Numerical Approximations of Hyperbolic Systems
with Source Terms and Applications

An engineering approach to the source terms treatment in shallow water equations

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Overview of recent years research activity in
Numerical Hydraulics at ENDIF, University of Ferrara:

- DFB technique (Divergence Form for Bed Slope Source Term) for 1D and 2D SWE

- Well-balancing in CWENO schemes for SWE on fixed bed
- Well-balancing in CWENO schemes for SWE on movable bed

- Well-balanced bottom discontinuities treatment for high-order SWE in a WENO context
- A digression: some useful analytical tools for SW Flows

- Towards compact schemes: a balanced HWENO scheme for SWE
- Towards compact schemes: Balancing RKDG methods on domains with curved boundaries

Divergence Form for Bed Slope Source Term in Shallow Water Equations

Valiani & Begnudelli, 2006, ASCE Journal of Hydraulic Engineering, 132 (7), 652–665
Valiani & Begnudelli, 2008, Closure, ASCE Journal of Hydraulic Engineering, 2008, 134 (5), 680-682

SWE are extensively used in numerical modeling of rivers, estuaries, lakes and coastal areas, as well as river flooding due to dam break or banks failure.
In real life, complex geometry and uneven topography are commonly encountered.

The proper numerical integration of bed slope source term is fundamental to obtain correct results.

Fundamental literature:
- Fractional step method (Toro, 1999)
- Upwind Method (Bermudez & Vázquez, 1994, Hubbard & Garcia-Navarro, 2000)
- Wave-propagation method (LeVeque, 1998)

In the present method (DFB), the bed slope source term is expressed as the divergence of a proper tensor, just like it happens for the momentum flux tem.

2D SWE: $$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{S} \qquad \left[\ \mathbf{F} = \left(\mathbf{E}, \mathbf{G}\right)\ \right]$$

$$\mathbf{U} = \begin{bmatrix} h \\ Uh \\ Vh \end{bmatrix}; \quad \mathbf{E} = \begin{bmatrix} Uh \\ U^2 + g\dfrac{h^2}{2} \\ UVh \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} Vh \\ UVh \\ V^2 h + g\dfrac{h^2}{2} \end{bmatrix}; \quad \mathbf{S} = \begin{bmatrix} 0 \\ gh\left(S_{0x} - S_{fx}\right) \\ gh\left(S_{0y} - S_{fy}\right) \end{bmatrix}$$

Bed Slope and friction
Source terms:

$$\begin{cases} S_{0x} = -\dfrac{\partial z_b}{\partial x} \\ S_{0y} = -\dfrac{\partial z_b}{\partial y} \end{cases} \qquad \begin{cases} S_{fx} = \dfrac{U\sqrt{U^2 + V^2}}{C^2 gh} \\ S_{fy} = \dfrac{V\sqrt{U^2 + V^2}}{C^2 gh} \end{cases}$$

SIMPLE IDEAS inspiring the method

- CONSERVATION is easier to obtain if we express complex quantities as the divergence of some other quantities:
Complicated things = div (different things)

- No matter how complex the bed elevation is:
Null velocity ↔ water elevation is a constant

- Hydraulic jump over flat bed must satisfy:
Total force conservation AND mass conservation

- Uniform flow must satisfy:
Bed Slope = Friction Slope

BED SLOPE SOURCE TERMS:

$$\begin{cases} ghS_{0x} = -gh\dfrac{\partial z_b}{\partial x} \\ ghS_{0y} = -gh\dfrac{\partial z_b}{\partial y} \end{cases}$$

May be rewritten as:

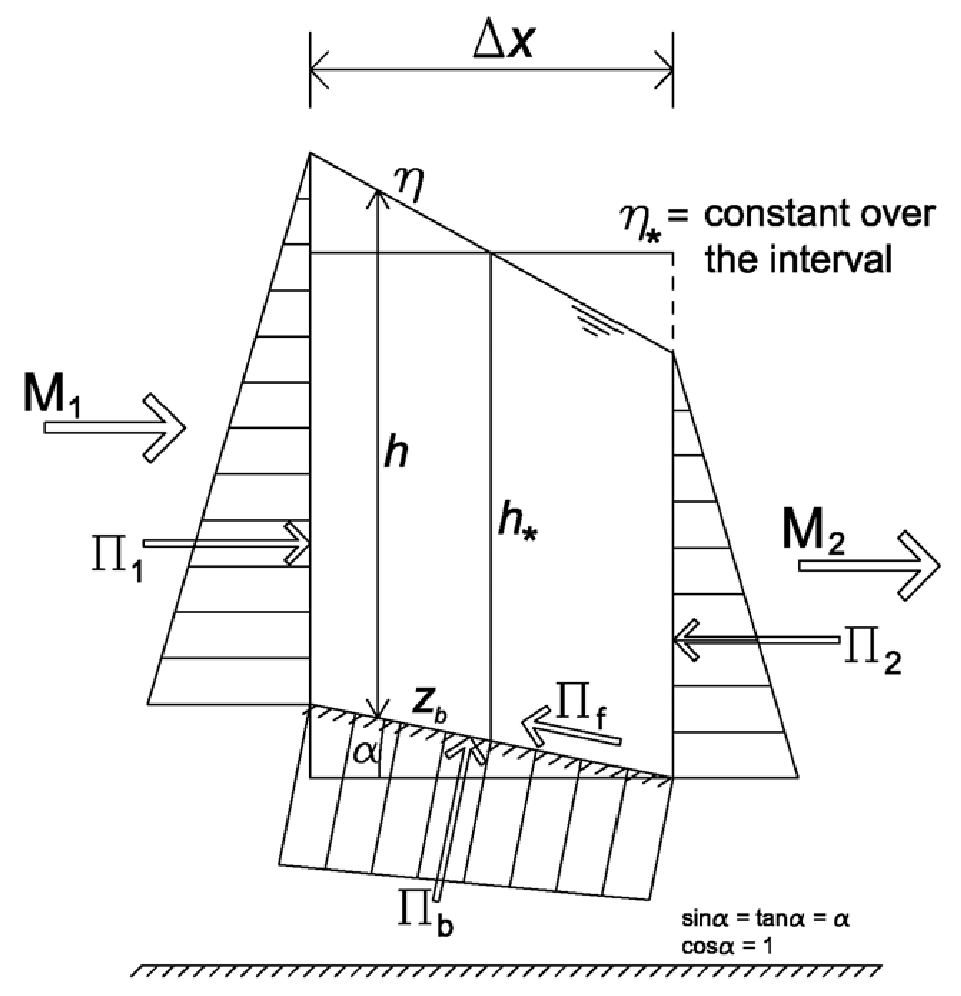
$$\begin{aligned}
& gh\left(-\frac{\partial z_b}{\partial x}\right) = \\
& = g\left(\eta - z_b\right)\left(-\frac{\partial z_b}{\partial x}\right) = g\eta\left(-\frac{\partial z_b}{\partial x}\right) + gz_b\left(\frac{\partial z_b}{\partial x}\right) = \\
& = -g\frac{\partial}{\partial x}\left(\eta z_b\right)\bigg|_{\eta=\text{const}} + g\frac{\partial}{\partial x}\left(\frac{z_b^2}{2}\right) = g\frac{\partial}{\partial x}\left(-\eta z_b + \frac{z_b^2}{2}\right)\bigg|_{\eta=\text{const}} \\
& = g\frac{\partial}{\partial x}\left(\frac{\eta^2}{2} - \eta z_b + \frac{z_b^2}{2}\right)\bigg|_{\eta=\text{const}} = g\frac{\partial}{\partial x}\left[\frac{1}{2}\left(\eta - z_b\right)^2\right]\bigg|_{\eta=\text{const}} \\
& = \frac{\partial}{\partial x}\left[\frac{1}{2}gh^2\right]\bigg|_{\eta=\text{const}}
\end{aligned}$$

⇨ $$gh\left(-\frac{\partial z_b}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right)\bigg|_{\eta=\text{const}}$$

BED SLOPE SOURCE TERMS:
PHYSICAL MEANING

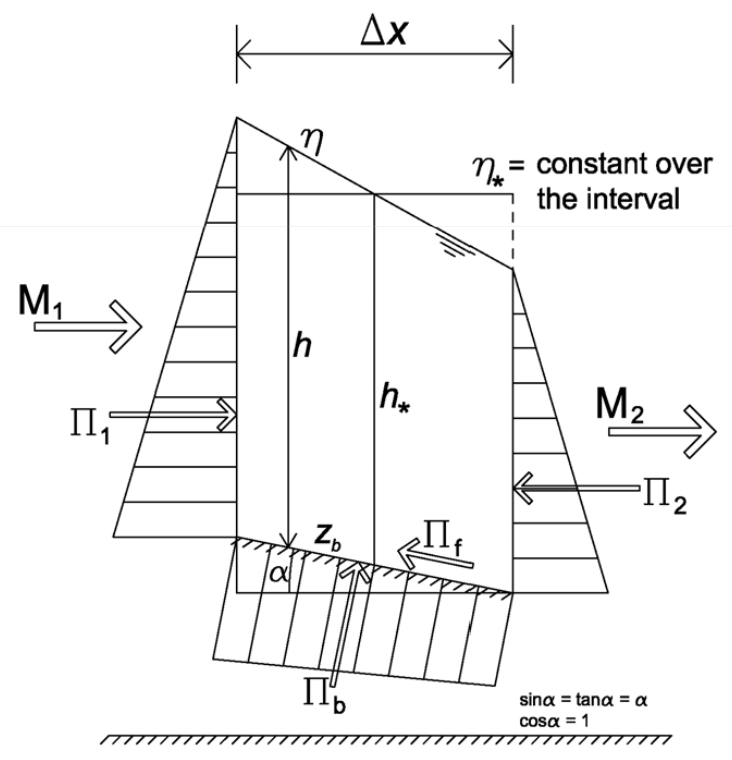
$$\Pi_{1x} - \Pi_{2x} + \Pi_{bx} - \Pi_{fx} = M_{1x} - M_{2x}$$

$$\begin{cases} \Pi_{1x} \ \Pi_{2x} & = \text{static forces over surfaces 1, 2} \\ \Pi_{bx} & = x \text{ comp. of bottom pressure forces} \\ \Pi_{fx} & = \text{bottom friction} \\ M_{1x} \ M_{1x} & = \text{momentum fluxes over surfaces 1, 2} \end{cases}$$



BED SLOPE SOURCE TERMS: PHYSICAL MEANING

$$\Pi_{1x} - \Pi_{2x} + \Pi_{bx} - \Pi_{fx} = M_{1x} - M_{2x}$$



$$\begin{cases} \Pi_{1x} = \frac{1}{2}\rho g\, h_1^2\,; \quad \Pi_{2x} = \frac{1}{2}\rho g\, h_2^2 \\ \Pi_{bx} = \rho g\, h_*\, \Delta x \sin\alpha\,; \quad \Pi_{fx} = \rho g\, h_*\, S_f\, \Delta x \\ \mathrm{M}_{1x} = \rho\, U_1^2 h_1\,; \quad \mathrm{M}_{2x} = \rho\, U_2^2 h_2 \end{cases}$$

$$\Rightarrow \frac{1}{\Delta x}\left[\left(\frac{1}{2}gh_2^2 + U_2^2 h_2\right) - \left(\frac{1}{2}gh_1^2 + U_1^2 h_1\right)\right] = $$

$$= gh_*\left(-\frac{\Delta z_b}{\Delta x}\right) - gh_*\, S_f$$

$$\lim_{\Delta x \to 0} -gh_* \frac{\Delta z_b}{\Delta x} = \lim_{\Delta x \to 0} -g\frac{\Delta z_b}{\Delta x}\left(\eta_* - \frac{z_{b2} + z_{b1}}{2}\right) =$$

$$= \lim_{\Delta x \to 0} \frac{g}{2} \frac{\left(\eta_* - z_{b2}\right)^2 - \left(\eta_* - z_{b1}\right)^2}{\Delta x} = \frac{\partial}{\partial x}\left(\frac{1}{2} g h^2\right)\bigg|_{\eta = \eta_*}$$

BED SLOPE SOURCE TERMS: TWO-DIMENSIONAL APPROACH

$$\left.\begin{aligned} gh\left(-\frac{\partial z_b}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\right)_{\eta=\text{cost}} \\ gh\left(-\frac{\partial z_b}{\partial y}\right) &= \frac{\partial}{\partial y}\left(\frac{1}{2}gh^2\right)_{\eta=\text{cost}} \end{aligned}\right\} -gh\,\nabla z_b = \nabla\left.\left(\frac{1}{2}g\,h^2\right)\right|_{\eta=\text{cost}=\eta_*}$$

H matrix defined as: $\mathbf{H}=\left[\mathbf{E}_b \quad \mathbf{G}_b\right]=\begin{bmatrix} 0 & 0 \\ \frac{1}{2}gh^2 & 0 \\ 0 & \frac{1}{2}gh^2 \end{bmatrix} \Rightarrow \mathbf{S_0}=\begin{bmatrix} 0 \\ -gh\frac{\partial z_b}{\partial x} \\ -gh\frac{\partial z_b}{\partial y} \end{bmatrix}=\nabla\cdot\mathbf{H}\big|_{\eta=\eta_*}$

$\Rightarrow$ 2D SWE: $\dfrac{\partial \mathbf{U}}{\partial t}+\nabla\cdot\mathbf{F}=\nabla\cdot\mathbf{H}\big|_{\eta=\eta_*}+\mathbf{S}_{\mathrm{f}}$

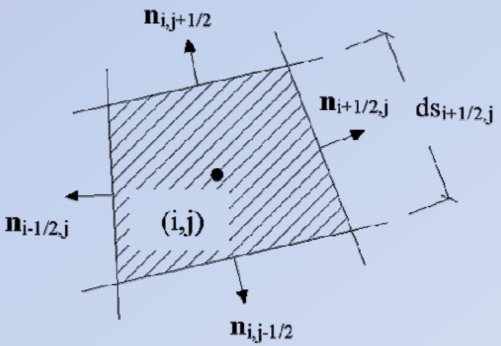
FINITE VOLUME DISCRETISATION

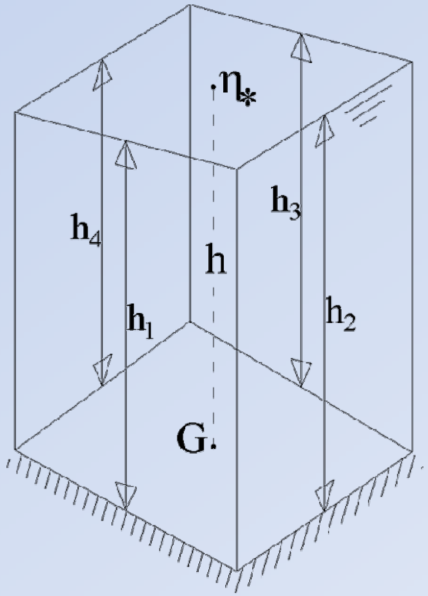
$$\Delta\mathbf{U} = -\frac{\Delta t}{\Delta V}\left[\int_{\Delta V} \nabla\cdot\mathbf{F}\,dV + \int_{\Delta V} \nabla\cdot\mathbf{H}\big|_{\eta=\eta_*}\,dV + \int_{\Delta V} \mathbf{S}_{\mathrm{f}}\,dV\right]$$

$$\Downarrow$$

$$\int_{\Delta V} \nabla\cdot\mathbf{H}\big|_{\eta=\eta_*}\,dV = \int_{\partial V} \mathbf{H}\cdot\mathbf{n}\big|_{\eta=\eta_*}\,dV = \begin{bmatrix} 0 \\ \sum_{r=1}^{4}\frac{1}{2}g\,h_r^2\Big|_{\eta=\eta_*} n_{xr}dS_r \\ \sum_{r=1}^{4}\frac{1}{2}g\,h_r^2\Big|_{\eta=\eta_*} n_{yr}dS_r \end{bmatrix}$$

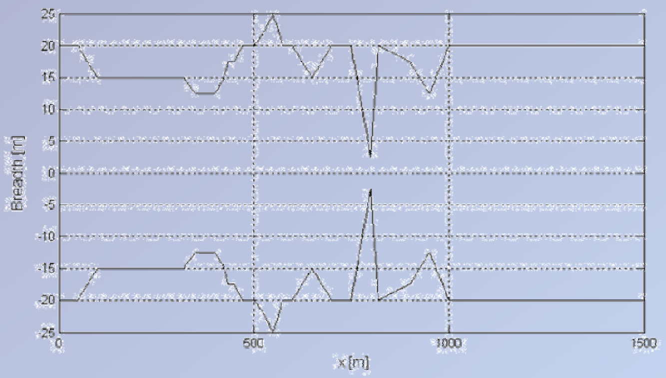
two estimations required:
$$\begin{cases} \eta_* = z_G + h \\ h_r = \eta_* - z_r \end{cases}$$

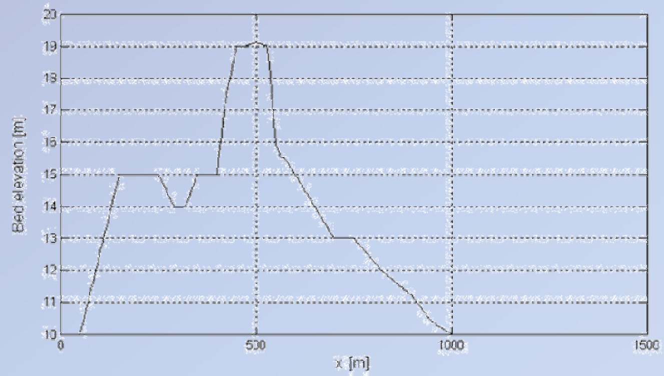




NUMERICAL SIMULATIONS

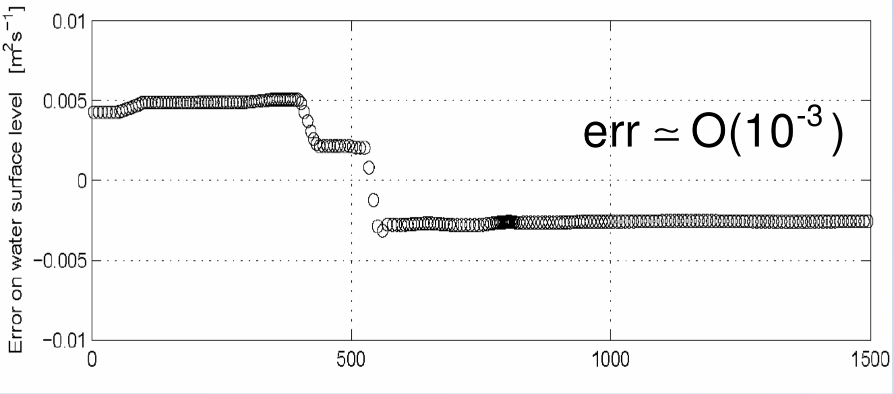
Still water in irregular channel



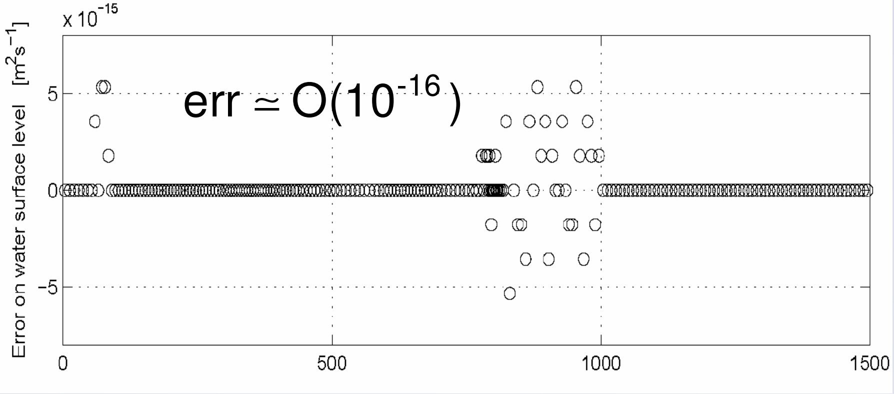


Error on water elevation (IC = SW)

without DFB



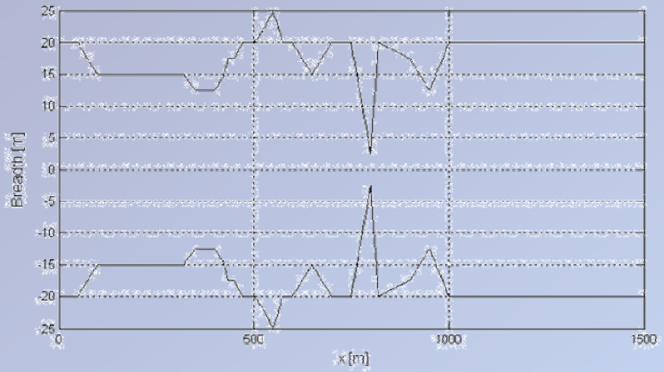
with DFB

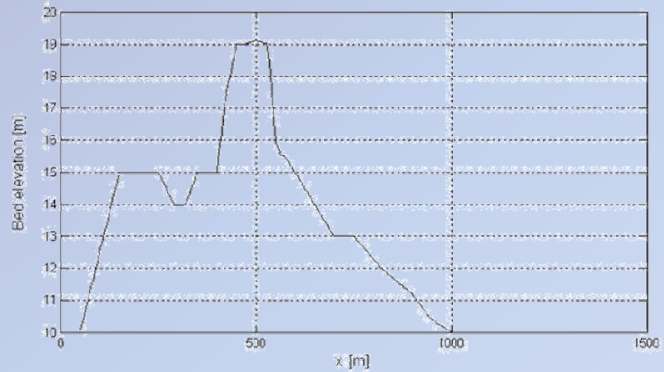


Numerical Approximations of Hyperbolic Systems with Source Terms and Applications

