Models & numerical schemes for free surface flows

Beyond the Saint-Venant system

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Also with
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NumHyp
Roscoff - September 2011

Navier-Stokes vs. Saint-Venant

2D Navier-Stokes $H, \mathbf{u} = (u, w)$

$$\begin{cases} \operatorname{div} \underline{\mathbf{u}} = 0, \\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)\underline{\mathbf{u}} + \nabla p = \mathbf{G} + \operatorname{div} \underline{\underline{\Sigma}}, \end{cases}$$

Saint-Venant $H, \bar{u}$

$$\begin{cases} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0, \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} - \kappa(\bar{u}) \end{cases}$$

Advantages of Saint-Venant compared to Navier-Stokes

But more accurate approximation of the Navier-Stokes system are needed

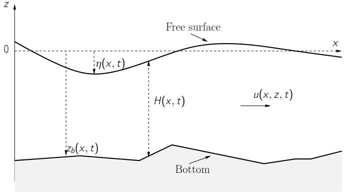
Outline & Main ideas

Beyond the Saint-Venant system

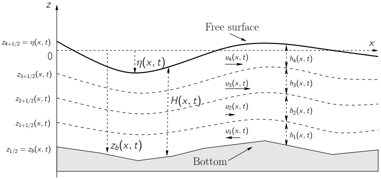
- Multilayer approximations of the Navier-Stokes system
  - valid for miscible fluids with variable density
  - with non-hydrostatic terms
- Numerical schemes & simulations
  - kinetic description
  - discretization of the source terms
- Hydrodynamics & couplings
  - biological or erosion models

Beyond the Saint-Venant system

Objective



$\Rightarrow$



- Many contributions (Audusse, Bouchut, LeVeque, Pares, . . .)
- Valid for non-miscible fluids
- Pb. with underlying physics,. . .

Key idea

Saint-Venant
$u(x, z, t) \approx \bar{u}(x, t)$

$\Rightarrow$

Multilayer Saint-Venant
$u(x, z, t) \approx \sum_{\alpha=1}^{N} \mathbf{1}_{z \in L_\alpha(x,t)} u_\alpha(x, t)$

Model derivation

Starting point

$$\text{(Euler hydro)} \quad \left\{ \begin{array}{c} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial p}{\partial z} = -g \end{array} \right.$$

Obtained model

- Weak form ($\mathbb{P}_0^t$) of the continuity equation with $\mathbb{P}_0^t = \left\{ \mathbf{1}_{z \in L_\alpha(x,t)},\ 1 \leq \alpha \leq N \right\}$

$$\frac{\partial h_\alpha}{\partial t} + \frac{\partial h_\alpha u_\alpha}{\partial x} = G_{\alpha+1/2} - G_{\alpha-1/2}$$

$$G_{\alpha+1/2} = \frac{\partial z_{\alpha+1/2}}{\partial t} + u_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} - w_{\alpha+1/2} = \sum_{j=1}^{\alpha} \frac{\partial h_j}{\partial t} + \frac{\partial h_j u_j}{\partial x}$$

$G_{1/2} = G_{N+1/2} = 0$ (kinematic boundary conditions)

Model derivation

Starting point

$$\text{(Euler hydro)} \quad \left\{ \begin{array}{l} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial p}{\partial z} = -g \end{array} \right.$$

Obtained model

- Weak form $(\mathbb{P}_0^t)$ of the Euler system $\mathbb{P}_0^t = \left\{ \mathbf{1}_{z \in L_\alpha(x,t)},\ 1 \leq \alpha \leq N \right\}$

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{N} \frac{\partial}{\partial x}(h_\alpha u_\alpha) = 0 \\ \frac{\partial (h_\alpha u_\alpha)}{\partial t} + \frac{\partial}{\partial x}\left(h_\alpha u_\alpha^2 + \frac{g}{2} h_\alpha f(\{h_j\}_{j \geq \alpha})\right) = F_{\alpha+1/2} - F_{\alpha-1/2} \\ \frac{\partial E_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(u_\alpha \left(E_\alpha + \frac{g}{2} h_\alpha H\right)\right) = E_{\alpha+1/2} - E_{\alpha-1/2} \end{array} \right.$$

