive mesh refinement for hyperbolic partial differential equations.
J. Comput. Phys., 53(3):484–512.

Berger-Calhoun-Helzel-LeVeque (2009).
Logically rectangular finite volume methods with adaptive refinement on the sphere.
Phil. Trans. R. Soc. A, 367:4483–4496.

Bermudez, A. and Vazquez, M. E. (1994).
Upwind methods for hyperbolic conservation laws with source terms.
Comput. & Fluids, 23(8):1049–1071.

Bouchut, F. (2004).

Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources.

Frontiers in Mathematics. Birkhäuser Verlag, Basel.

Bouchut, F. and Morales de Luna, T. (2010).

A subsonic-well-balanced reconstruction scheme for shallow water flows.

SIAM J. Numer. Anal., 48(5):1733–1758.

Caselles-Donat-Haro (2009).

Flux-gradient and source-term balancing for certain high resolution shock-capturing schemes.

Comput. & Fluids, 38(1):16–36.

Castro-Fernández-Nieto-Ferreiro-García-Rodríguez-Parés (2009).

High order extensions of roe schemes for two-dimensional nonconservative hyperbolic systems.

J. Sci. Comput., 39:67–114.

Cohen, A., Kaber, S. M., Müller, S., and Postel, M. (2003).

Fully adaptive multiresolution finite volume schemes for conservation laws.

Math. Comp., 72(241):183–225 (electronic).

Donat, R. and Martínez-Gavara, A. (2011).

A hybrid second order scheme for shallow water flows.
to appear in APNUM.

Gascón, L. and Corberán, J. M. (2001).
Construction of second-order TVD schemes for nonhomogeneous hyperbolic conservation laws.
J. Comput. Phys., 172(1):261–297.

George, D. L. (2011).
Adaptive finite volume methods with well-balanced Riemann solvers for modeling floods in rugged terrain: Application to the Malpasset dam-break flood (France, 1959).
INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, 66(8):1000–1018.

Greenberg, J. M. and Leroux, A. Y. (1996).
A well-balanced scheme for the numerical processing of source terms in hyperbolic equations.
SIAM J. Numer. Anal., 33(1):1–16.

LeVeque, R. J. (1998).
Balancing source terms and flux gradients in high-resolution godunov methods: the quasi-steady wave-propagation algorithm.

J. Comput. Phys., 146:346–365.

Müller, S. and Stiriba, Y. (2007).
Fully adaptive multiscale schemes for conservation laws employing locally varying time stepping.
J. Sci. Comput., 30(3):493–531.

Quirk, J. (1996).
A parallel adaptive grid algorithm for computational shock hydrodynamics.
APPLIED NUMERICAL MATHEMATICS, 20(4):427–453.