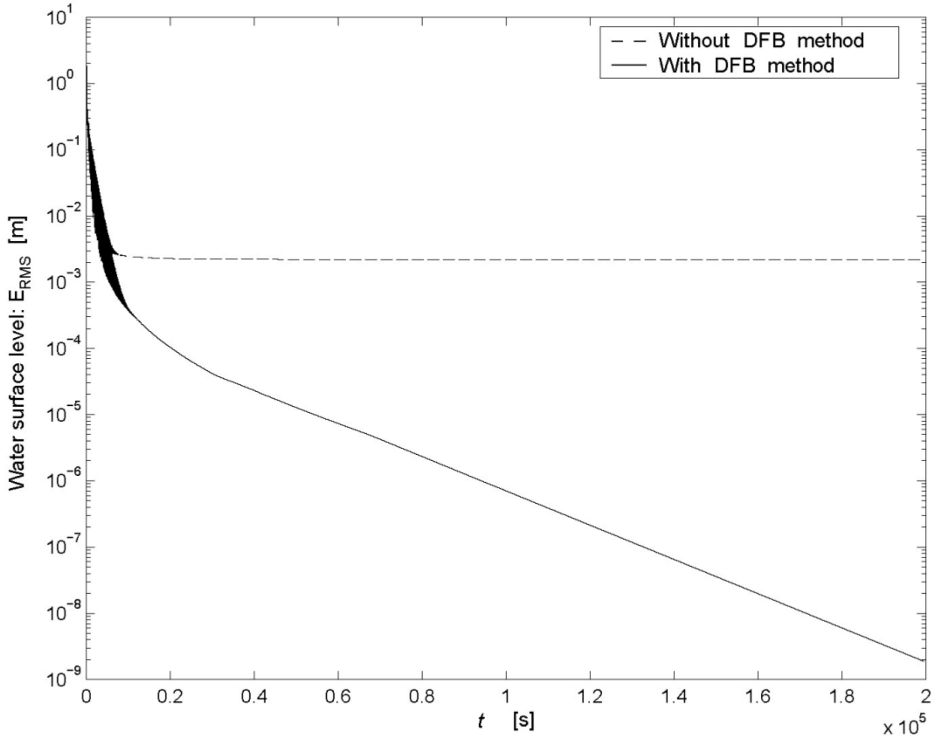
NUMERICAL SIMULATIONS

Still water in irregular channel





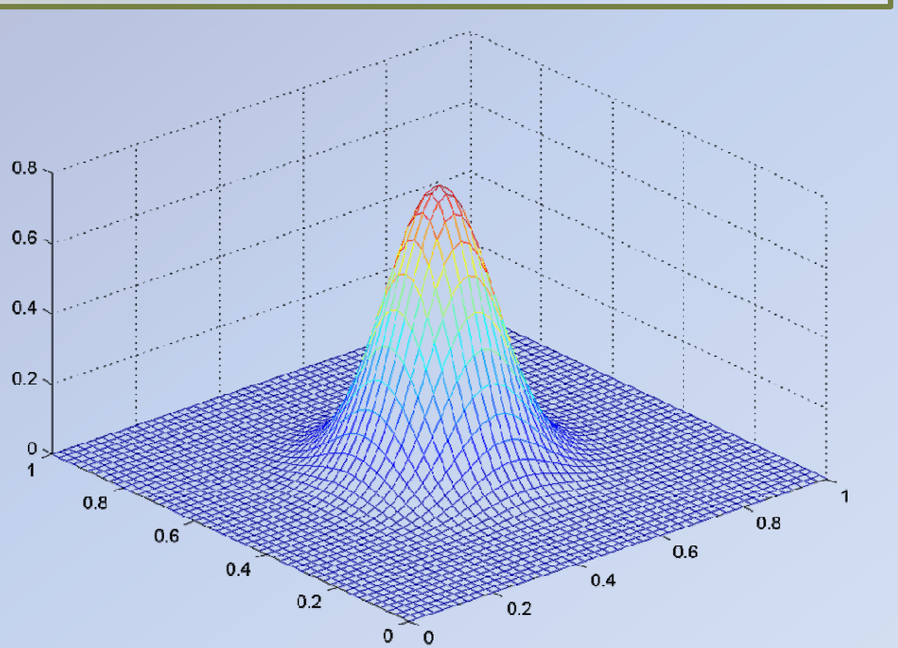
Error on water elev. (IC = perturb.)



SGM (Surface Gradient Method), Zhou et al. (2001), is applied.

NUMERICAL SIMULATIONS

Still water over Gaussian bump (Leveque, 1998)

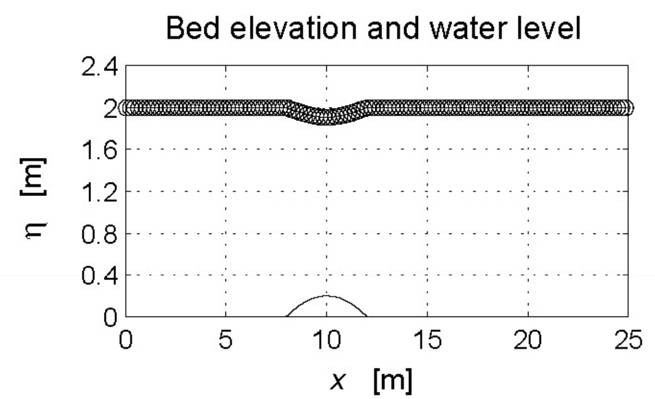


Comparison:

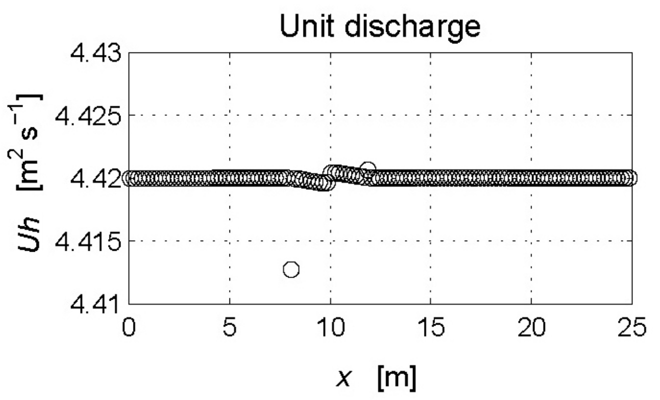
DFB vs. Leveque (1998)

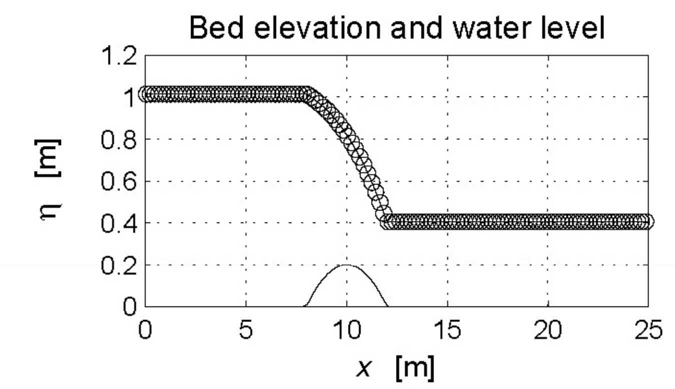
Griglia (N x N)	Leveque (1998) $t = 0.1$ s	DFB $t = 0.1$ s	DFB $t = 10$ s
50	1×10^{-3}	2.2×10^{-16}	4.4×10^{-16}
100	2.5×10^{-4}	4.4×10^{-16}	4.4×10^{-16}
200	6.3×10^{-5}	4.4×10^{-16}	4.4×10^{-16}

NUMERICAL SIMULATIONS: steady motion parabolic bump

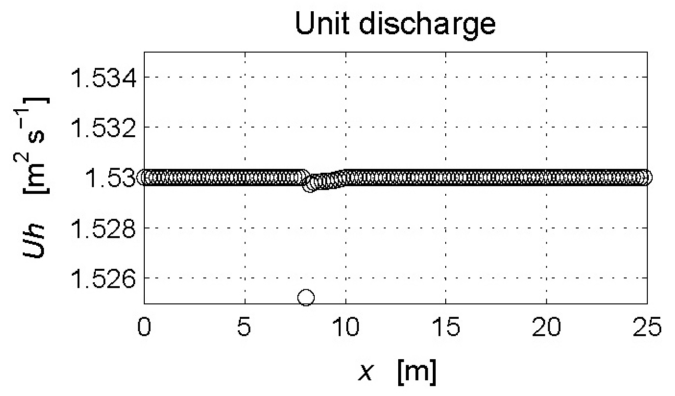


Subcritical flow

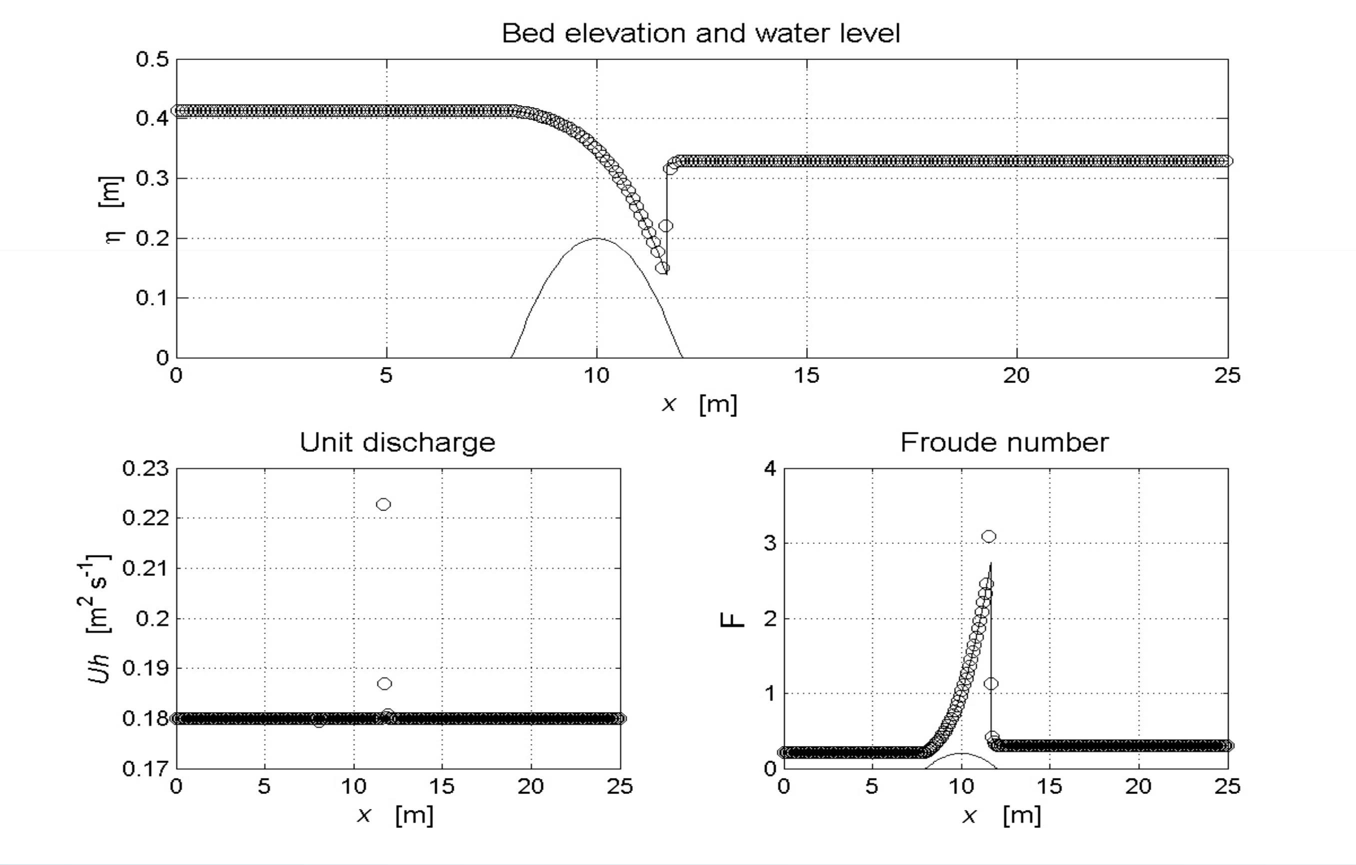




Transcritical flow



NUMERICAL SIMULATIONS: steady motion parabolic bump
Transcritical flow with shock



NUMERICAL SIMULATIONS

Run-up of a solitary wave over a conic island (Briggs et al. 1995)

Rectangular basin 26 x 27.6m

Smooth bottom

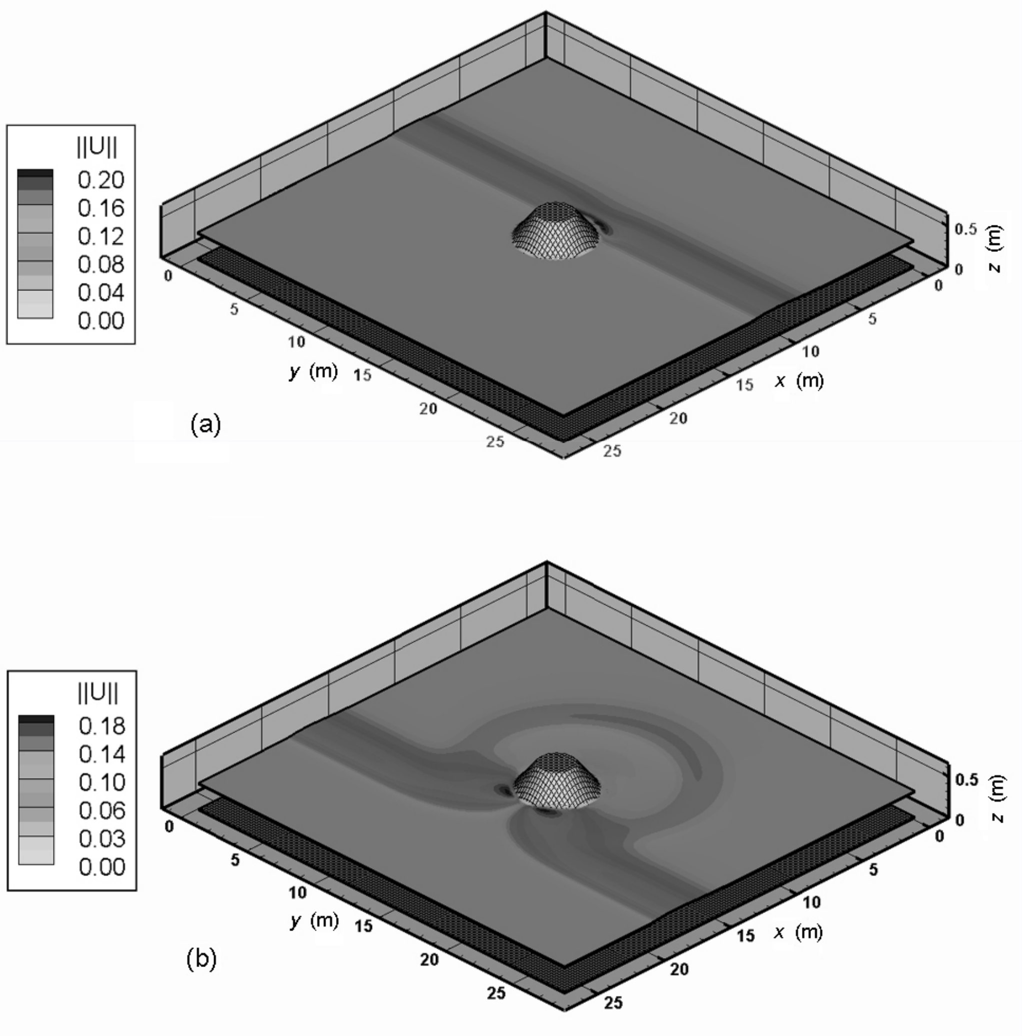
Island: base radius 3.6m, top radius 1.1m, height 0.625 m.

Solitary wave elevation:

$$\eta(t)=\eta_0+H_w\text{sech}^2\left(\frac{C_s(t-T)}{l_s}\right)$$

being: $c_s=\sqrt{gh_0}\left[1+H_w/(2h_0)\right]$

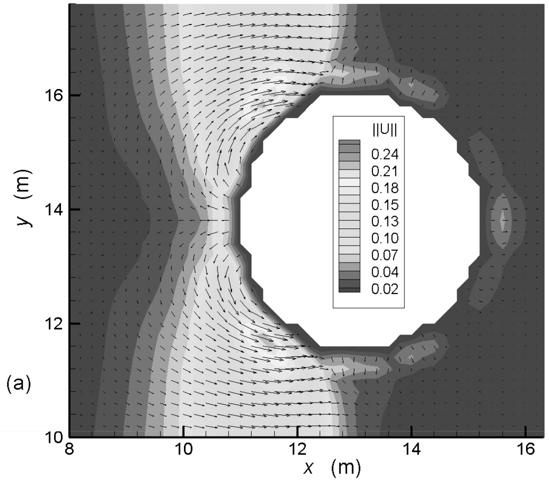
$$l_s=h_0\sqrt{4c_sh_0/\left(3H_w\sqrt{gh_0}\right)} \qquad H_w=0.032\,\text{m}$$



NUMERICAL SIMULATIONS

Run-up of a solitary wave over a conic island (Briggs et al. 1995)

(A)



(B)

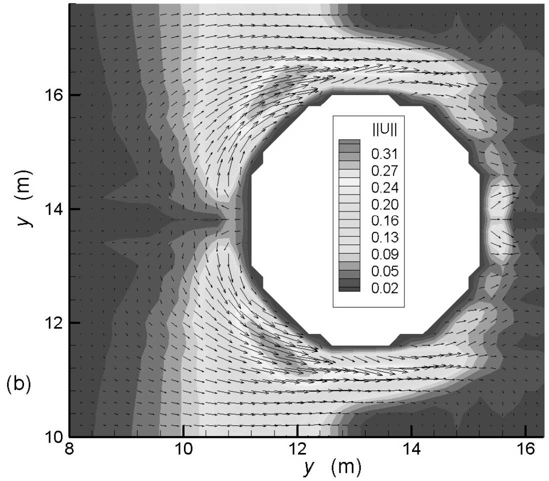


Fig. A) Velocity field using DFB.

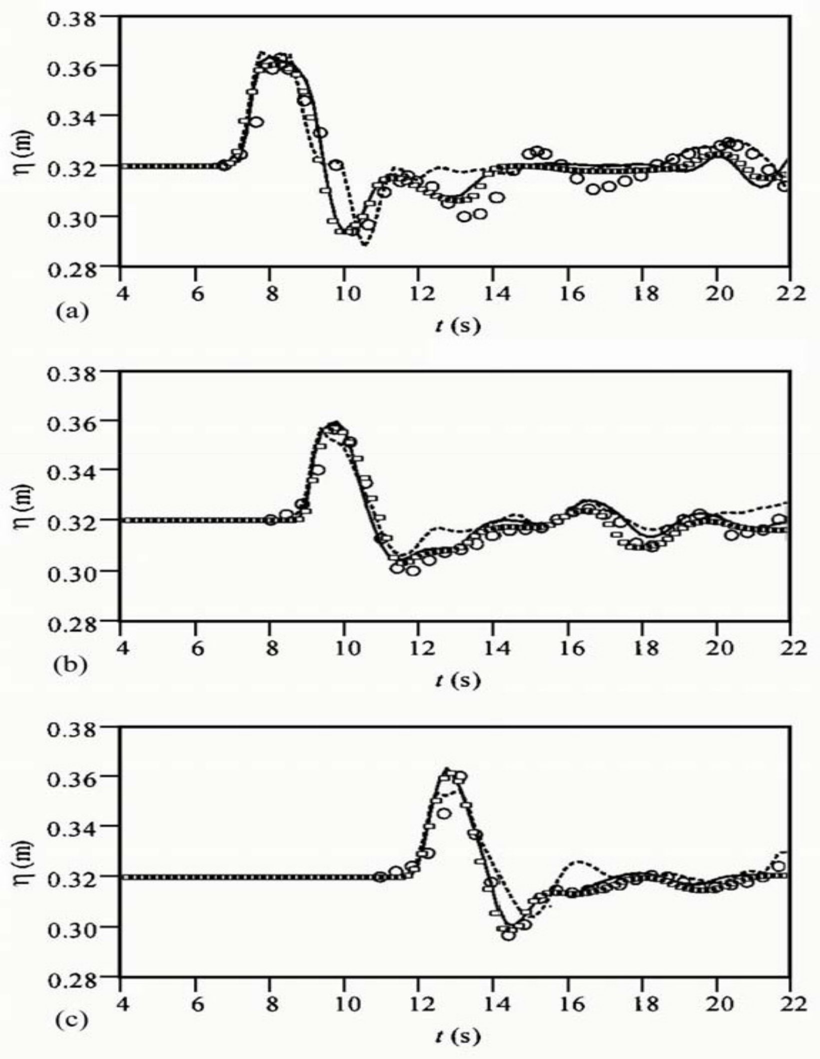
Spurious results are highly reduced both downstream and at the side of the island.

Fig. B) Velocity field without DFB. Source term computed using centered finite differences.

Spurious results are clearly observable near the island. Momentum flux and source term due to bed slope are not completely balanced.