- Only one "global" continuity equation, $H = \sum h_\alpha$
- Exchange terms $G_{\alpha+1/2}$, $F_{\alpha+1/2} = u_{\alpha+1/2} G_{\alpha+1/2} + P_{\alpha+1/2}$,

$$E_{\alpha+1/2} = \frac{u_{\alpha+1/2}^2}{2} G_{\alpha+1/2} + u_{\alpha+1/2} P_{\alpha+1/2}$$

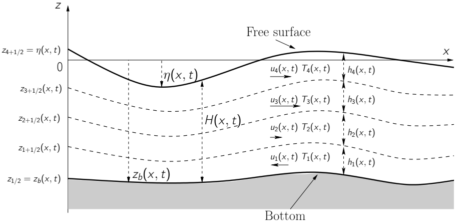
- If $G_{\alpha+1/2} \equiv 0$ non-miscible fluids, N cont. equations

Hydrostatic Euler (NS) with varying density (Aud.,Brist.,Pela.,JSM JCP 2010)

Starting point

$$\begin{cases} \dot{\rho} + \mathrm{div}\,(\rho \underline{\mathbf{u}}) = 0, \\ \dot{\overline{\rho \underline{\mathbf{u}}}} + (\underline{\mathbf{u}}.\nabla)(\rho \underline{\mathbf{u}}) + \nabla p = \rho \mathbf{G}, \\ \dot{\overline{\rho T}} + \mathrm{div}\,(\rho T \underline{\mathbf{u}}) = \mu_T \Delta T, \end{cases}$$

with $\rho = \rho(\{T, S\}) \quad (= \rho(T, S, H))$



Obtained model

$$u(x,z,t) \approx \sum_{\alpha=1}^{N} \mathbf{1}_{z \in L_\alpha} u_\alpha(x,t)$$

$$\begin{cases} \frac{\partial}{\partial t} \sum_{\alpha=1}^{N} (\rho_\alpha h_\alpha) + \sum_{\alpha=1}^{N} \frac{\partial}{\partial x} (\rho_\alpha h_\alpha u_\alpha) = 0, \\ \frac{\partial(\rho_\alpha h_\alpha u_\alpha)}{\partial t} + \frac{\partial}{\partial x} \left(\rho_\alpha h_\alpha u_\alpha^2 + \frac{g}{2} h_\alpha f(\{\rho_j h_j\}_{j \geq \alpha}) \right) = \\ \qquad + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} + \textit{Sc. Terms}, \\ \frac{\partial(\rho_\alpha h_\alpha T_\alpha)}{\partial t} + \frac{\partial}{\partial x} (\rho_\alpha h_\alpha u_\alpha T_\alpha) = T_{\alpha+1/2} G_{\alpha+1/2} - T_{\alpha-1/2} G_{\alpha-1/2}, \end{cases}$$

Properties of the model

- **Hyperbolicity ?**
  quasi-linear form $X = (H, u_1, \dots, u_N, E_1, \dots, E_N)^T$

$$M(X)\dot{X} + F(X)\frac{\partial X}{\partial x} = 0$$

  - "often" hyperbolic
  - strictly hyperbolic for $N = 2$
  - for $N > 2$, "arrow matrices" and interlacing of eigenvalues

$$\frac{1}{N}\sum_{1}^{N} u_i^2 \leq gH \qquad (generalized\ \ Froude\ \ number)$$

  - family of entropies
- When $N \to \infty$?

Non-hydrostatic terms

The Euler system

$$\left\{\begin{array}{l} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{\partial p}{\partial z} = -g \end{array}\right.$$

Two main approaches

- Irrotational flows
  - many contributors (Bona, Dutyk, Lannes, Saut,. . .)
- Non-hydrostatic Saint-Venant system
  - mainly $\frac{\partial w}{\partial t}$ (Peregrine,. . .)
  - for shallow water flows

Approximation of the Euler (NS) system (JSM M3AS'11)

Starting point

$$(\mathcal{S}) \begin{cases} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{\partial p}{\partial z} = -g \end{cases}$$

- For Shallow Water systems : simple vertical mean
- **Key Idea** : vertical momenta $\int_{z_b}^{\eta} z^i \, (\mathcal{S}) \, dz, \; i = 0, 1$
- Compact form

$$\frac{\partial X}{\partial t} + \frac{\partial}{\partial x} F(X, \bar{p}) - R(X, \bar{p}) = S(X, \bar{p})$$

with $X = (H, \bar{u}, \bar{w}, \ldots)$

- Possibly vertical semi-discretization (multilayer)

$$u(x, z, t) \approx \sum_{\alpha=1}^{N} \mathbf{1}_{z \in L_\alpha}(z) u_\alpha(x, t)$$