NUMERICAL SIMULATIONS

Run-up of a solitary wave over a conic island (Briggs et al. 1995)



Water elevation was measured by 3 probed:
A) Upstream the island
B) At a side of the island

Comparison with:

- Briggs et al. (1995), experimental, circles
- Liu et a. (1995), numerical, continuous
- Bradford and Sanders (2002), numerical, dashed
- Proposed approach (DFB), squared

CONCLUSIONS on DFB

Vantaggi del metodo proposto:

- ☻ 3 Reference cases are exactly satisfied:
  - a) Total force conservation and mass conservation in hydraulic jump over flat bottom
  - b) C - Property (constant water surface elevation) is satisfied for any uneven topography
  - c) Bottom Slope = Friction Slope for uniform flow
- ☻ Extreme simplicity and low computational cost
- ☻ Superconvergence
- ☻ Very good results in practical test cases
- ☻ Applicability to different numerical techniques

Fourth-order balanced source term treatment in central WENO schemes for shallow water equations

Caleffi V., Valiani A., Bernini A., 2006,
Journal of Computational Physics, 218(1), 228-245

Numerical modeling of Shallow Water Equation with Source Terms.
Focusing on two particular aspects:

- *Increasing computational efficiency of numerical methods, applying to high order schemes, used on quite coarse grids.*

- *Reducing numerical errors, due to the treatment of source term due to bed slope, using well-balanced approaches:*
  i.e. source term treatment must satisfy the so-called C-property, that is still water and null velocity must be preserved "exactly".

Main features of the numerical model

- *Mathematical model consists of SWE system with BED SLOPE SOURCE TERM;*

- *Space and time accuracy is of 4° order:*
- *Accuracy in space is obtained by WENO reconstruction of variables;*
- *Original treatment of source term (high order and well-balanced).*

- *A STAGGERED GRID is used, to obtain a simple computation of fluxes;*

- *Accuracy in time is obtained using a scheme Runge-Kutta schemed, coupled with the NCE (Natural Continuous Extension) technique.*

Mathematical Model of the Shallow Water Equations

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=s(u,x), \qquad u(x,0)=u_0(x);$$

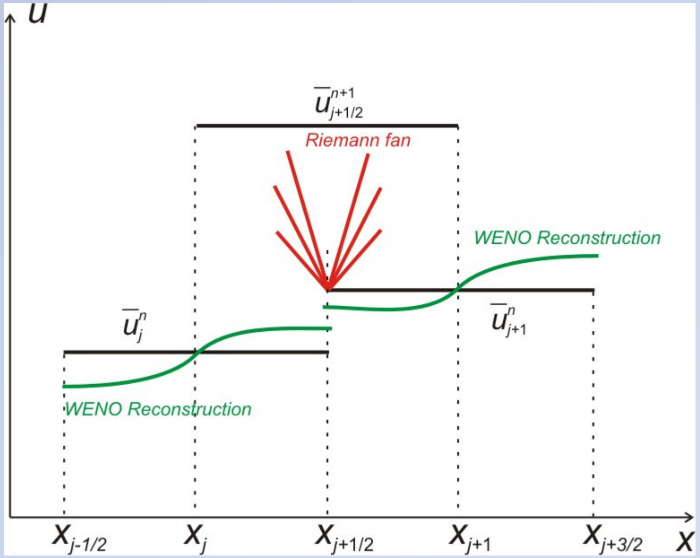
$$u(x,t)=\begin{bmatrix} h \\ Uh \end{bmatrix}; \qquad f(u)=\begin{bmatrix} Uh \\ U^2h+g\frac{h^2}{2} \end{bmatrix}; \qquad s(u,x)=\begin{bmatrix} 0 \\ -gh\frac{\partial z}{\partial x} \end{bmatrix};$$

Numerical Approximations of Hyperbolic Systems with Source Terms and Applications

Numerical model

Integration procedure:

- *At the current time, t^n, cell-averaged values of solution are known;*
- *A half-cell staggered grid is introduced;*



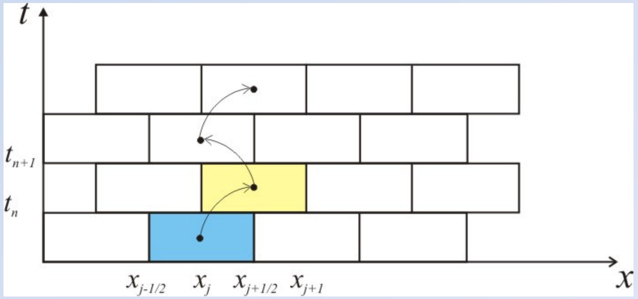
Integrating between $\left[x_j\, ,x_{j+1}\right]$ and $\left[t^n\, ,t^{n+1}\right]$

The balance law becomes:

$$\overline{u}_{j+1/2}^{n+1} = \overline{u}_{j+1/2}^{n} - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f\left(u\left(x_{j+1},t\right)\right) - f\left(u\left(x_j,t\right)\right)\mathrm{d}t + \int_{t_n}^{t_{n+1}} S_{j+1/2}^{*}\left(t\right)\mathrm{d}t\right]$$

where:

$$\overline{u}_{j+1/2}^{n} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u\left(x,t_n\right)dx \quad ; \; S_{j+1/2}^{*}\left(t\right) = \int_{x_j}^{x_{j+1}} s\left(x,t\right)dx$$



- At the following time step t^{n+1} the solution is known in terms of cell-averages on the staggered grid.

Numerical model

Integration in time: Simpson rule (4th order)

$$\overline{u}_{j+1/2}^{n+1}=\overline{u}_{j+1/2}^{n}-\frac{\Delta t}{\Delta x}\sum_{l=1}^{3}N_l\left[f\left(\hat{u}\left(x_{j+1},t^n+\beta_l\Delta t\right)\right)-f\left(\hat{u}\left(x_j,t^n+\beta_l\Delta t\right)\right)+\Delta x s_{j+1/2}^{n+\beta_l}\right]$$

$$s_{j+1/2}^{n+\beta_l}=\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}}s\left(x,t^n+\beta_l\Delta t\right)\mathrm{d}x;\qquad N=\left[1/6,2/3,1/6\right]^T;\quad \beta=\left[0,1/2,1\right]^T$$

t^{n+1} Runge-Kutta (4th order)

$$\hat{u}^{n+1}=\hat{u}^n+\Delta t\sum_{i=1}^{4}b_i k^{(i)}\quad \text{with}\quad \hat{u}^{(i)}=\hat{u}^n+\Delta t\sum_{j=1}^{i}a_{ij}k^{(j)}\qquad b=\left[1/6,1/3,1/3,1/6\right]^T;$$

$$a_{mn}=0\ \text{ except for }\ a_{21}=a_{32}=1/2\quad \text{and}\quad a_{43}=1$$

$t^{n+1/2}$ NCE (4th order)

$$\hat{u}^{n+1/2}=\hat{u}^n+\Delta t\sum_{i=1}^{4}B_i k^{(i)}\qquad B=\left[5/24,1/6,1/6,-1/24\right]^T$$

Kutta fluxes: $k_j=\left(-f_x+s\right)_j$

Numerical model

The key problem is:
How can we obtain balancing between
the gradient of momentum flux and the bed slope source term?

1. *Using SGM techniques (Surface Gradient methods- Zhou et al. 2001)*
2. *Balanced reconstruction of Kutta flux*
3. *Balanced integration of Bed Slope Source Term*

Numerical Model

SGM (Surface Gradient method) Technique:

The reconstructed vector variable is μ :

$$\mu = \begin{bmatrix} \eta \\ vh \end{bmatrix} = \begin{bmatrix} h + z_b \\ vh \end{bmatrix}$$

← *Surface water elevation*

← *Specific discharge*

Without SGM $\quad \overline{u}_j^n \xrightarrow{\text{reconstruction}} \overline{u}_{j+1/2}^n$

With SGM $\quad \overline{u}_j^n \xrightarrow{\overline{\eta}=\overline{h}+\overline{z}_b} \overline{\mu}_j^n \xrightarrow{\text{reconstruction}} \overline{\mu}_{j+1/2}^n \xrightarrow{\overline{h}=\overline{\eta}-\overline{z}_b} \overline{u}_{j+1/2}^n$

Numerical Model

Balanced reconstruction of the flux derivative, coupled with the bed source term

$$K_j\left(x,u\left(x,t\right)\right)=-\begin{bmatrix}\left(vh\right)-\left(vh\right)_j\\ \left(v^2h+1/2g\left(\eta-z\right)^2\right)-\left(v^2h+1/2g\left(\eta-z\right)^2\right)_j\end{bmatrix}+\begin{bmatrix}0\\ 1/2g\left(\left(\eta_j-z\right)^2-\left(\eta_j-z_j\right)^2\right)\end{bmatrix}$$

It satisfies two analytical conditions:

1. $\left.\dfrac{\partial K_j}{\partial x}\right|_{x=x_j}=k_j=\left(-f_x+s\right)_j$ 2. $K_j\left(x\right)=0 \quad \forall x \quad \text{when} \quad \eta=\text{const.} \;\; \text{and} \;\; hv=0$

When a reconstruction is defined, such that, starting from the point values K_j , it gives an approximation of its derivatives on x_j (see, e.g., the standard reconstruction of flux derivatives), then:

- eq. 1. ensures that such derivative is an approximation of Kutta fluxes.
- eq. 2. ensures the balancing between flux derivative and bed source term, in case of quiescent fluid.

Numerical Approximations of Hyperbolic Systems with Source Terms and Applications

Numerical model

Balanced reconstruction of the integral in space of bed source term

$$\boxed{\overline{s}^{[2]}_{j+1/2} = -\int_{x_j}^{x_{j+1}} gh\frac{\mathrm{d}z_b}{\mathrm{d}x}\mathrm{d}x}$$

→ Integration by parts + $h = \eta - z$ →

$$\overline{s}^{[2]}_{j+1/2} = \frac{1}{2}g\left[z_{j+1}^2 - z_j^2\right] - g\left[\eta_{j+1}z_{j+1} - \eta_j z_j\right] + \int_{x_j}^{x_{j+1}} gz\frac{\mathrm{d}\eta}{\mathrm{d}x}\mathrm{d}x$$

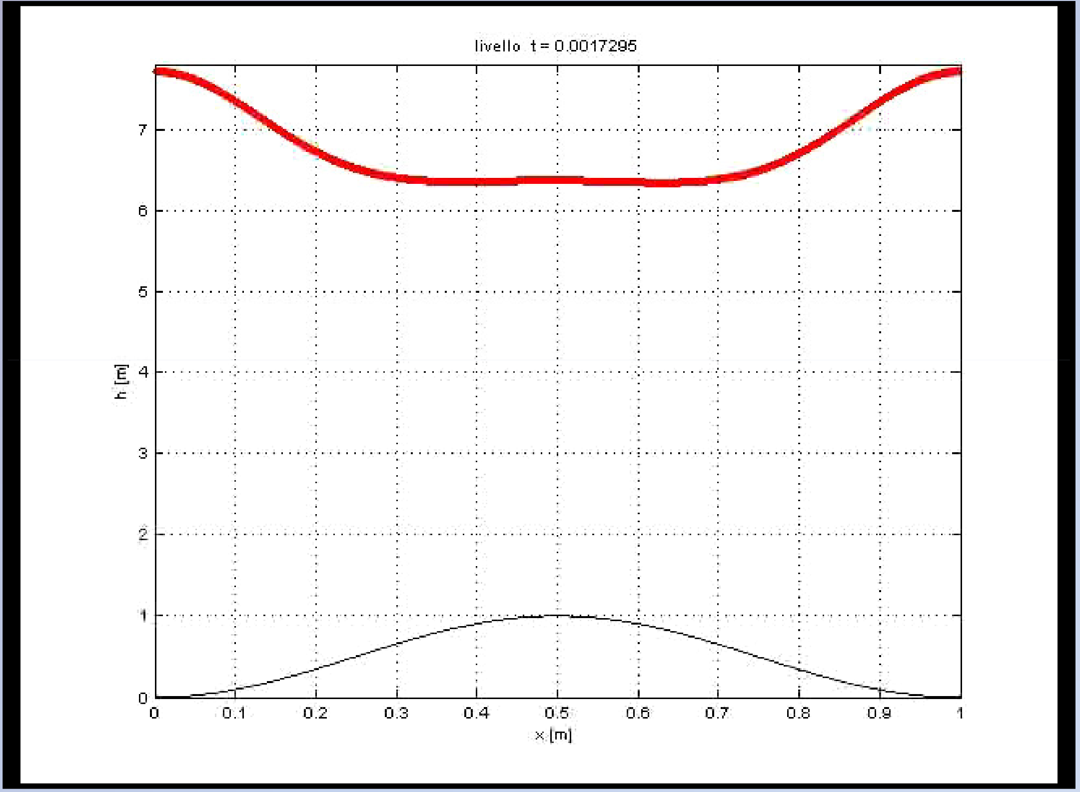
→ $\xi(x) = gz\frac{\mathrm{d}\eta}{\mathrm{d}x}$ →

$$\boxed{\overline{s}^{[2]}_{j+1/2} = \frac{1}{2}g\left[z_{j+1}^2 - z_j^2\right] - g\left[\eta_{j+1}z_{j+1} - \eta_j z_j\right] + \int_{x_j}^{x_{j+1}} \xi(x)\mathrm{d}x}$$

The two forms are equivalent from the analytical point of view, but they are not from the numerical point of view. The latter form is convenient because in practical problems water elevation is a much more regular surface than bottom elevation. Two WENO reconstructions are needed:

$$\hat{\eta}_j \xrightarrow{\text{reconstruction}} \hat{\eta}'_j \qquad \hat{\xi}_j = gz_j\hat{\eta}'_j \xrightarrow{\text{reconstruction}} \overline{\xi}_{j+1/2}$$

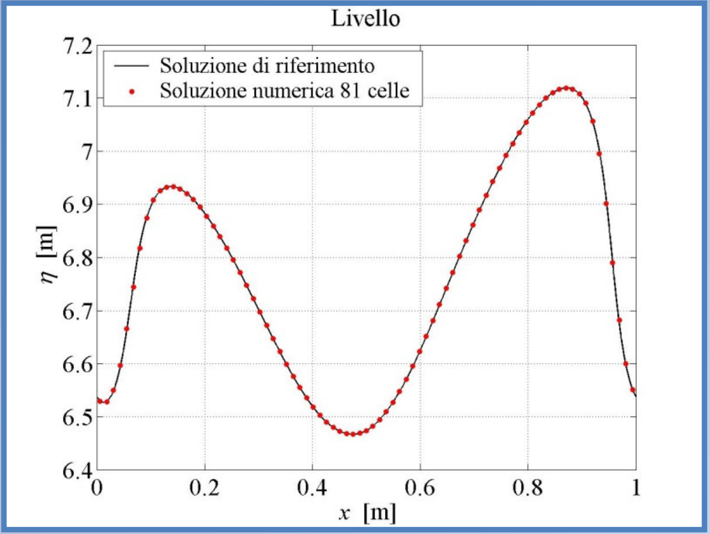
Accuracy test case



Unsteady flow over sinusoidal bump

- Bottom: $z(x) = \sin^2(\pi x)$
- I. C: $h(x,0) = h_0 + e^{\cos(2\pi x)};$
  $vh(x,0) = \sin(\cos(2\pi x)).$
- Periodic Boundary Conditions
- Results a $t = 0.1$ s.

Accuracy test case



Unsteady flow over sinusoidal bump

- Bottom: $z(x) = \sin^2(\pi x)$
- Initial conditions: $h(x,0) = h_0 + e^{\cos(2\pi x)};$
$$vh(x,0) = \sin(\cos(2\pi x)).$$
- Periodic Boundary Conditions
- Results a t = 0.1 s.

Centered WENO						
N of cells	L^1	order	L^2	order		order
81	7.2854E–04		1.9615E–03		9.7843E–03	
243	1.3648E–05	**3.6205**	5.0621E–05	**3.3288**	4.0506E–04	**2.8987**
729	1.1443E–07	**4.3522**	4.3949E–07	**4.3205**	4.2323E–06	**4.1519**
2187	1.2328E–09	**4.1240**	4.5753E–09	**4.1552**	4.3143E–08	**4.1743**

Test case to verify balancing (C-property)

h			vh		
L^1	L^2		L^1	L^2	
3.50E–15	1.45E–15	8.E–16	3.4E–13	1.3E–13	1.0E–13

Still water over smooth bottom

- bottom elevation: $z(x) = 5e^{-2/5(x-5)^2}$ m
- Initial conditions: $\eta(x,0) = 10 \quad m$

  $vh(x,0) = 0.$

Steady flow over parabolic bump

- Parabolic Bump

  H_{max} = 0.2 m

  Axis x = 10 m

- Boundary conditions:

  upstream q = 0.18 m²/s

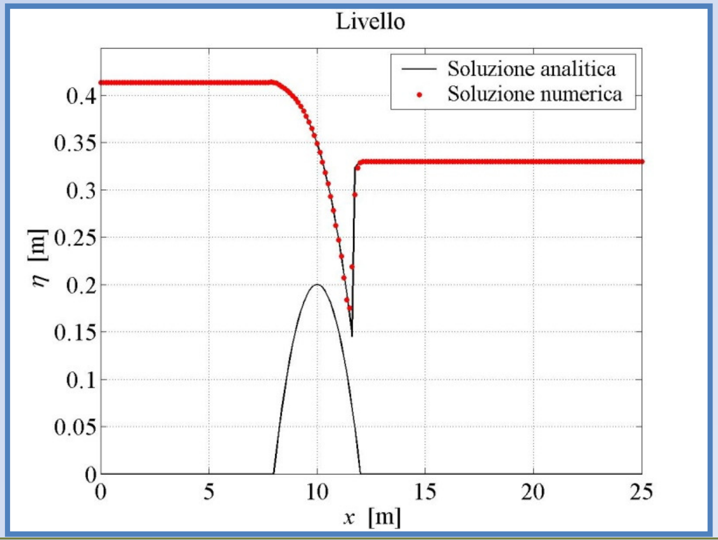
  downstream h = 0.33 m

- Physical domain dimensions:

  L = 25 m

  B = 1 m

  250 cells



Test case on unsteady flow: rectangular pulse over bump

- Initial condition:

Bump

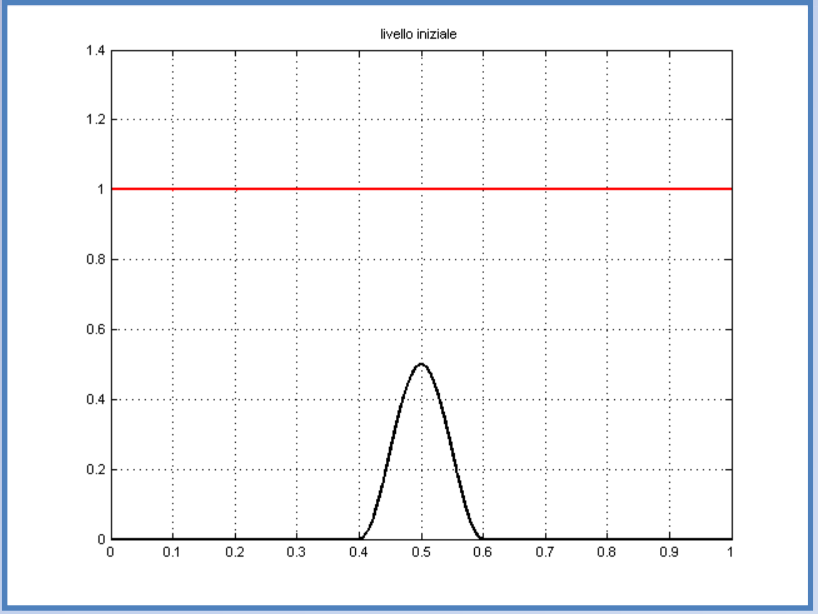
$$z(x)=\begin{cases}0.25\left[\cos\left(10\pi\left(x-1/2\right)\right)+1\right] & \text{m} \quad \text{if } \left|x-1/2\right|\le 0.1\,\text{m}\\ 0 & \text{elsewhere}\end{cases}$$

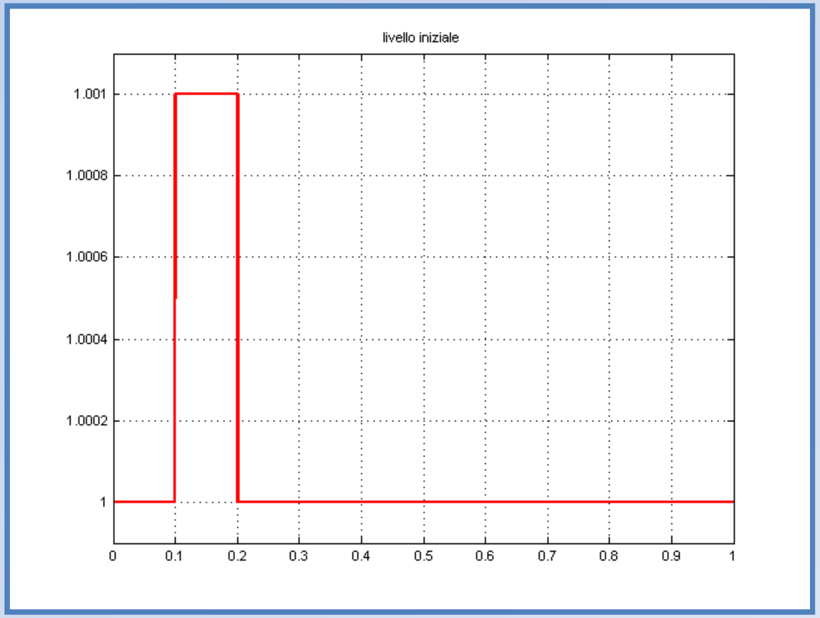
$$\eta(x)=\begin{cases}1+\varepsilon \quad \text{m} & \text{se } \left|x-0.15\right|\le 0.1\,\text{m}\\ 1 & \text{elsewhere}\end{cases} \qquad vh=0$$

- Computational domain:

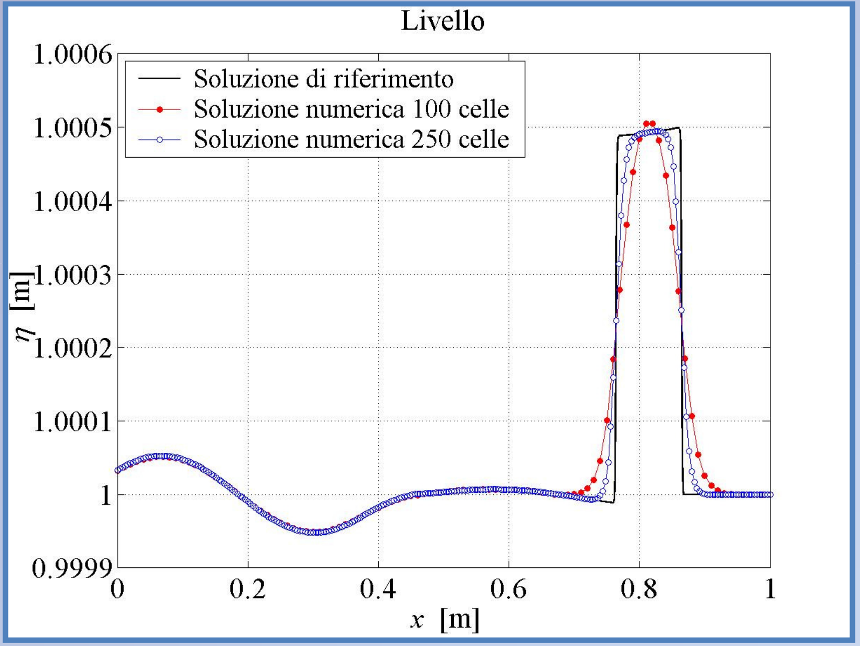
L = 1 m
B = 1 m

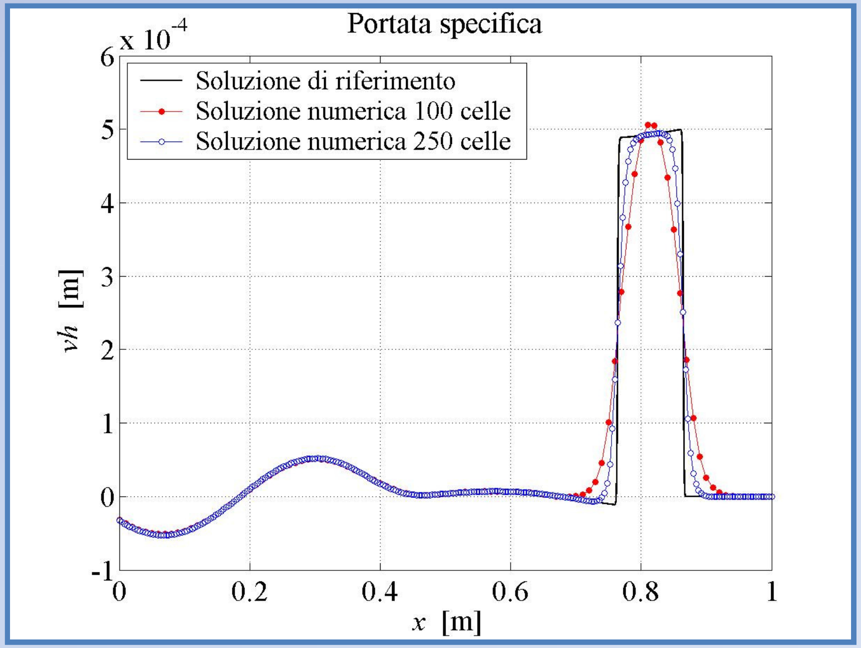
$$\boxed{\varepsilon=1.0^{-3}m}$$





Test case on unsteady flow: rectangular pulse over bump





Conclusions on high order CWENO for SWE with Source Terms

Central WENO schemes are versatile, efficient and robust tools to numerically integrate hyperbolic system of conservation laws: as a consequence, they are particularly attractive for hydraulic engineering application of SWE with bed source terms.

Concerning our original contribution, we propose *a new approach for the treatment of bed slope source term*, which obtains the *C-property* preservation in the case of quiescent fluid over uneven bottom.
Such an approach preserves the *4° order accuracy in time and space*, and can be theoretically applied to any WENO scheme (i.e. upwind …).

Different test cases are analyzed, to verify the previously described properties: accuracy, C-property preservation, good discontinuities resolution.

The natural following step consists of considering the mobile bed case.

High-order balanced CWENO scheme for movable bed shallow water equations

Caleffi V., Valiani A., Bernini A., 2007, Advances in Water Resources, 30(4), 730-741

Aim of the research:

Extend the CWENO scheme for the application to movable bed problems

Advantage of CWENO schemes:

CWENO schemes do not need either Riemann problem resolution at the interface of neighbouring cells or the knowledge of the eigenstructure of the adopted system of conservation laws.

→ Simple structure of the code.

→ Strong reduction of computational time.

→ Simplification in the management of applicative cases (interaction between solid and liquid phase, secondary currents in curvilinear channels, etc.) in which the eigenstructure is strongly different from the standards available in literature.

Mathematical Model of the Shallow Water Equations

1D shallow water equations for a movable bed:

$$\frac{\partial u}{\partial t}+\frac{\partial f}{\partial x}=s$$

with: u conservative variables vector; f flux vector; s source terms vector.

$$u=\begin{bmatrix}\eta\\ vh\\ z\end{bmatrix};\qquad f=\begin{bmatrix}vh+\left[1/(1-\lambda)\right]q_s\\ v^2h+gh^2/2\\ \left[1/(1-\lambda)\right]q_s\end{bmatrix};\qquad s=\begin{bmatrix}0\\ -gh\dfrac{\partial z}{\partial x}\\ 0\end{bmatrix};$$

h	water depth	q_s	bed load sediment transport
v	vertically averaged velocity	λ	bed porosity
z	bottom elevation	η	water level

Mathematical Model of the sediment transport

The sediment transport evaluation is case dependent. Two models are used in this study:

➡ $$q_s = Av^3$$

This simple power law is selected in order to allow the comparison with numerical results proposed by Črnjarić-Žic et al., 2004 in previous numerical works carried out on the same topic.

➡ $$q_s = \beta\left(v - v_{cr}\right)^r$$

To allow the comparison with analytical solutions by Lyn & Altinakar 2002.

Numerical Model

The numerical model is very similar to the previously discussed one, only the function used to achieve a balanced reconstruction of the flux derivative (coupled with the bed source term) is quite different:

$$K_j\left(x,u(x,t)\right)=-\begin{bmatrix}(vh)-(vh)_j\\ \left(v^2h+1/2g(\eta-z)^2\right)-\left(v^2h+1/2g(\eta-z)^2\right)_j\end{bmatrix}+\begin{bmatrix}0\\ 1/2g\left((\eta_j-z)^2-(\eta_j-z_j)^2\right)\end{bmatrix}$$

↓ Becomes…

$$K_j\left(x,u(x,t)\right)=-\begin{bmatrix}\left(vh+\left[1/(1-\lambda)\right]q_s\right)-\left(vh+\left[1/(1-\lambda)\right]q_s\right)_j\\ \left(v^2h+gh^2/2\right)-\left(v^2h+gh^2/2\right)_j\\ \left(\left[1/(1-\lambda)\right]q_s\right)-\left(\left[1/(1-\lambda)\right]q_s\right)_j\end{bmatrix}+\begin{bmatrix}0\\ 1/2g\left((\eta_j-z)^2-(\eta_j-z_j)^2\right)\\ 0\end{bmatrix}$$

Test Cases

Observations:

- *Simulations are carried out following 2 steps*:

  1. A preliminary run is made considering fixed bed until the hydrodynamic stationary conditions are achieved;
  2. A second run with movable bed follows the first. The previously obtained hydrodynamic stationary conditions are adopted as initial conditions for the second run.

- *Boundary conditions:*

  Hydrodynamic variables in subcritical flows are imposed at boundary cells by decoupling water flow from sediment flow and following the procedure suggested by Sanders (2002), in order to avoid reflection of non physical undulations. Note that the decoupling hypothesis is necessary only at the boundaries and not elsewhere. A discharge is imposed at the inlet and a depth is imposed at the outlet.

Accuracy test case (Xing & Shu 2005, Unsteady flow over sinusoidal bump modified)

- Bottom: $z\left(x\right)=\sin^2\left(\pi x\right)$
- Initial conditions: $h\left(x,0\right)=h_0+e^{\cos\left(2\pi x\right)};$

  $vh\left(x,0\right)=\sin\left(\cos\left(2\pi x\right)\right).$
- Solid discharge: $\boxed{q_s = Av^3 \quad \text{with: } A=0.2\ [s^2/m] \quad \text{and: } \lambda=0.2;}$

- Periodic Boundary Conditions
- Results a *t* = 0.1 s.

Water Level						
N of cells	L^1	order	L^2	order		order
81	1.2583e−03		4.1051e−03		2.3033e−02	
243	7.0796e−05	2.62	3.2769e−04	2.30	2.7863e−03	1.92
729	9.0952e−07	3.96	6.1453e−06	3.62	8.8593e−05	3.14
2187	8.4739e−09	4.26	5.5608e−08	4.28	8.7511e−07	4.20

Bottom						
N of cells	L^1	order	L^2	order		order
81	4.3993e−04		9.6021e−04		4.1836e−03	
243	4.0183e−06	4.27	1.1419e−05	4.03	8.4622e−05	3.55
729	2.7727e−08	4.53	1.5476e−07	3.92	2.2255e−06	3.31
2187	2.2301e−10	4.39	1.3607e−09	4.31	2.0880e−08	4.25

Subcritical flow over movable bed test case (Lyn & Altinakar, 2002)

An analytical solution is available, in nearly critical flows, i.e. $1-\mathrm{Fr}^2 = O\left(\psi_U^{1/2}\right)$, starting from:

$$\frac{\partial \tilde{u}}{\partial t} + J_U \frac{\partial \tilde{u}}{\partial x} = 0 \qquad \text{with:} \quad J_U = \begin{bmatrix} v_U & h_U & 0 \\ g & v_U & g \\ 0 & h_U \psi_U & 0 \end{bmatrix}; \quad \tilde{u} = \begin{bmatrix} h \\ v \\ z \end{bmatrix} \quad \text{and} \quad \psi_U = \frac{1}{(1-\lambda) h_U} \frac{\partial q_b}{\partial v_U}$$

using a perturbation analysis in the small parameter ψ_U, the following eigensystem is achieved:

$$l_1 = \left[\frac{3}{2} + \frac{1}{2\mathrm{Fr}_U^2}\right] v_U ; \quad l_{2,3} = \left[\frac{1}{4}\left(1 - \frac{1}{\mathrm{Fr}_U^2}\right) \pm \frac{1}{4}\sqrt{\left(1 - \frac{1}{\mathrm{Fr}_U^2}\right)^2 + \frac{8\psi_U}{\mathrm{Fr}_U^2}}\right] v_U ;$$

$$L_i = \left[1, \left(l_i - v_U\right)/g, 1 - v_U / g\right] \quad i = 1, \ldots, 3$$

then, the system can be analytically solved using the classical characteristics method.

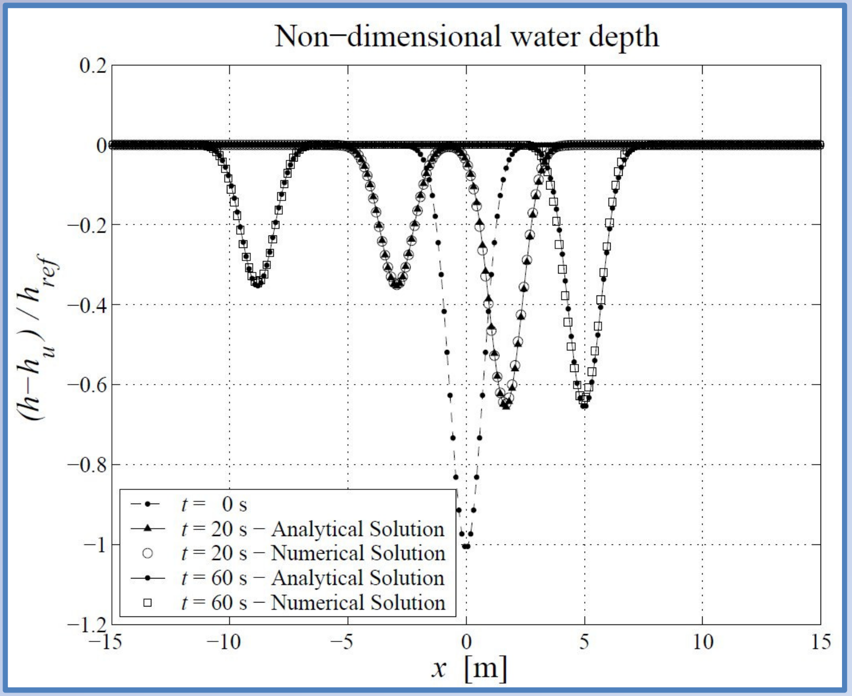
Subcritical flow over movable bed test case

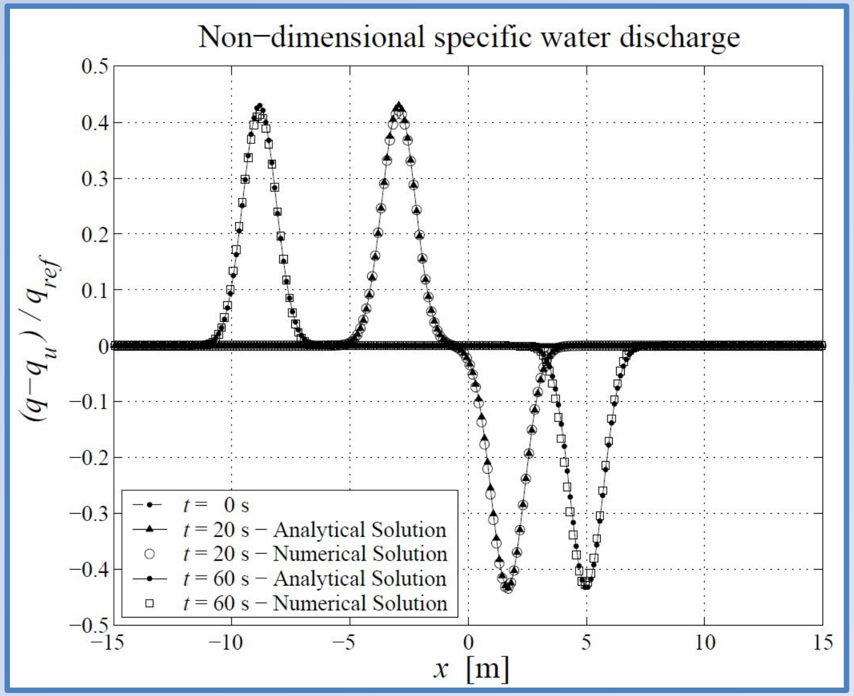
$$z(x) = z_{\max} e^{-x^2} \quad \text{with} \quad -15m < x < 15m$$

$$h_U = 1m; \quad \lambda = 0.4; \quad \psi_U = 2.5\times10^{-3};$$

$$q_s = \beta\left(v - v_{cr}\right)^r \quad \text{with} \quad \beta = 3.4\times10^{-4}; \quad r = 2.65;$$

$$\boxed{\mathrm{Fr}_U = 0.96;}$$


Non−dimensional water depth


Non−dimensional specific water discharge

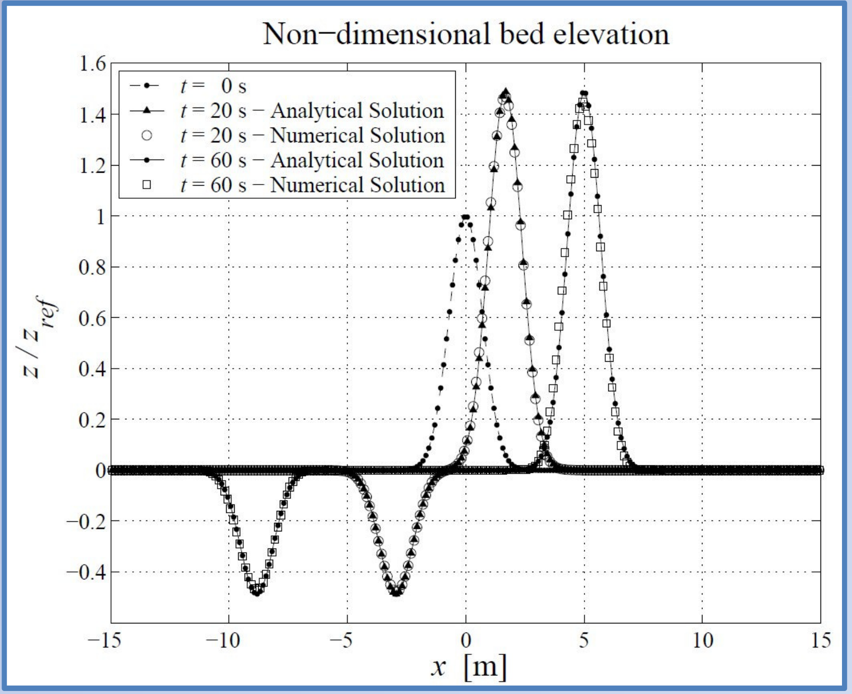
Subcritical flow over movable bed test case

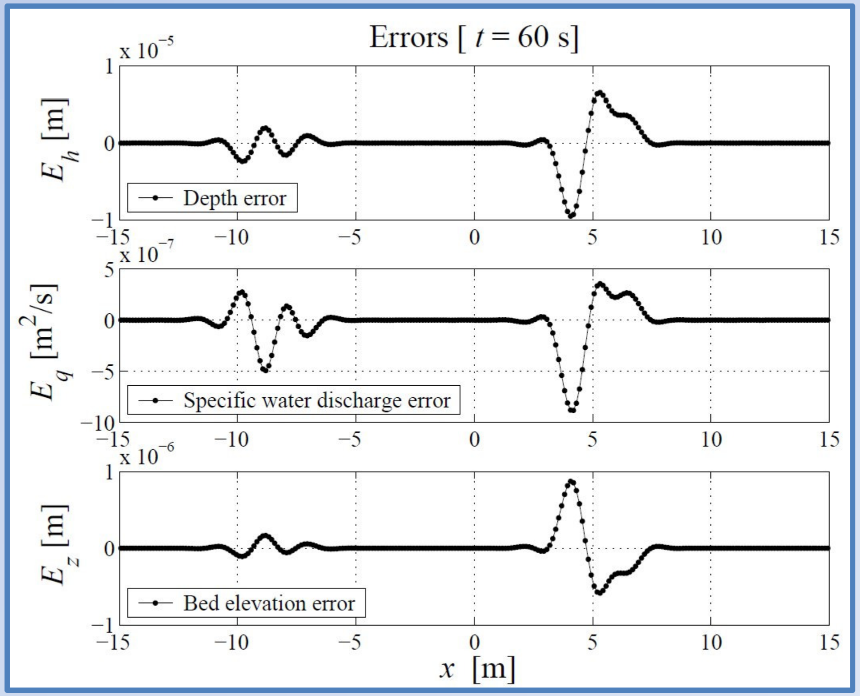
$$z(x) = z_{\max} e^{-x^2} \quad \text{with} \quad -15m < x < 15m$$

$$h_U = 1m; \quad \lambda = 0.4; \quad \psi_U = 2.5\times10^{-3};$$

$$\boxed{\mathrm{Fr}_U = 0.96;}$$

$$q_s = \beta(v - v_{cr})^r \quad \text{with} \quad \beta = 3.4\times10^{-4}; \quad r = 2.65;$$





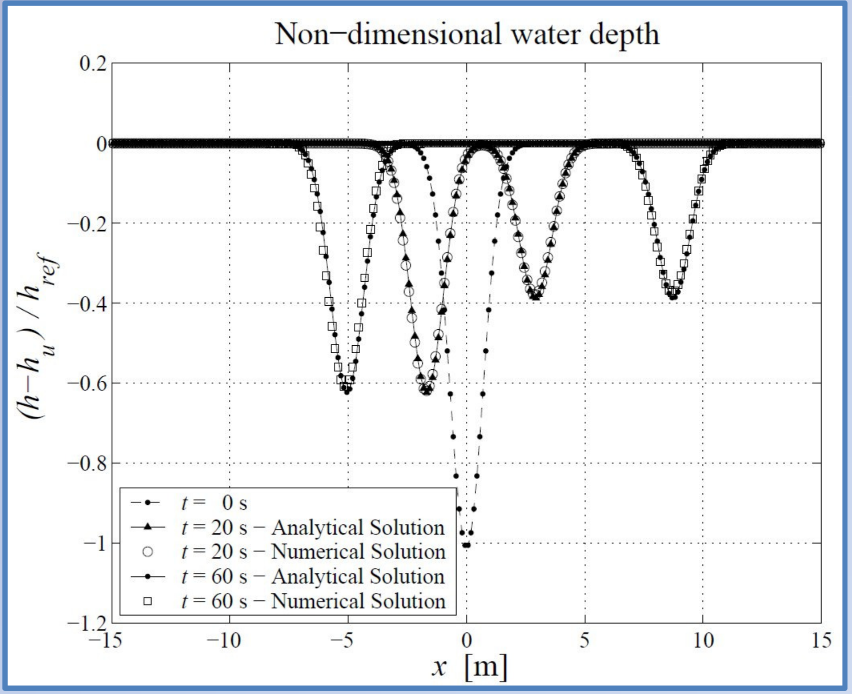
Supercritical flow over movable bed test case