Averaged Euler system (non-hydrostatic Saint-Venant)

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\overline{u}) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\eta^2 - z_b^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{\eta^2 - z_b^2}{2}\overline{u}\right) - H\overline{w} = 0$$

$$\frac{\partial}{\partial t}(H\overline{u}) + \frac{\partial}{\partial x}\left(H\overline{u}^2 + H\overline{p}\right) = -p_b\frac{\partial z_b}{\partial x}$$

$$\frac{\partial}{\partial t}\left(\frac{\eta^2 - z_b^2}{2}\overline{u}\right) + \frac{\partial}{\partial x}\left(\frac{\eta^2 - z_b^2}{2}\overline{u}^2 + \int_{z_b}^{\eta} zp\ dz\right) - H\overline{w}\,\overline{u} = -\frac{1}{2}\frac{\partial z_b^2}{\partial x}p_b$$

$$\frac{\partial}{\partial t}(H\overline{w}) + \frac{\partial}{\partial x}(H\overline{wu}) = p_b - gH$$

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} z\hat{w}\ dz + \frac{\partial}{\partial x}\overline{u}\int_{z_b}^{\eta} z\hat{w}\ dz - \int_{z_b}^{\eta} \hat{w}^2\ dz - H\overline{p} = z_b p_b - g\frac{\eta^2 - z_b^2}{2}$$

- Only 5 unknowns H, $\overline{u}$, $\overline{w}$, $\overline{p}$, p_b... but only 5 independant equations

What to do with such models ?

- A family of models approximating the Navier-Stokes system
  - only formal convergence as $N \to \infty$
  - math. analysis ?
- Good candidates
  - rigourous derivation process
  - energy balance, entropies
- Simpler than the corresponding Navier-Stokes system
  - independant of z, **fixed meshes**
- But not so simple to analyse & discretize !

Kinetic approach for conservation laws

- A fantastic tool for
  - physical understanding (upwinding)
  - mathematical analysis
  - numerical analysis and schemes

- Basis : adopt a microscopic description (Boltzmann)

$$\textit{Cont. model} \quad \Leftrightarrow \quad \frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} - g \frac{\partial z_b}{\partial x} \frac{\partial M}{\partial \xi} = Q(x,t,\xi)$$

  - $M(x,t,\xi)$ particle density, $Q(x,t,\xi)$ collision term ($= 0$ a.e.)
  - $\int_{\mathbb{R}} \xi^p M \, d\xi$ gives the macroscopic variables
  - linear transport equation + Vlasov

- Only kinetic representations and not kinetic formulations

Kinetic representation of the Saint-Venant system

- Gibbs equilibrium $M(x,t,\xi) = \frac{H}{c}\chi\left(\frac{\xi-\bar{u}}{c}\right)$ with $c = \sqrt{gH/2}$

where $\chi(\omega) = \chi(-\omega) \geq 0$, $\text{supp}\,(\chi) \subset \Omega$, $\int_{\mathbb{R}} \chi(\omega) = \int_{\mathbb{R}} \omega^2\chi(\omega) = 1$

Proposition (Audusse, Bristeau, Perthame)

The functions $(H,\bar{u},E)(t,x)$ are strong solutions of the Saint-Venant system if and only if $M(x,t,\xi)$ is solution of the kinetic equation

$$(\mathcal{B}), \qquad \frac{\partial M}{\partial t} + \xi\frac{\partial M}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi} = Q(x,t,\xi)$$

where $Q(t,x,\xi)$ is a "collision term".