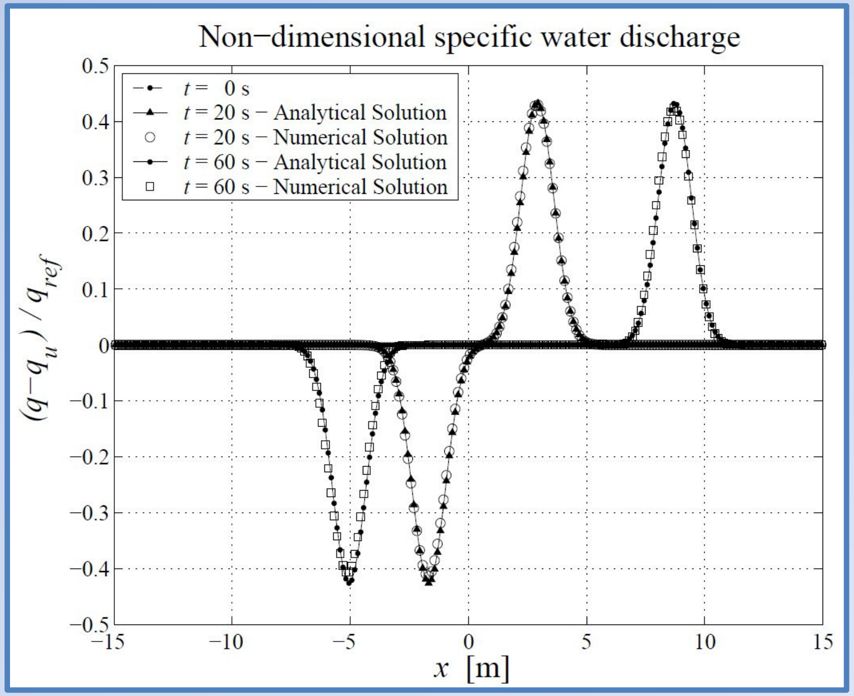
$$z(x) = z_{\max} e^{-x^2} \quad \text{with} \quad -15m < x < 15m$$

$$h_U = 1m; \quad \lambda = 0.4; \quad \psi_U = 2.5\times10^{-3};$$

$$q_s = \beta(v - v_{cr})^r \quad \text{with} \quad \beta = 3.4\times10^{-4}; \quad r = 2.65;$$

$$\boxed{\mathrm{Fr}_U = 1.04;}$$





Numerical Approximations of Hyperbolic Systems with Source Terms and Applications

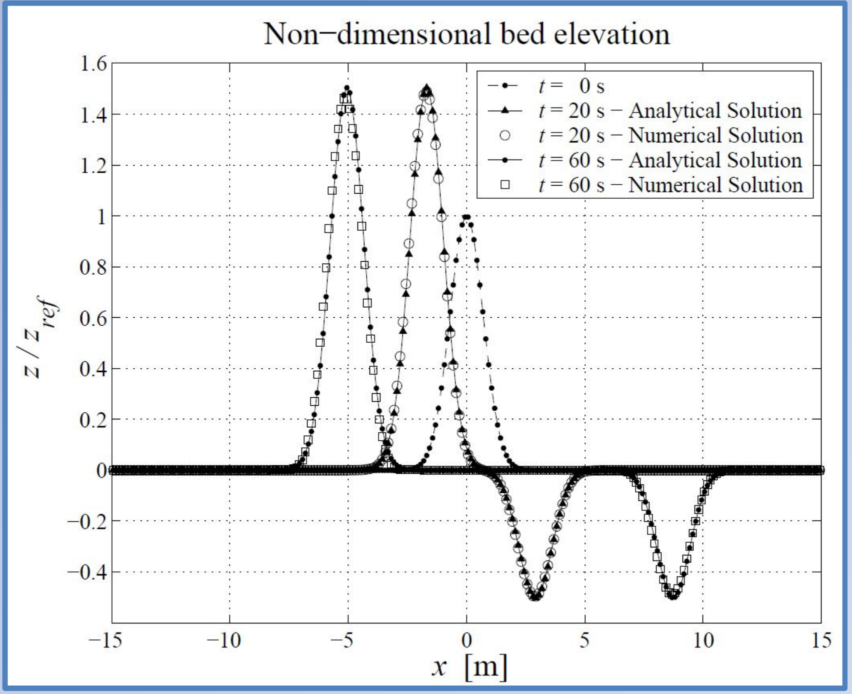
Supercritical flow over movable bed test case

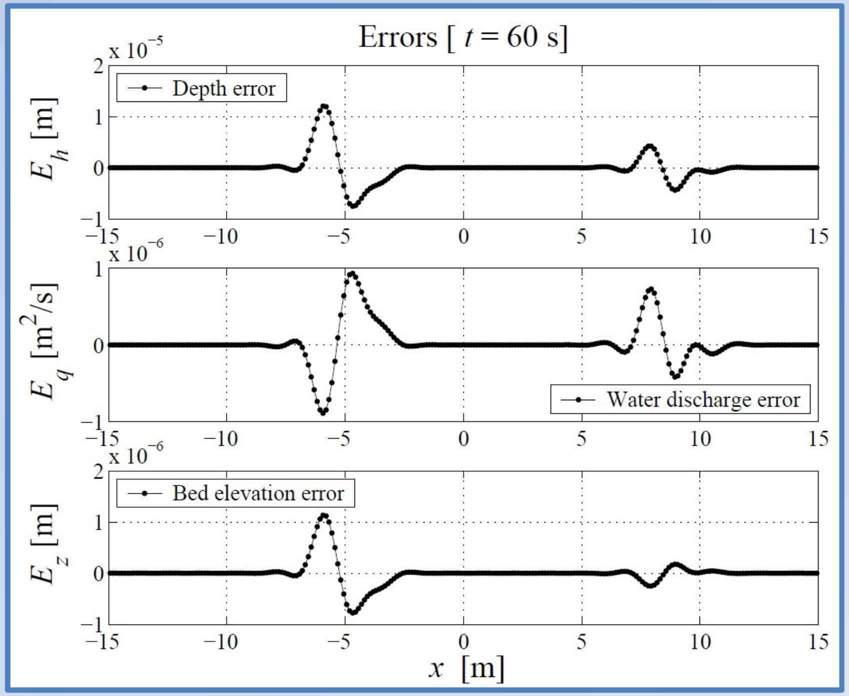
$$z(x) = z_{\max} e^{-x^2} \quad \text{with} \quad -15m < x < 15m$$

$$h_U = 1m; \quad \lambda = 0.4; \quad \psi_U = 2.5\times10^{-3};$$

$$q_s = \beta(v - v_{cr})^r \quad \text{with} \quad \beta = 3.4\times10^{-4}; \quad r = 2.65;$$

$$\mathrm{Fr}_U = 1.04;$$





Numerical Approximations of Hyperbolic Systems with Source Terms and Applications

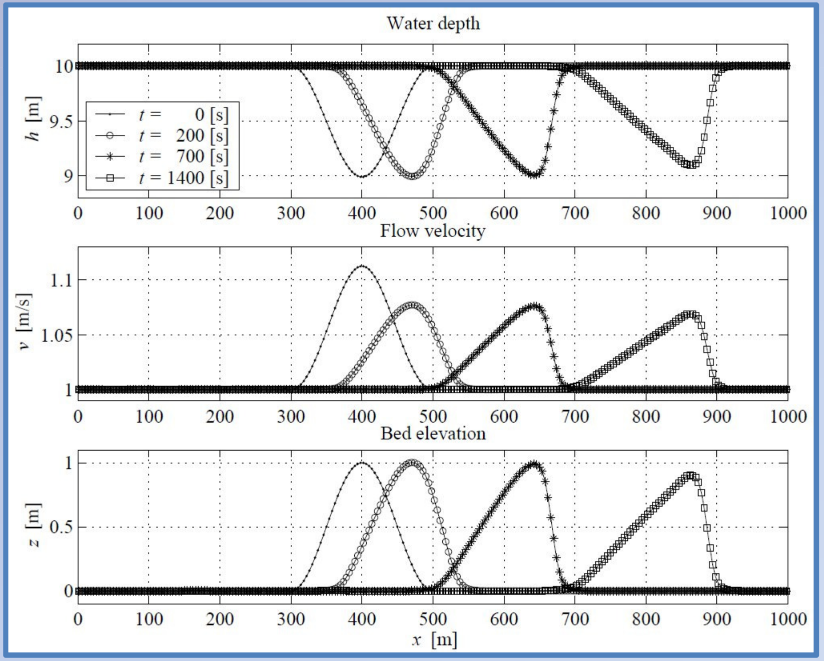
Convection-diffusion of a bump on movable bed with low/high solid discharge

Initial conditions: q = 10 m^2/s; h = 10 m; L = 1000 m; B = 1 m; 250 cells;

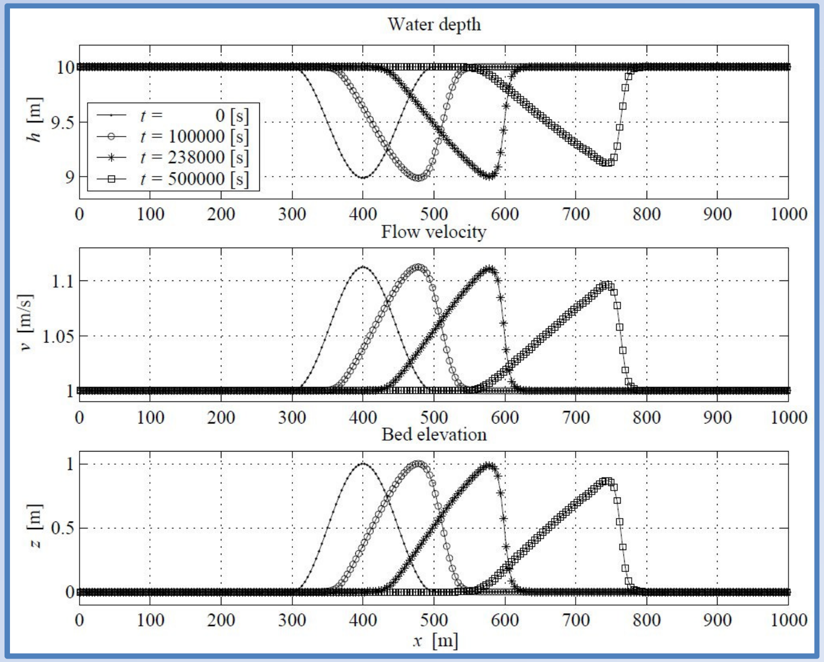
$$z_b(x) = \begin{cases} \sin^2\left[\pi(x-300)/200\right] & \text{if} \quad 300m \le x \le 500m \\ 0 & \text{otherwise} \end{cases}$$

$$q_s = Av^3$$

Intense sediment transport $A = 1\ m^{-1}s^2$



Moderate sediment transport $A = 0.001\ m^{-1}s^2$

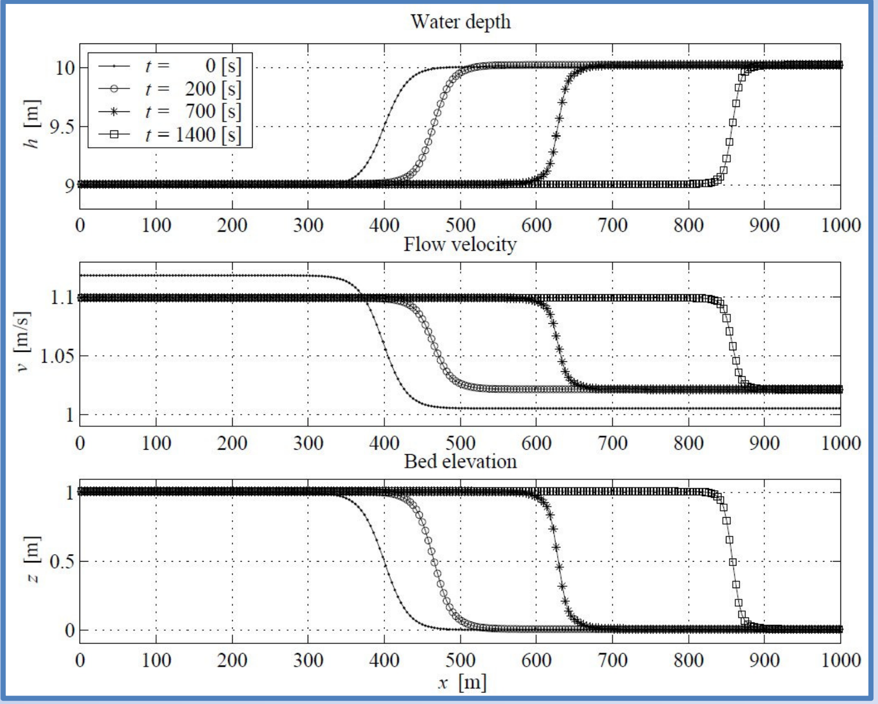


Convection-diffusion of a step on movable bed with a low discharge

Initial conditions: q = 10 m^2/s; h = 10 m; A = 1 m^{-1}s^2 (intense sed. tr.)

Computational domain geometry: same configuration of the previous cases, with bottom

$$z_b(x) = \left[1 + \exp\left(\left(x - x_0\right)/5\,\pi\right)\right]^{-1}, \quad x_0 = 400\ m$$



Conclusion

1. In this work the applications of a CWENO scheme to shallow water equations over a moving bed is presented.
2. The method is competitive with respect to other high resolution schemes for solving complex system of conservation laws, mainly for its computational efficiency and for its generality.
3. The code is validated using test cases on movable bed, obtaining very good results in terms of:
    - accuracy;
    - well-balancing property;
    - suitability to correctly reproduce the amplitude and the celerity of water and sand waves.

Well-balanced bottom discontinuities treatment for high-order shallow water equations WENO scheme

Caleffi & Valiani, 2009, ASCE Journal of Engineering Mechanics,
135 (7), 684–696

The context

This work is part of the extensive study on high order WENO schemes for SWE with source terms. The final aim is numerical modeling for engineering problems.

Available work

Reliable WENO schemes are considered as applicable in terms of:

- *very good efficiency, very good resolution of discontinuities, stability;*
- *Moreover, well balanced source terms treatment are available (i.e. satisfying C-property).*

Open problems

Strongly uneven geometries are typical of real problems; a lot of engineering problems are characterized by the presence of "singularities", such as bores propagating on wet bed, or sheer geometrical singularities, like bottom discontinuities.

In this work

A 1D FV WENO scheme is proposed, 4th order accurate in time and space, to numerically integrate SWE with source term due to the bottom slope, including bottom discontinuities.

The main original contribution consists in the new approach for managing bottom discontinuities. The method is (physically) based on a proper correction of numerical flux, obtained from the energy conservation principle and from a momentum balance which take the force on the bottom steps into account.

When the bottom is continuous, a high-order extension of the DFB method is used. This is a further original contribution.

Some classical test cases are used, to verify the order of accuracy, the C-property preservation, and the good resolution which can be obtained, in both cases of continuous and discontinuous bottom.

The numerical model - introduction

Classical Shallow Water Equations:

$$u_t + f_x = s \quad \Leftrightarrow \quad \frac{\partial}{\partial t}\begin{bmatrix} h \\ vh \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} vh \\ gh^2/2 + v^2 h \end{bmatrix} = \begin{bmatrix} 0 \\ -gh\frac{\mathrm{d}z}{\mathrm{d}x} \end{bmatrix},$$

Are:

integrated over each cell $I_j=[x_{j-1/2}, x_{j+1/2}]$

discretized in space

The numerical model – the WENO reconstruction of variables

The spatial discretization requires the estimate of point values of conservative variables at the cell boundaries, starting form the cell-averaged values of such variables. Similarly, to compute the source term, half-cell averaged values of water elevation are necessary: left and right half cell are considered.
To compute such values, WENO reconstructions are used (see Qiu & Shu (2002) and references herein)

SGM method (Zhou et al., 2001) is applied, so that the water surface elevation (= piezometric head), $\eta = z+h$ *, and the specific discharge,* $q = vh$*, are chosen as fundamental quantities to perform variable reconstructions.*

The numerical model - introduction

Formally, SWE in a semi-discretized form can be written as:

$$\boxed{\frac{d\overline{u}_j}{dt} = -\frac{1}{\Delta x}\left[\mathcal{F}_j^R - \mathcal{F}_j^L\right] + \overline{\mathcal{S}}_j = \mathcal{L}\left(\overline{u}, x, t\right),}$$

1) $\overline{\mathcal{S}}_j$ *Integral on a cell of the source term*

2) $$\left.\begin{aligned} \mathcal{F}_j^R &= \mathcal{F}_j^R\left(\hat{u}_j^R, \hat{u}_{j+1}^L, \hat{z}_j^R, \hat{z}_{j+1}^L\right) \\ \mathcal{F}_j^L &= \mathcal{F}_j^L\left(\hat{u}_{j-1}^R, \hat{u}_j^L, \hat{z}_{j-1}^R, \hat{z}_j^L\right) \end{aligned}\right\}$$

Numerical fluxes, to be better specified in the following, relative to the right interface (R) and to the left interface (L) of the j-th cell.

It is worth noting that not only conservative variables, but also bed elevation values, relative to the same interface, but to different adjacent cells can be different. THE BED ELEVATION CAN BE DISCONTINUOUS.

The numerical model – time integration

Several methods, available in literature, concerning integration in time of the semi-discretized equation are tested.
Finally, the Strong Stability Preserving Runge-Kutta SSPRK(5,4) scheme [five steps, fourth order accurate], by Spiteri & Ruuth (2002) is selected.
Coupled with a proper spatial WENO discretization, such a method allows to obtain very efficient non oscillatory schemes.