- Macroscopic variables $(H,\bar{u},E) = \int_{\mathbb{R}}(1,\xi,\xi^2/2)M\,d\xi$
- A linear transport equation ...easy to upwind

Kinetic interpretation for the multilayer system (hydrostatic)

- Gibbs equilibria
  - $M_\alpha(x,t,\xi) = \frac{h_\alpha}{c_\alpha}\chi\left(\frac{\xi-u_\alpha}{c_\alpha}\right)$, with $c_\alpha = \sqrt{gf(\{h_j\}_{j\geq\alpha})}$
  - $N_{\alpha+1/2}(x,t,\xi) = G_{\alpha+1/2}\,\delta\left(\xi - u_{\alpha+1/2}\right)$,

Proposition (Audusse, Bristeau, Perthame, JSM 2009)

The functions $(h_\alpha, u_\alpha, E_\alpha)(t,x)$ are strong solutions of the multilayer Saint-Venant system if and only if the set $\{M_j(x,t,\xi)\}_{j=1}^N$ is solution of the kinetic equations

$$\frac{\partial M_\alpha}{\partial t} + \xi\frac{\partial M_\alpha}{\partial x} - N_{\alpha+1/2} + N_{\alpha-1/2} = Q_\alpha(x,t,\xi)$$

- source terms (exchanges, pressure)
- also for variable density case

Kinetic interpretation for the Euler system (JSM, M3AS'11)

- Starting point

(Euler) $$\left\{\begin{array}{l} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{\partial p}{\partial z} = -g \end{array}\right.$$

- Kinetic interpretation for the averaged Euler system

$$(B_M) \quad \frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} - g \frac{\partial M}{\partial \gamma} = Q_1,$$

$$(B_{R,M}) \quad \frac{\partial R}{\partial t} + \xi \frac{\partial R}{\partial x} - \gamma M = Q_2,$$

with the Gibbs equilibria

$$M(x,t,\xi,\gamma) = \frac{H}{c_1 c_2} \chi\left(\frac{\xi - \overline{u}}{c_1}\right) \psi\left(\frac{\gamma - \overline{w}}{c_2}\right)$$

$$R(x,t,\xi,\gamma) = \frac{\eta^2 - z_b^2}{2c_3} \chi\left(\frac{\xi - \overline{u}}{c_3}\right) \delta\left(\gamma - \check{w}\right)$$

- Proof: $\int_{\mathbb{R}^2} (1, \xi, \gamma, |\xi|^2/2) \mathcal{B}_M d\xi d\gamma$, $\int_{\mathbb{R}^2} (1, \xi, \gamma) \mathcal{B}_{R,M} d\xi d\gamma$
- Also multilayer approximation

Main properties of the schemes (hydrostatic models)

- $f_i^{n+1}(\xi) = M_i^n(\xi) - \xi\sigma_i^n \left(M_{i+1/2}^n(\xi) - M_{i-1/2}^n(\xi) \right) + \Delta t^n S_i^n(\xi)$
- Positive schemes (CFL=1 but more complex)
- 2^{nd} order schemes (space & time)
- Maximum principle (tracer)
- Well balanced (with hydrostatic reconstruction)
- The source terms (pressure)

$$gH\frac{\partial z_b}{\partial x} \quad \Rightarrow \quad g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi}, \qquad \frac{\partial z_{\alpha+1/2}}{\partial x}\frac{\partial M_\alpha}{\partial \xi}$$

  - kinetic interpretation useful for discretization
- Numerical cost $N \times SW$, possibly mesh adaptation

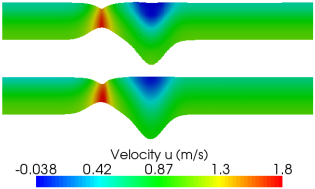
Analytical validation (I) (Boulanger, JSM 2011)

- Analytical solutions to the Euler system
  - 2D and 3D solutions, for any bottom topography $z_b(x,y)$
  - with entropic shocks
  - not necessarily free surface flows
  - In 2D (continuous solutions), u and H characterized by

$$u = \alpha\beta \frac{\cos\beta(z-z_b)}{\sin\beta H}, \qquad \left(g(H+z_b) + \frac{\alpha^2\beta^2}{2\sin(\beta H)^2}\right)_x = 0$$

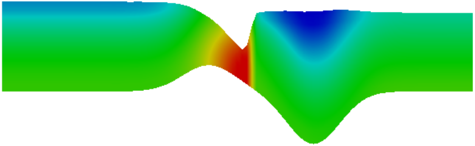
- Recovered by the 2D and 3D codes

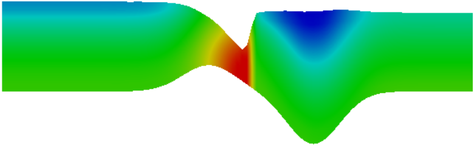
  - analytic
  - simulated



Analytical validation (II)

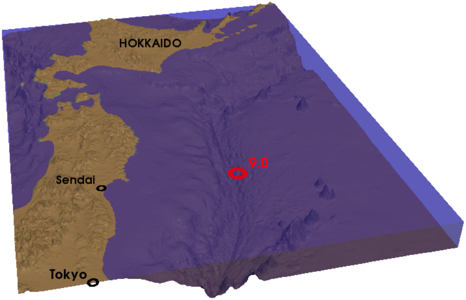
- With shocks

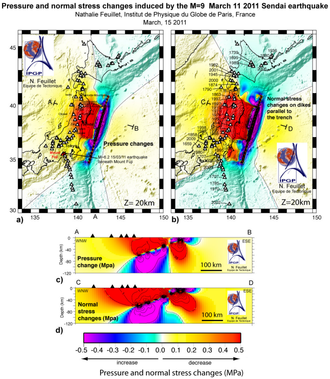




- Also without free surface

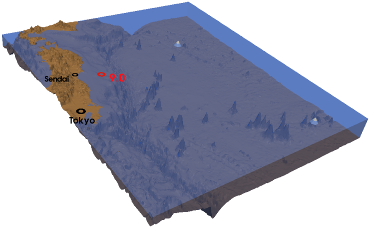
Seism in Japan, march 2011

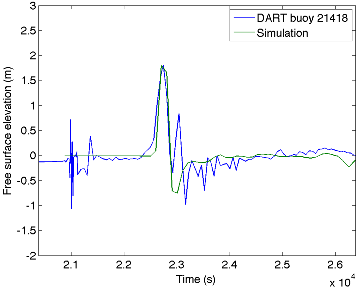


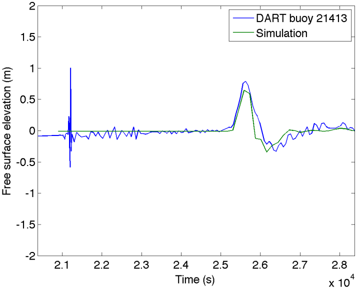


seism source IPGP (A. Mangeney)

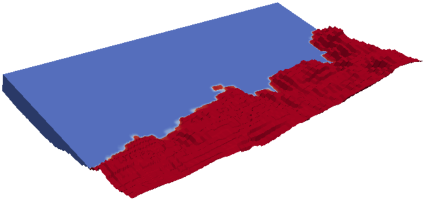
"Validation" with DART buoys







The next tsunami : Roscoff 6.0 !

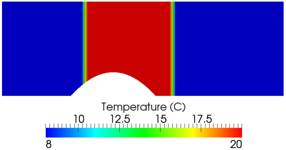


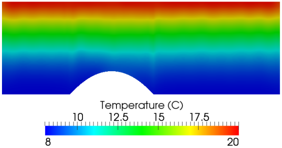
Archimedes law

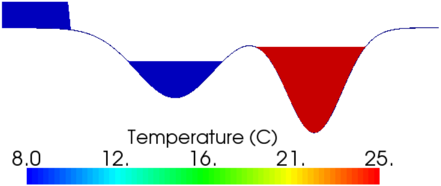
- Well-balanced schemes

$$\frac{\partial}{\partial x}\left(\int_z^\eta \rho g \; dz\right) = 0 \qquad \neq \qquad \frac{\partial \eta}{\partial x} = 0$$

- Stability

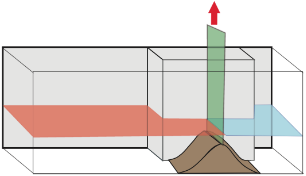




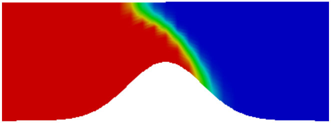


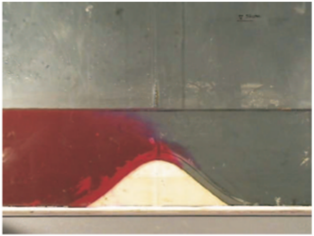
Archimedes law (with C. Pares)

- Experimental set up (simulation)



- Velocity field (6 s)





Hydrodynamics-biology coupling

Description : algae pool driven into motion by a paddle wheel

Goal : optimize the biomass production by playing on the nutrients supply, water depth, agitation,. . .