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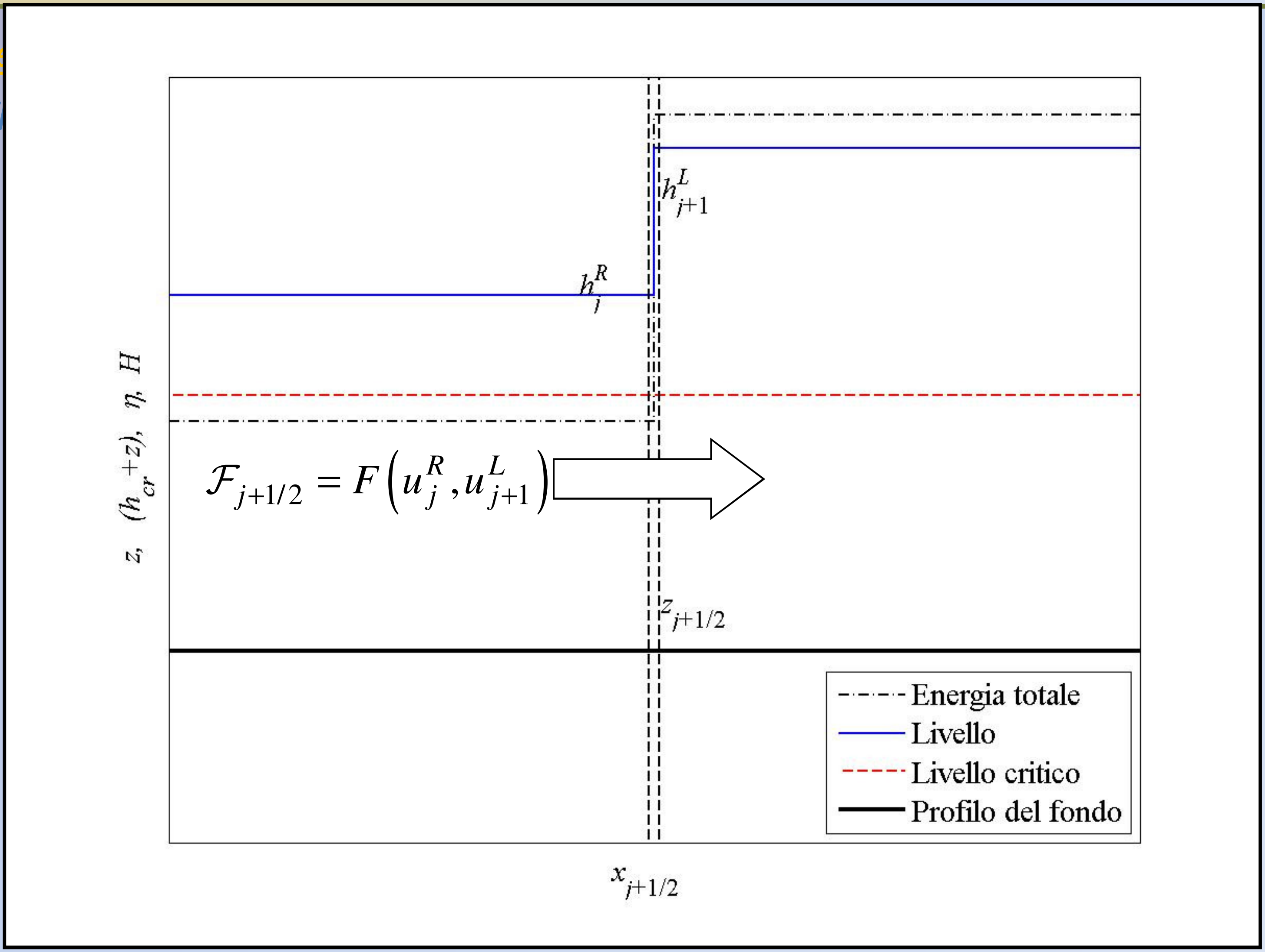
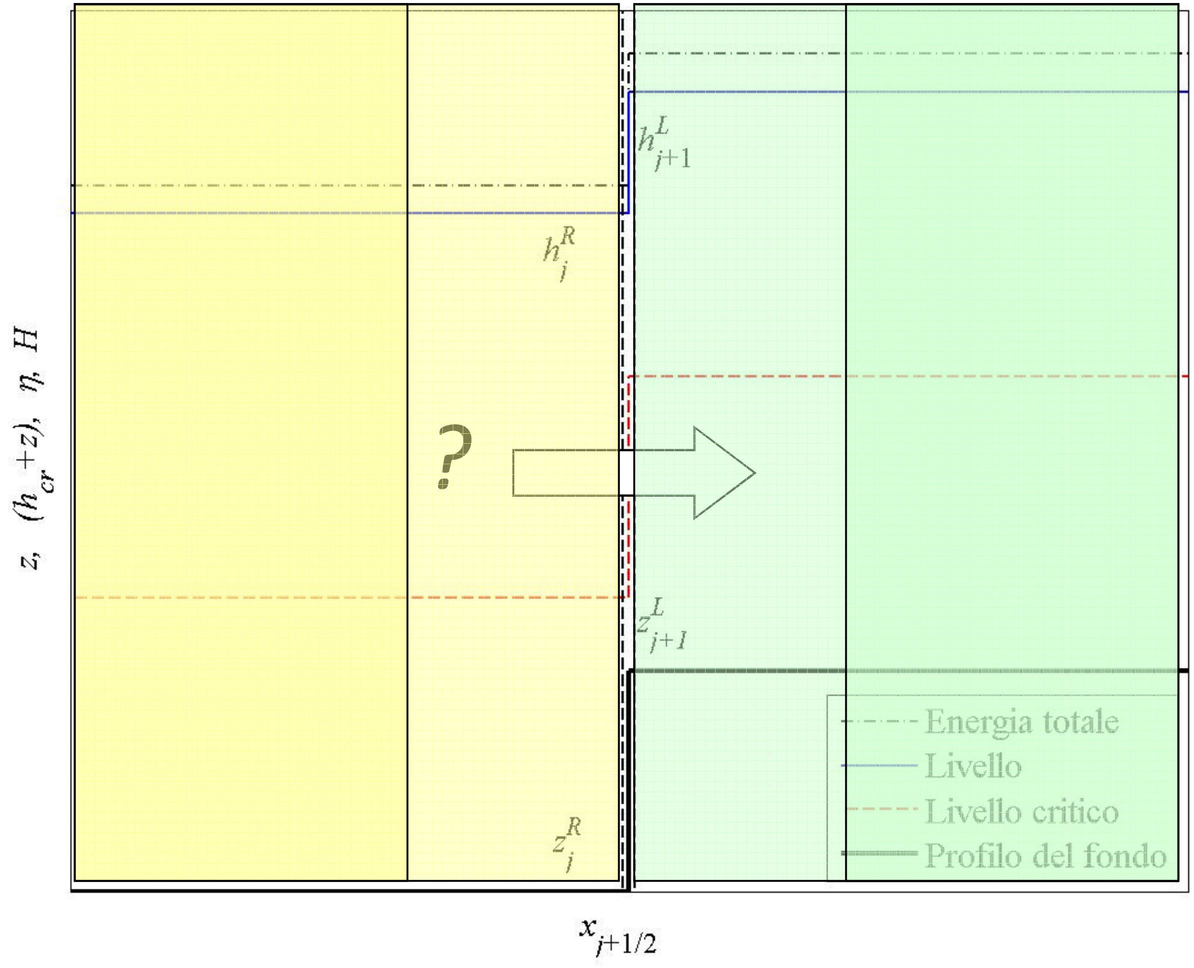
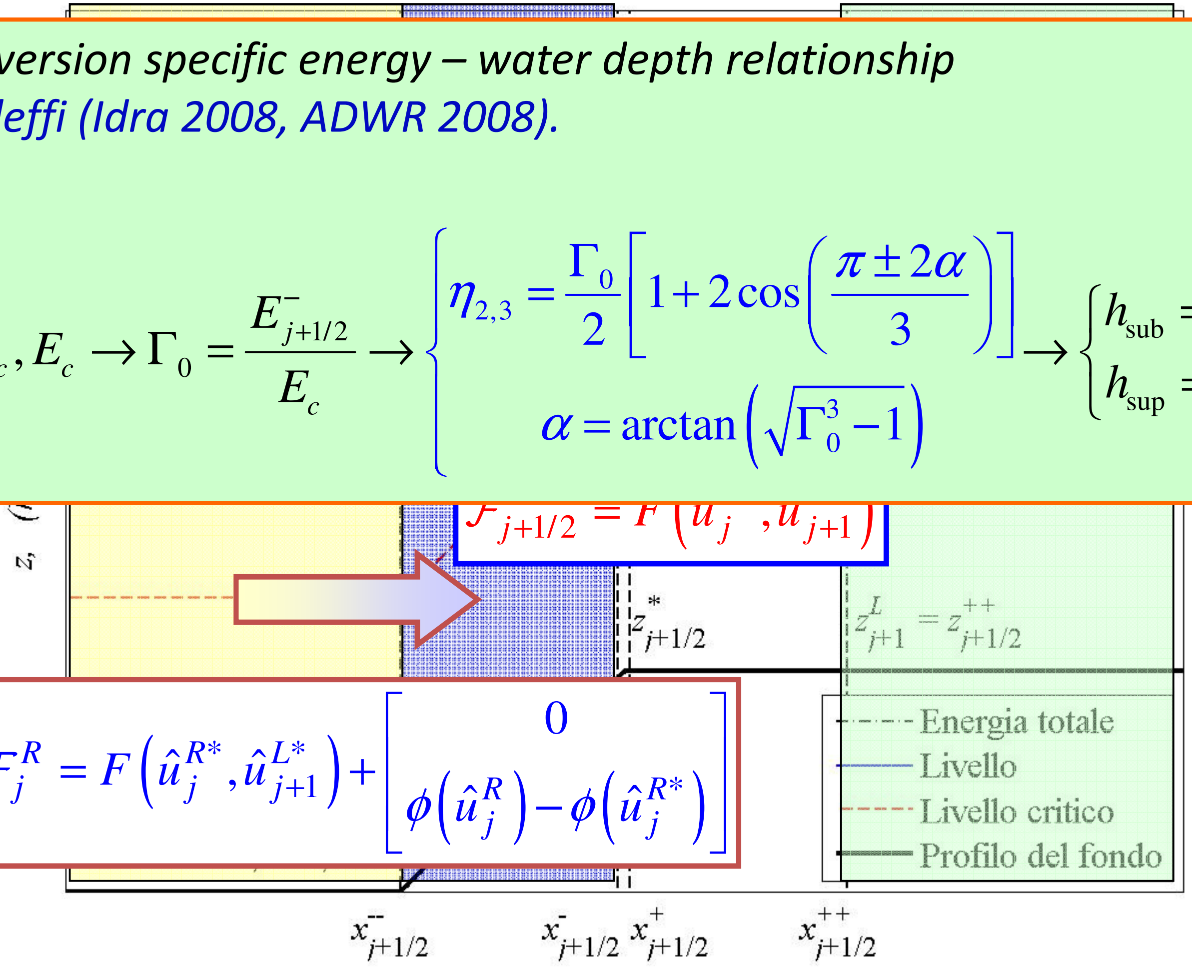

h_{j+1}^{L}
h_{j}^{R}
$z,\ (h_{cr}+z),\ \eta,\ H$
$\mathcal{F}_{j+1/2} = F\left(u_j^R, u_{j+1}^L\right)$
$z_{j+1/2}$
$x_{j+1/2}$
Energia totale
Livello
Livello critico
Profilo del fondo


$z,\ (h_{cr}+z),\ \eta,\ H$
h^L_{j+1}
h^R_j
?
z^L_{j+1}
z^R_j
Energia totale
Livello
Livello critico
Profilo del fondo
$x_{j+1/2}$

Analytical inversion specific energy – water depth relationship

Valiani & Caleffi (Idra 2008, ADWR 2008).

$$\hat{q}^-_{j+1/2} \to h_c, E_c \to \Gamma_0 = \frac{E^-_{j+1/2}}{E_c} \to \begin{cases} \eta_{2,3} = \dfrac{\Gamma_0}{2}\left[1 + 2\cos\left(\dfrac{\pi \pm 2\alpha}{3}\right)\right] \\ \alpha = \arctan\left(\sqrt{\Gamma_0^3 - 1}\right) \end{cases} \to \begin{cases} h_{\text{sub}} = \eta_2 h_c \\ h_{\text{sup}} = \eta_3 h_c \end{cases}$$



$$\mathcal{F}_j^R = F\left(\hat{u}_j^{R*}, \hat{u}_{j+1}^{L*}\right) + \begin{bmatrix} 0 \\ \phi\left(\hat{u}_j^R\right) - \phi\left(\hat{u}_j^{R*}\right) \end{bmatrix}$$

The numerical model – computation of fluxes

For the j-th cell, numerical fluxes have the following expression:

$$\mathcal{F}_j^R = F\left(\hat{u}_j^{R*}, \hat{u}_{j+1}^{L*}\right) + \begin{bmatrix} 0 \\ \phi\left(\hat{u}_j^R\right) - \phi\left(\hat{u}_j^{R*}\right) \end{bmatrix}; \quad \mathcal{F}_j^L = F\left(\hat{u}_{j-1}^{R*}, \hat{u}_j^{L*}\right) + \begin{bmatrix} 0 \\ \phi\left(\hat{u}_j^L\right) - \phi\left(\hat{u}_j^{L*}\right) \end{bmatrix}.$$

Where: $\phi(u) = gh^2/2 + v^2 h$ *TOTAL FORCE (divided by ρ)*

A similar approach is proposed by Audusse et al. (2004) and Noelle et al. (2006): in these previous works fluxes corrections physically correspond to static forces balances only. On the contrary, we suggest to consider a complete momentum balance, including dynamic forces.

The numerical model – the balanced discretization of the source term

- *The 4° order extension of 2° order DFB method (Valiani & Begnudelli, 2006) is used.*

$$\bar{S}_j = \frac{g}{2\Delta x}\left[\left(\bar{\eta}_j - \hat{z}_j^R\right)^2 - \left(\bar{\eta}_j - \hat{z}_j^L\right)^2\right];$$

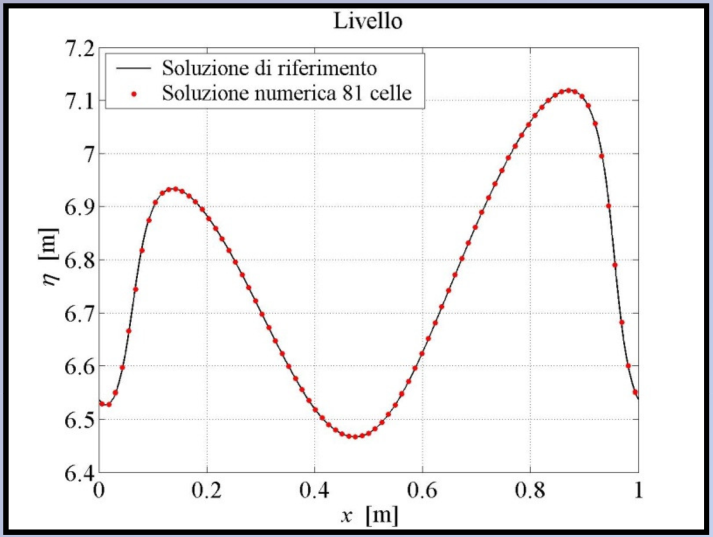
- *The extension to fourth order accuracy is obtained using a proper numerical extrapolation (Noelle et al., 2006). Each cell is considered as a whole, and then it is divided into two half-cells, obtaining the following quadrature expressions:*

$$\bar{S}_j = \frac{4\bar{S}_j^{(2)} - \bar{S}_j^{(1)}}{3} \quad \text{with:}$$

$$\bar{S}_j^{(1)} = \frac{g}{2\Delta x}\left[\left(\bar{\eta}_j - \hat{z}_j^R\right)^2 - \left(\bar{\eta}_j - \hat{z}_j^L\right)^2\right];$$

$$\bar{S}_j^{(2)} = \frac{g}{2\Delta x}\left[\left(\bar{\eta}_j^L - \hat{z}_j^C\right)^2 - \left(\bar{\eta}_j^L - \hat{z}_j^L\right)^2\right] + \frac{g}{2\Delta x}\left[\left(\bar{\eta}_j^R - \hat{z}_j^R\right)^2 - \left(\bar{\eta}_j^R - \hat{z}_j^C\right)^2\right];$$

Accuracy test case (Xing & Shu, 2005)



Second order “standard” model:

- FVM
- Balanced
- Godunov type
- HLL flux
- MUSCL
- predictor corrector

Accuracy analysis						
cells	L^1	order	L^2	order		order
81	5.8497E–04		1.5917E–03		8.1106E–03	
243	9.0554E–06	3.7941	3.4204E–05	3.4955	2.5933E–04	3.1338
729	4.8619E–08	4.7579	2.0434E–07	4.6607	1.7308E–06	4.5599
2187	2.8989E–10	4.6625	1.1671E–09	4.7016	1.0222E–08	4.6712

Test case for accuracy (Xing & Shu, 2005) - Efficiency

4^{th} order

n° cells	L^1	order	L^2	order		order
81	5.8497E–04		1.5917E–03		8.1106E–03	
729	4.8619E–08	**4.7579**	2.0434E–07	**4.6607**	1.7308E–06	**4.5599**
2187	2.8989E–10	**4.6625**	1.1671E–09	**4.7016**	1.0222E–08	**4.6712**

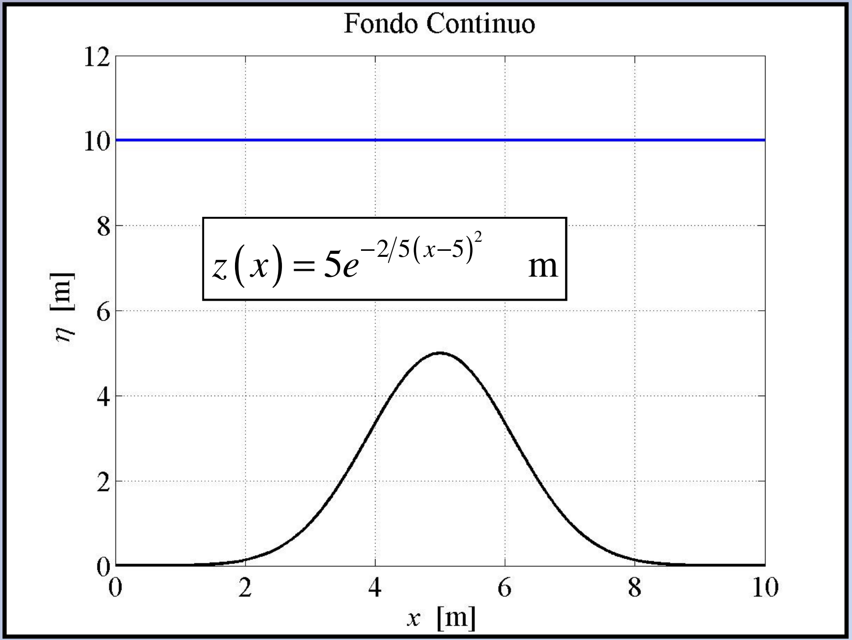
2^{sd} order

n° cells	L^1	order	L^2	order		order
243	1.7760E–03		3.2702E–03		1.5352e-002	
2187	2.6099E–05	**1.9322**	6.0169E–05	**1.8642**	4.2968e-004	**1.7422**
6561	3.0075E–06	**1.9668**	7.1506E–06	**1.9388**	5.2428e-005	**1.9148**

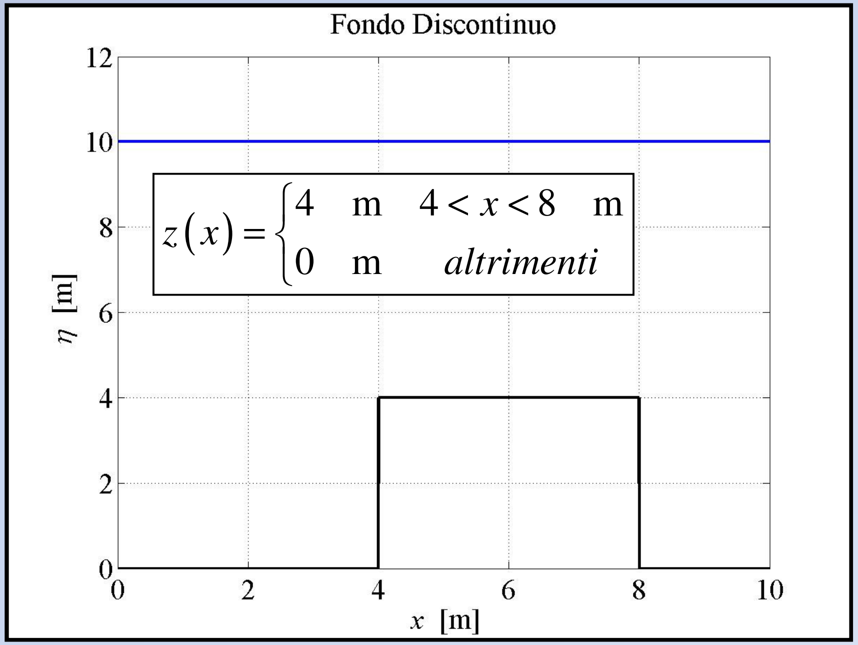
Fourth order - 729 cells -> 7.03 [s] 14%
Second order - 6561 cells -> 50.13 [s] 100%

Test case for C-property (balancing)

Still water preservation on continuous and discontinuous bottom



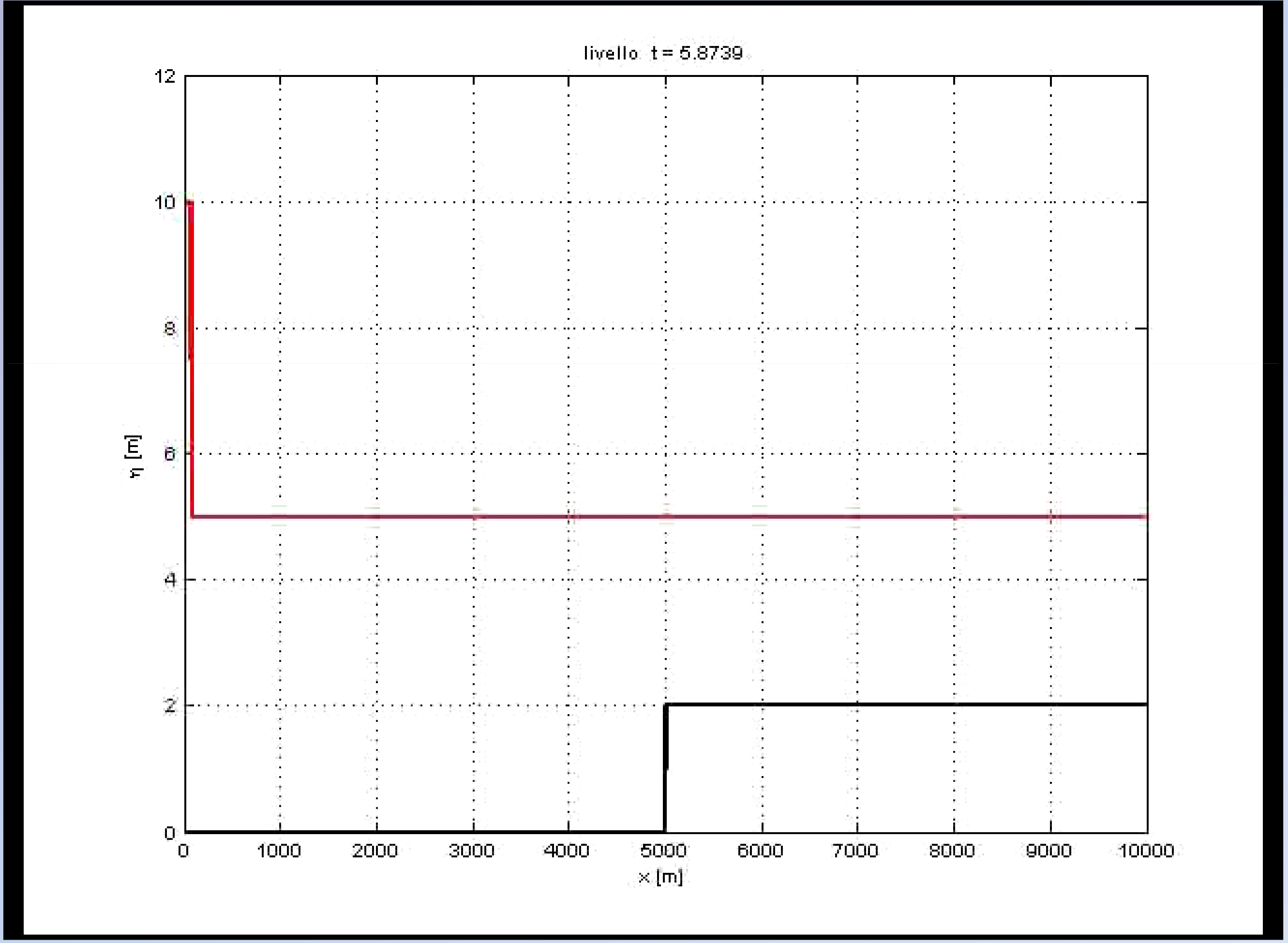
$$z(x) = 5e^{-2/5(x-5)^2} \quad \text{m}$$



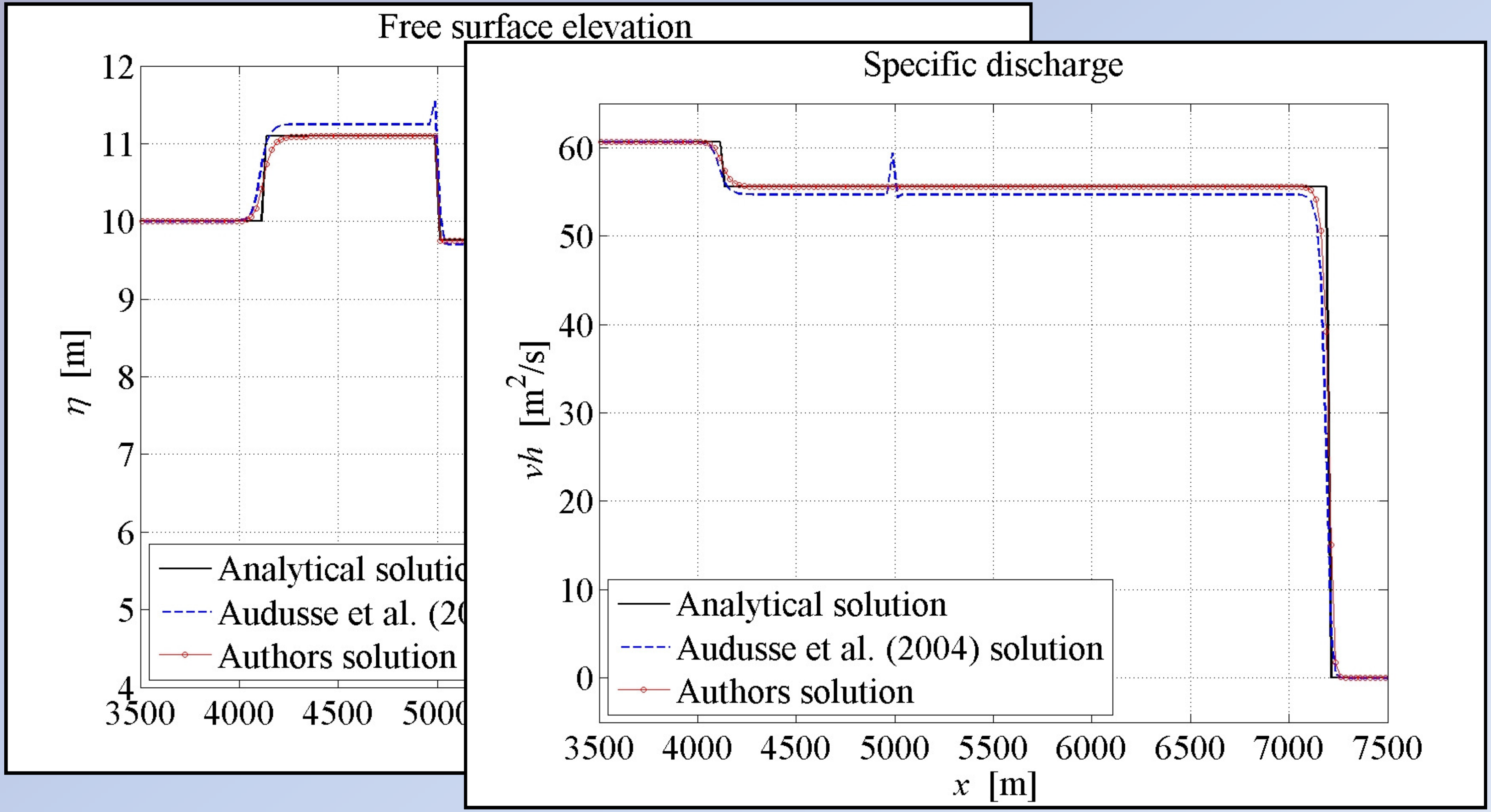
$$z(x) = \begin{cases} 4 \quad \text{m} & 4 < x < 8 \quad \text{m} \\ 0 \quad \text{m} & \textit{altrimenti} \end{cases}$$

	h			vh		
Bottom	L^1	L^2	L^∞	L^1	L^2	L^∞
Continuos	3.20E–15	1.4E–15	8.9E–16	4.2E–16	1.3E–15	4.1E–15
Discontinuous	1.80E–15	5.6E–15	1.8E–15	1.3E–13	4.4E–14	3.6E–11

Test case: shock wave moving towards a step



Test case: shock wave moving towards a step



Conclusions on bottom discontinuities

Concerning SWE with source term due to bed elevation variability,

A well balanced FV WENO method is presented.

It is 4° order accurate in time and space.

Time integration and conservative variable reconstruction are performed using well stated techniques.

Two original ingredients are introduced:

1. *A numerical flux correction, based on physical reasoning (momentum balance), to treat bottom discontinuities;*
2. *A high order (4°) extension of DFB* 2^{nd} *order method to treat source term, in case of continuous bottom elevation;*

Model validation against significant test cases gives very good responses.

SOME USEFUL ANALYTICAL RESULTS on FREE SURFACE FLOWS

Depth–energy and depth–force relationships in open channel flows: Analytical findings

Valiani & Caleffi, AWR,
Advances in Water Resources 31 (2008) 447–454

We make reference to the work:
Shallow water equations with variable topography in the resonance regime
By Philippe G. LeFloch and Mai Duc Thanh

Concerning section 2.2, **Equilibrium states**, such authors
Consider the problem of finding a certain RHS, given a prescribed LHS.

$$\text{State} \quad U_0 = (h_0 , \quad u_0 , \quad a_0) \quad [\text{left hand state}]$$

$$\text{State} \quad U = (h, \quad u, \quad a), \quad a \neq a_0 \quad [\text{right hand state}]$$

We want to find out h and u in terms of U_0 and a, solving:

$$a + h + \frac{u^2}{2\,g} = a_0 + h_0 + \frac{u_0^2}{2\,g}$$

applying also mass conservation as:

$$u\,h = u_0\,h_0$$

According classical hydraulic engineering language, we use the terms

$Y \leftrightarrow h$ as the water depth

$u\,h = u_0\,h_0 \leftrightarrow q$ as the unit width discharge

$E = h + \frac{u^2}{2\,g}$ as the specific energy

For the U_0 state, we call critical depth the value:

$$Y_c \leftrightarrow h_{\min}\left(U_0\right) = \left(\frac{u_0^2\,h_0^2}{g}\right)^{1/3}$$

and, finally, we make any variable as non dimensional,

by dividing for the corresponding critical value:

$$\eta = \frac{Y}{Y_c}; \quad \Gamma = \frac{E}{E_c}$$

SOME USEFUL ANALYTICAL RESULTS on FREE SURFACE FLOWS

Water Depth (Y), Specific Energy (E), Total Force (F)

Q = q b , discharge (prescribed)

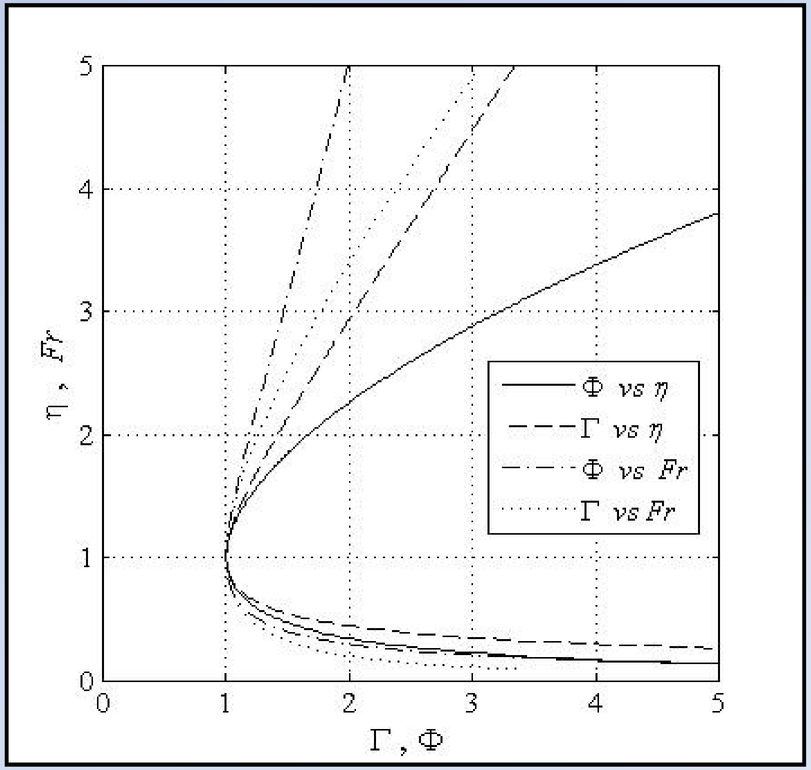
The subscript c means critical conditions:

Minimum E , for the given Q

Minimum F , for the given Q

$$ND\begin{cases}\text{depth}\\ \text{energy}\\ \text{force}\end{cases}\qquad \eta=\frac{Y}{Y_c},\quad \Gamma=\frac{E}{E_c},\quad \Phi=\frac{F}{F_c}$$

$$\Gamma=\frac{2}{3}\eta+\frac{1}{3}\left(\frac{1}{\eta}\right)^2;\quad \Phi=\frac{1}{3}\eta^2+\frac{2}{3}\left(\frac{1}{\eta}\right)$$



Analytical inversion of the Energy-Depth relationship

It can be shown that, given a prescribed value of the nd specific energy Γ_0 , greater then the critical value ($\Gamma_0 \geq 1$), three values of nd depth exist, satisfying the relation $\Gamma=\Gamma_0$. Such three values are the following:

$$\eta_1 = \frac{\Gamma_0}{2}\left[1-2\cos\left(\frac{2}{3}\alpha\right)\right]; \quad \boxed{\eta_{2,3} = \frac{\Gamma_0}{2}\left[1+2\cos\left(\frac{\pi\pm 2\alpha}{3}\right)\right]}$$

Being: $\alpha = \arctan\left(\sqrt{\Gamma_0^3 - 1}\right)$ *a real number.*

It is simple to show that:

The 1° root is negative (to discard)

The 2° root is positive and ≤ 1 (supercritical flow)

The 3° root is positive and ≥ 1 (subcritical flow)

Analytical inversion of the Total Force-Depth relationship

It can be shown that, given a prescribed value of the nd total force Φ_0 , greater then the critical value ($\Phi_0 \geq 1$), three values of nd depth exist, satisfying the relation $\Phi = \Phi_0$. Such three values are the following:

$$\eta_1 = -2\sqrt{\Phi_0}\cos\left(\frac{\theta}{3}\right); \quad \boxed{\eta_{2,3} = 2\sqrt{\Phi_0}\cos\left(\frac{\pi \pm \theta}{3}\right)}$$

Being: $\theta = \arctan\left(\sqrt{\Phi_0^3 - 1}\right)$ *a real number.*

It is simple to show that:

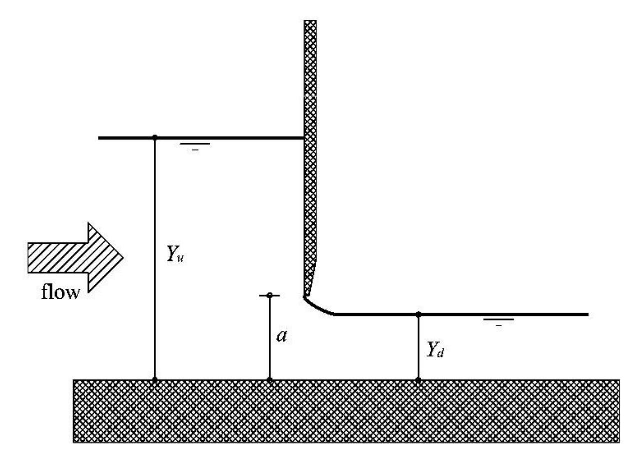
The 1° root is negative (to discard)

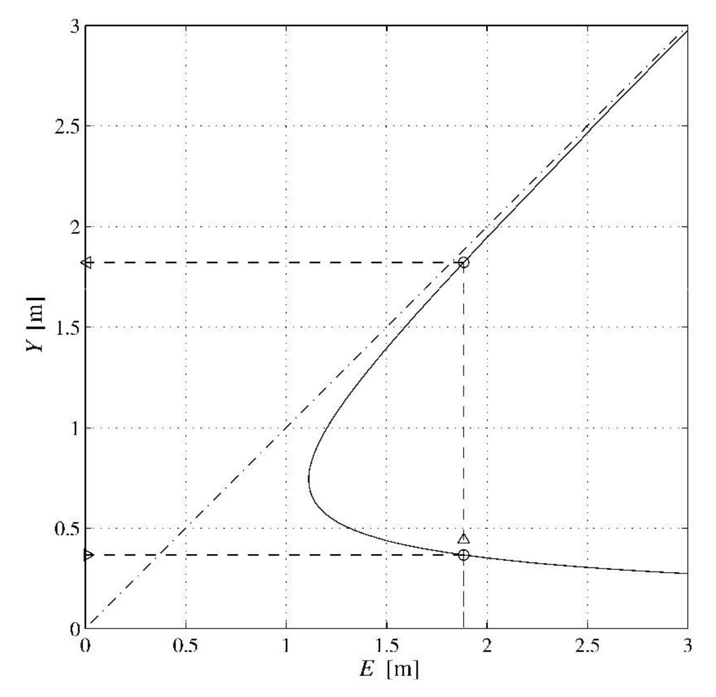
The 2° root is positive and ≤ 1 (supercritical flow)

The 3° root is positive and ≥ 1 (subcritical flow)

A classical example: the sluice gate

For a given discharge, the downstream flow depth is imposed by gate opening. Computing the corresponding specific energy and using the proper inverse relationship allows to find the upstream depth.

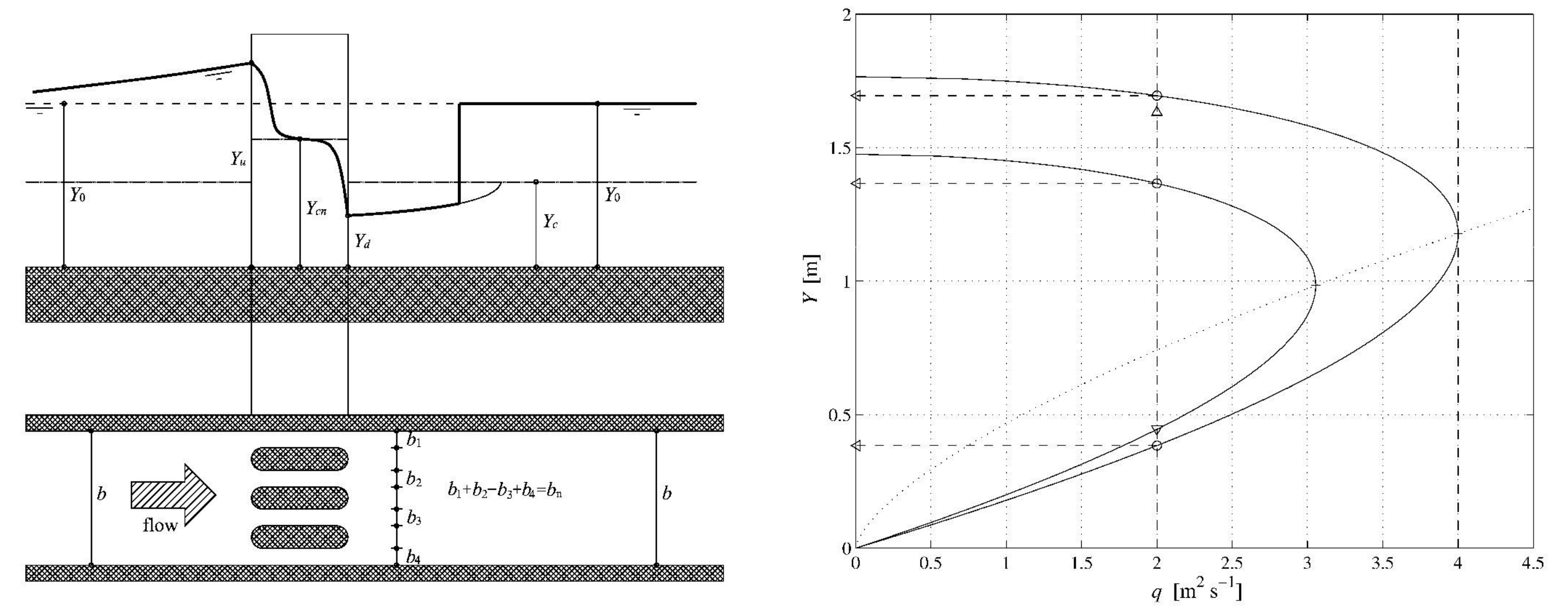




A classical example: the width reduction due to bridge piers

For a given discharge, and given width ratio ("strong" contraction), a certain specific energy is necessary to cross the piers. Upstream and downstream flow depth, corresponding to such energy, can be computed by the inverse relationship.

Total force inversion gives immediately the depth upstream the jump.



HWENO

A new well-balanced Hermite weighted essentially non-oscillatory scheme for shallow water equations

Caleffi, 2010, International Journal For Numerical Methods In Fluids, DOI: 10.1002/fld.2410

Numerical Approximations of Hyperbolic Systems with Source Terms and Applications

The context

This work is developed in the context of the high-order accuracy schemes for the shallow water equations with bottom slope source term. In particular, the class of well-balanced and compact methods is take into account.

The state of the art

The state of the art is represented by the Runge-Kutta Discontinuous Galerkin schemes (e.g. Cockburn & Shu, 2001) and the PNPM methods (Dumbser et al., 2008), while the Hermitian WENO schemes are recently proposed as an interesting alternative (e.g. Qiu & Shu, 2004).

In this work

The attention is focused on the well balancing of a fourth-order accurate HWENO schemes and a satisfying source term treatment is achieved.

- Cockburn, B. & Shu, C.W., 2001, *Runge-Kutta discontinuous Galerkin methods for convection-dominated problems*, Journal of Scientific Computing, 16(3), pp. 173-261.
- Qiu, J.X. & Shu, C.W. , 2004, *Hermite WENO schemes and their application as limiters for Runge-Kutta discontinuous Galerkin method: one-dimensional case*. J. Comput. Phys. ,193,115-135.
- Dumbser M, Balsara D, Toro EF, Munz CD. A unified framework for the construction of one-step finite-volume and discontinuous Galerkin schemes. J Comput Phys 2008;227:8209-53

Mathematical Model for the HWENO Scheme

To obtain the compactness of the HWENO schemes both the conservative variables and their derivatives are evolved in time, while in the original WENO schemes only the conservative variables are evolved.

Classical shallow water equations

$$u_t + f(u)_x = s(u,x), \qquad u(x,0) = u_0(x);$$

$$u(x,t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \qquad f(u) = \begin{bmatrix} u_2 \\ \dfrac{u_2^2}{u_1} + g\dfrac{u_1^2}{2} \end{bmatrix}; \qquad s(u,x) = \begin{bmatrix} 0 \\ -g u_1 z_x \end{bmatrix};$$

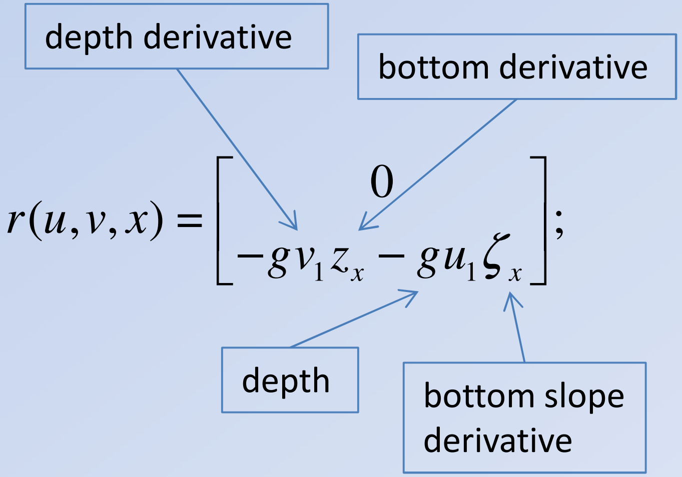
with: $u_1 = h; \qquad u_2 = Uh;$

Mathematical Model for the HWENO Scheme

Derivative of the shallow water equations (with respect to x)

$$v_t + g(u,v)_x = r(u,v,x), \qquad v(x,0) = v_0(x);$$

$$v(x,t) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}; \qquad g(u,v) = \begin{bmatrix} v_2 \\ \left[gu_1 - \left(\dfrac{u_2}{u_1} \right)^2 \right] v_1 + 2\dfrac{u_2}{u_1} v_2 \end{bmatrix};$$



$$r(u,v,x) = \begin{bmatrix} 0 \\ -gv_1 z_x - gu_1 \zeta_x \end{bmatrix};$$

depth derivative

bottom derivative

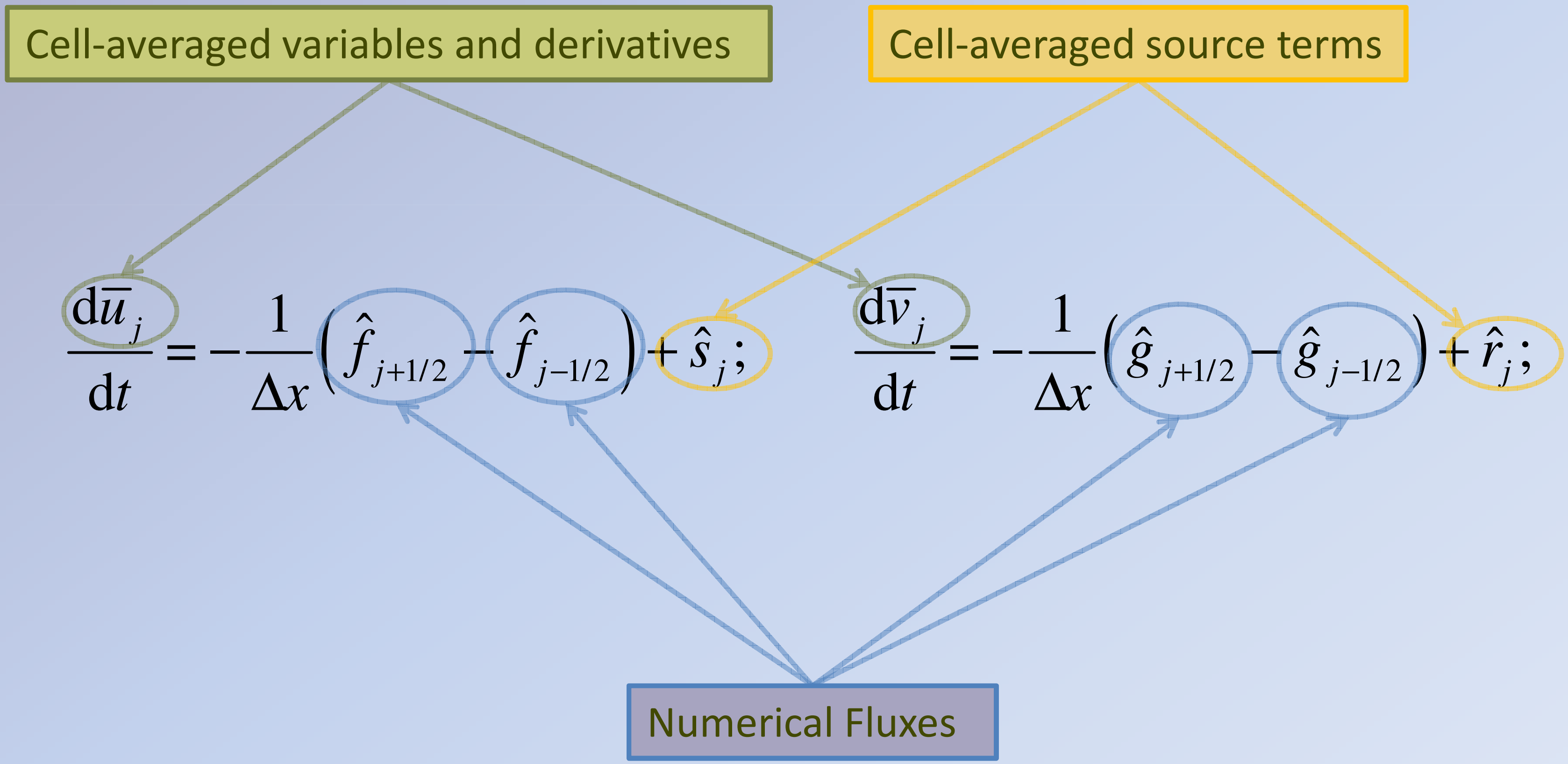
depth

bottom slope derivative

with: $v_1 = \partial u_1 / \partial x; \qquad v_2 = \partial u_2 / \partial x; \qquad \zeta = \partial z / \partial x; \qquad g(u,v) = f'(u) u_x = f'(u) v$