The biological model

- Droop model (Droop 1983)

$$\begin{cases} \frac{dC_1}{dt} = \mu(\frac{C_2}{C_1}, I)C_1 - RC_1 \\ \frac{dC_2}{dt} = -\lambda(C_3, \frac{C_2}{C_1})C_1 \\ \frac{dC_3}{dt} = \lambda(C_3, \frac{C_2}{C_1})C_1 - RC_2 \end{cases}$$

with C_1 : phytoplanktonic carbon, C_2 residual nitrates and C_3 phytoplanktonic nitrogen, I light, R death rate

- Advection, reaction and diffusion PDE's

$$\frac{\partial X}{\partial t} + \nabla.(\mathbf{u}X) = F(X) + \nu\Delta X$$

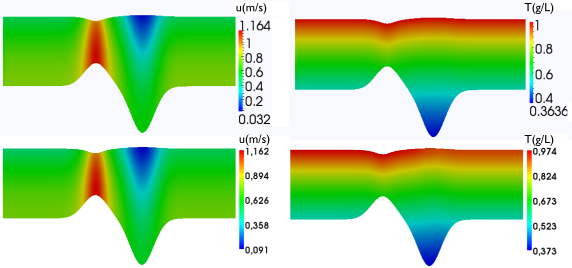
with $X = (C_1, C_2, C_3)^T$

- Hydrodynamics ($\mathbf{u}$) governed by the Navier-Stokes Eqs
- Multilayer discretization & kinetic interpretation

Numerical validation (Boulanger, JSM 2011)

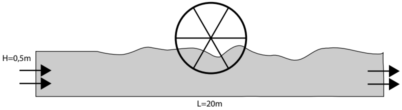
- Analytical solution

(Euler hydro + bio)
$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u^2}{\partial x} + \dfrac{\partial uw}{\partial z} + \dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial p}{\partial z} = -g \\ \dfrac{\partial T}{\partial t} + \dfrac{\partial uT}{\partial x} + \dfrac{\partial wT}{\partial z} = f(x,z)T \end{cases}$$

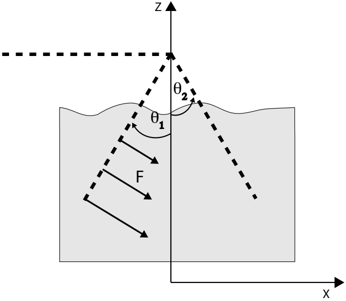


Raceway

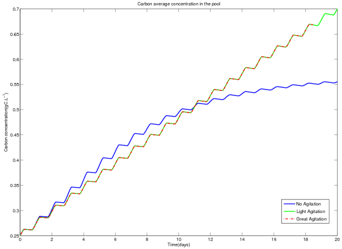
- A '1D' raceway



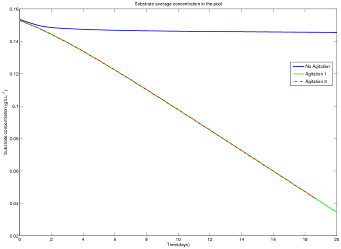
- The paddlewheel



Simulations results



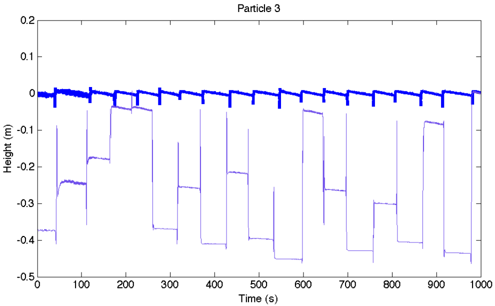
Carbon production



Substrate

Lagrangian trajectories

- $\frac{dM(t)}{dt} = \mathbf{u}(M(t), t)$



Poisition of a particle along time and free surface

- 3d animation

For the future

- Numerical analysis & schemes
  - for the full NS system
  - hyperbolicity, entropies,. . .
  - stability of the schemes
  - towards industrial codes
- Non-hydrostatic models
  - efficient schemes
- Hydrodynamics and couplings
  - erosion, carriage and associated problems
  - hydrodynamics-biology coupling, water quality management
- Control, stabilization, data assimilation,. . .
  - everything is more simple at the kinetic level