Numerical Model

Working in the classical framework of the Godunov type - Finite Volume methods (on uniform grid), the semi-discretized form of the mathematical model becomes:



Cell-averaged variables and derivatives

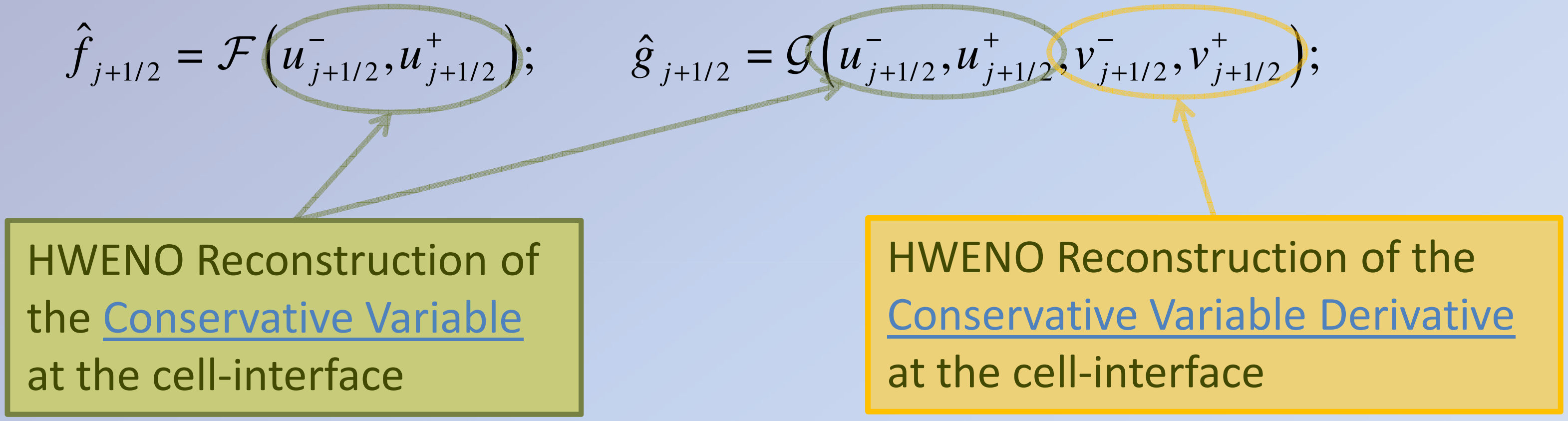
Cell-averaged source terms

$$\frac{\mathrm{d}\bar{u}_j}{\mathrm{d}t} = -\frac{1}{\Delta x}\left(\hat{f}_{j+1/2} - \hat{f}_{j-1/2}\right) + \hat{s}_j; \qquad \frac{\mathrm{d}\bar{v}_j}{\mathrm{d}t} = -\frac{1}{\Delta x}\left(\hat{g}_{j+1/2} - \hat{g}_{j-1/2}\right) + \hat{r}_j;$$

Numerical Fluxes

Numerical Model

The HLL Riemann solver is used to evaluate the numerical fluxes, together with the evaluation of the wave celerity based on the two rarefaction approximation.



$$\hat{f}_{j+1/2} = \mathcal{F}\left(u^{-}_{j+1/2}, u^{+}_{j+1/2}\right); \qquad \hat{g}_{j+1/2} = \mathcal{G}\left(u^{-}_{j+1/2}, u^{+}_{j+1/2}, v^{-}_{j+1/2}, v^{+}_{j+1/2}\right);$$

HWENO Reconstruction of the Conservative Variable at the cell-interface

HWENO Reconstruction of the Conservative Variable Derivative at the cell-interface

The selected time integration scheme, specific for conservation laws, is the five step, fourth-order accurate, Strong Stability Preserving Runge-Kutta, SSPRK(5,4), by Spiteri & Ruuth (2002).

The Surface Gradient Method (Zhou et al.,2001) is applied to simplify the well-balancing achievement, therefore the reconstructed quantities are:

$\eta = h + z$ water level

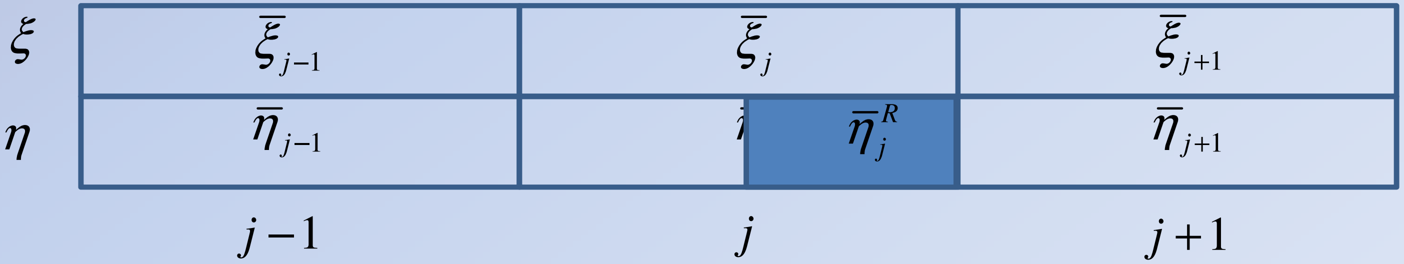
$\xi = h_x + z_x$ water level derivative

HWENO Reconstructions

Starting from: $\bar{u}_{j-1},\bar{u}_{j},\bar{u}_{j+1},\bar{v}_{j-1},\bar{v}_{j},\bar{v}_{j+1} \Longrightarrow \underbrace{u^{+}_{j+1/2},u^{-}_{j+1/2},v^{+}_{j+1/2},v^{-}_{j+1/2}}_{\textit{Point values}}$

Starting from: $\bar{\eta}_{j-1},\bar{\eta}_{j},\bar{\eta}_{j+1},\bar{\xi}_{j-1},\bar{\xi}_{j},\bar{\xi}_{j+1} \Longrightarrow \underbrace{\bar{\eta}^{R}_{j},\bar{\eta}^{L}_{j},\bar{\xi}^{R}_{j},\bar{\xi}^{L}_{j}}_{\textit{Half-cell averages}}$

For example, we consider the reconstruction for the half-cell average of water level.



ξ	$\bar{\xi}_{j-1}$	$\bar{\xi}_{j}$	$\bar{\xi}_{j+1}$
η	$\bar{\eta}_{j-1}$	$\bar{\eta}^{R}_{j}$	$\bar{\eta}_{j+1}$
	$j-1$	j	$j+1$

HWENO Reconstructions

Three 2nd order polynomials, $p_0(x), p_1(x), p_2(x)$ and a 4th order polynomial $q(x)$ are defined such that…

$$\frac{1}{\Delta x}\int_{I_{j+l}} p_0(x)\mathrm{d}x = \overline{\eta}_{j+l}; \quad l=-1,0; \qquad \frac{1}{\Delta x}\int_{I_{j-1}} p_0'(x)\mathrm{d}x = \overline{\xi}_{j-1};$$

$$\frac{1}{\Delta x}\int_{I_{j+l}} p_1(x)\mathrm{d}x = \overline{\eta}_{j+l}; \quad l=0,1; \qquad \frac{1}{\Delta x}\int_{I_{j+1}} p_1'(x)\mathrm{d}x = \overline{\xi}_{j+1};$$

$$\frac{1}{\Delta x}\int_{I_{j+l}} p_2(x)\mathrm{d}x = \overline{\eta}_{j+l}; \quad l=-1,0,1;$$

$$\frac{1}{\Delta x}\int_{I_{j+l}} q(x)\mathrm{d}x = \overline{\eta}_{j+l}; \quad l=-1,0,1; \qquad \frac{1}{\Delta x}\int_{I_{j+l}} q'(x)\mathrm{d}x = \overline{\xi}_{j+l}; \quad l=-1,1;$$

HWENO Reconstructions

where the solution is regular:

$$\left(\mathcal{I}_j^R\right)_l = \frac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}} p_l(x)\mathrm{d}x = \overline{\eta}_j^R + \mathcal{O}(\Delta x^2) \qquad l = 0,1,2;$$

$$\left(\mathcal{I}_j^R\right)_q = \frac{2}{\Delta x}\int_{x_j}^{x_{j+1/2}} q(x)\mathrm{d}x = \overline{\eta}_j^R + \mathcal{O}(\Delta x^4);$$

---

where the solution is smooth:

$$\overline{\eta}_j^R = \sum_{l=0}^{2} \overline{\gamma}_l \left(\mathcal{I}_j^R\right)_l = \left(\mathcal{I}_j^R\right)_q;$$

where $\overline{\gamma}_0, \overline{\gamma}_1, \overline{\gamma}_2.$ are linear weights.

where the solution is discontinuous:

we want evaluate $\overline{\eta}_j^R$ using the information coming from the more regular polynomials.

HWENO Reconstructions

This is obtained writing:

$$\boxed{\bar{\eta}_j^R = \sum_{l=0}^{2} \bar{\omega}_l \left(\mathcal{I}_j^R\right)_l .}$$

with, the index of smoothness is defined as usual by:

$$\beta_l = \sum_{i=1}^{2} \int_{I_j} \Delta x^{2i-1} \left(\frac{\partial^i p_l(x)}{\partial x^i} \right) \mathrm{d}x; \qquad \bar{\omega}_l = \frac{\bar{\alpha}_l}{\sum_k \bar{\alpha}_k}; \qquad \bar{\alpha}_k = \frac{\bar{\gamma}_k}{\left(\varepsilon + \beta_k\right)^2};$$

Source term treatment

Several well-balanced treatments of the classical SWE source term are available in literature. Here the approach presented in (Caleffi & Valiani, 2009) is used.

More interesting is the original well-balanced treatment of the source term related to the Shallow Water Equations Derivative:

$$\hat{r}_j^{(2)} = \frac{\mathcal{R}_j}{\Delta x} = \frac{1}{\Delta x}\int_{I_j}\left(-gh_x z_x - gh\zeta_x\right)\mathrm{d}x = \frac{1}{\Delta x}\int_{I_j}\left(-gv_1 z_x - gu_1\zeta_x\right)\mathrm{d}x,$$

The key idea is the introduction of a function of x:

$$\boxed{\phi(x;\eta^*,\xi^*) = g(\eta^* - z)(\xi^* - \zeta),}$$

and the following approximation:

$$\int_{I_j}(-gv_1 z_x - gu_1\zeta_x)\mathrm{d}x \cong \phi(x_{j+1/2};\eta^*,\xi^*) - \phi(x_{i-1/2};\eta^*,\xi^*)$$

We have analytically proved that this approximation is second-order accurate, consistent and well-balanced.

Source term treatment

Assuming: $\eta^* = \overline{\eta}_j$ and $\xi^* = \overline{\xi}_j$

The second-order accurate source term treatment becomes:

$$\mathcal{R}_j^{\mathrm{HDFB2}} = \phi(x_{j+1/2};\overline{\eta}_j,\overline{\xi}_j) - \phi(x_{j-1/2};\overline{\eta}_j,\overline{\xi}_j)$$

The fourth-order accurate formulation is obtained using a numerical extrapolation (Caleffi & Valiani, 2009):

$$\left.\begin{aligned} \mathcal{R}_j^{(a)} &= \phi(x_{j+1/2};\overline{\eta}_j,\overline{\xi}_j) - \phi(x_{j-1/2};\overline{\eta}_j,\overline{\xi}_j) \\ \mathcal{R}_j^{(b)} &= \Delta\phi^L + \Delta\phi^R \end{aligned}\right\} \Rightarrow \mathcal{R}_j^{\mathrm{HDFB4}} = \frac{4\mathcal{R}_j^{(b)} - \mathcal{R}_j^{(a)}}{3}.$$

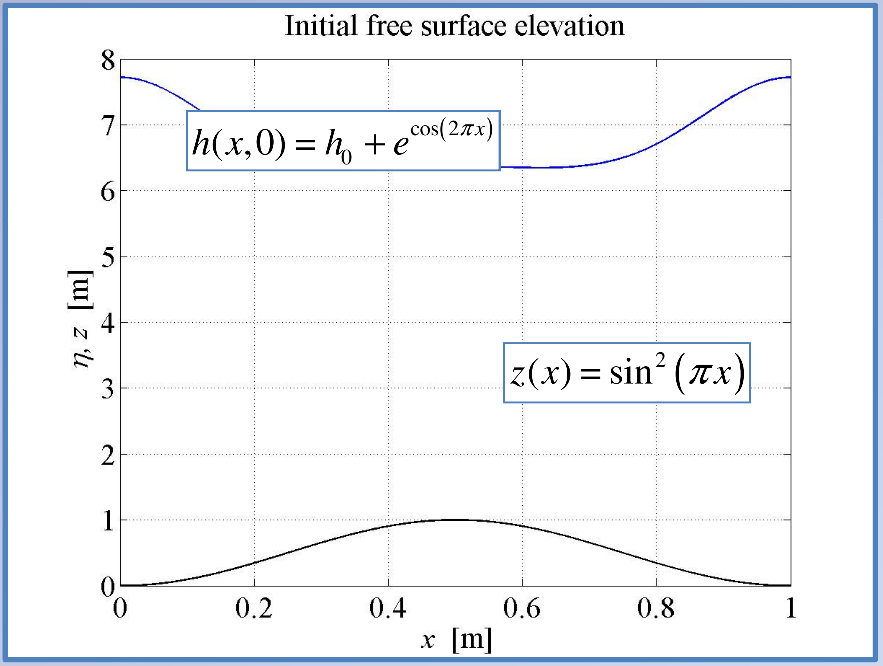
with:

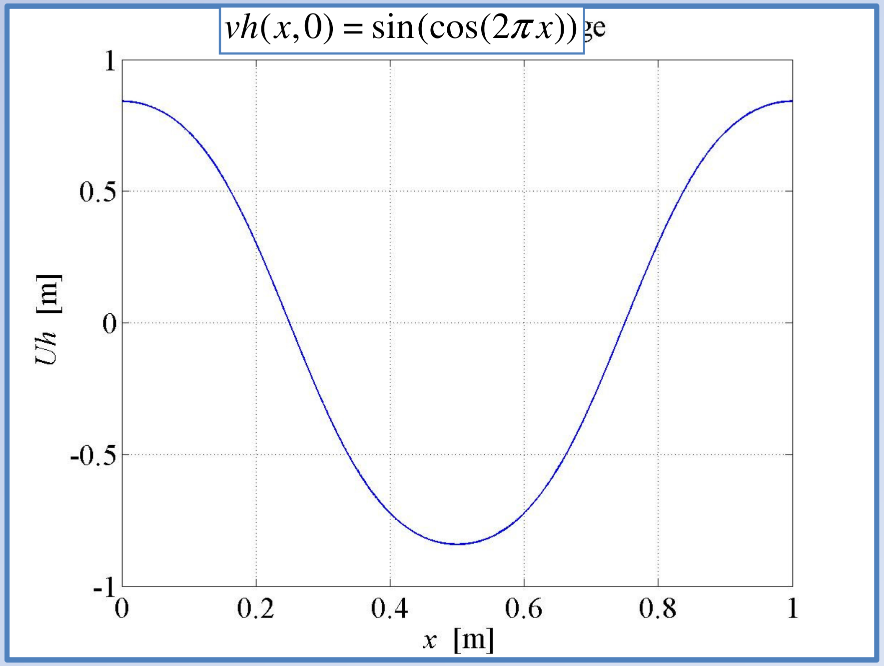
$$\Delta\phi^L = \phi(x_j;\overline{\eta}_j^L,\overline{\xi}_j^L) - \phi(x_{j-1/2};\overline{\eta}_j^L,\overline{\xi}_j^L)$$

$$\Delta\phi^R = \phi(x_{j+1};\overline{\eta}_j^R,\overline{\xi}_j^R) - \phi(x_j;\overline{\eta}_j^R,\overline{\xi}_j^R)$$

Applications

Accuracy analysis - Unsteady flow over a sinusoidal bump (Xing & Shu, 2005)





With periodic boundary conditions.

Applications

Accuracy analysis - Unsteady flow over a sinusoidal bump (Xing & Shu, 2005)

HWENO Scheme – free surface level.

Cells	time [s]	L^1	order	L^2	order	L^∞	order
81	0.0781	4.2689E-04		1.0322E-03		5.8588E-03	
243	0.5468	4.3563E-06	4.1734	1.7381E-05	3.7175	1.4420E-04	3.3720
729	6.6875	3.3439E-08	4.4326	1.3025E-07	4.4544	1.2196E-06	4.3443
2187	56.5313	3.3605E-10	4.1873	1.2591E-09	4.2226	1.1714E-08	4.2285

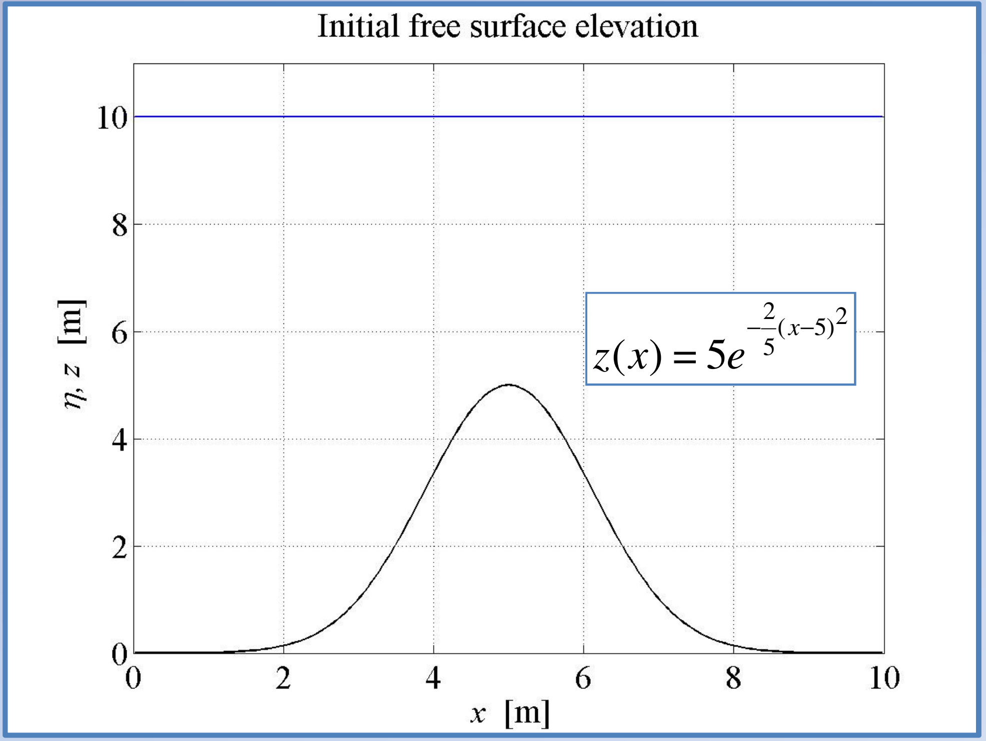
DGRK Scheme (Xing & Shu, 2006) – free surface level.

Cells	time [s]	L^1	order	L^2	order	L^∞	order
81	0.3750	9.9563E-07		3.6600E-06		2.2818E-05	
243	2.7344	9.9882E-09	4.1889	3.5285E-08	4.2251	2.5512E-07	4.0902
729	27.1875	1.2726E-10	3.9713	4.2604E-10	4.0202	3.0039E-09	4.0431
2187	234.734	1.1706E-11	2.1719	1.2603E-11	3.2046	5.1879E-11	3.6944

Xing ,Y. & Shu, C.W., 2006, A new approach of high order well-balanced finite volume WENO schemes and discontinuous Galerkin methods for a class of hyperbolic systems with source terms, Communications in Computational Physics, pp.100-134.

Applications

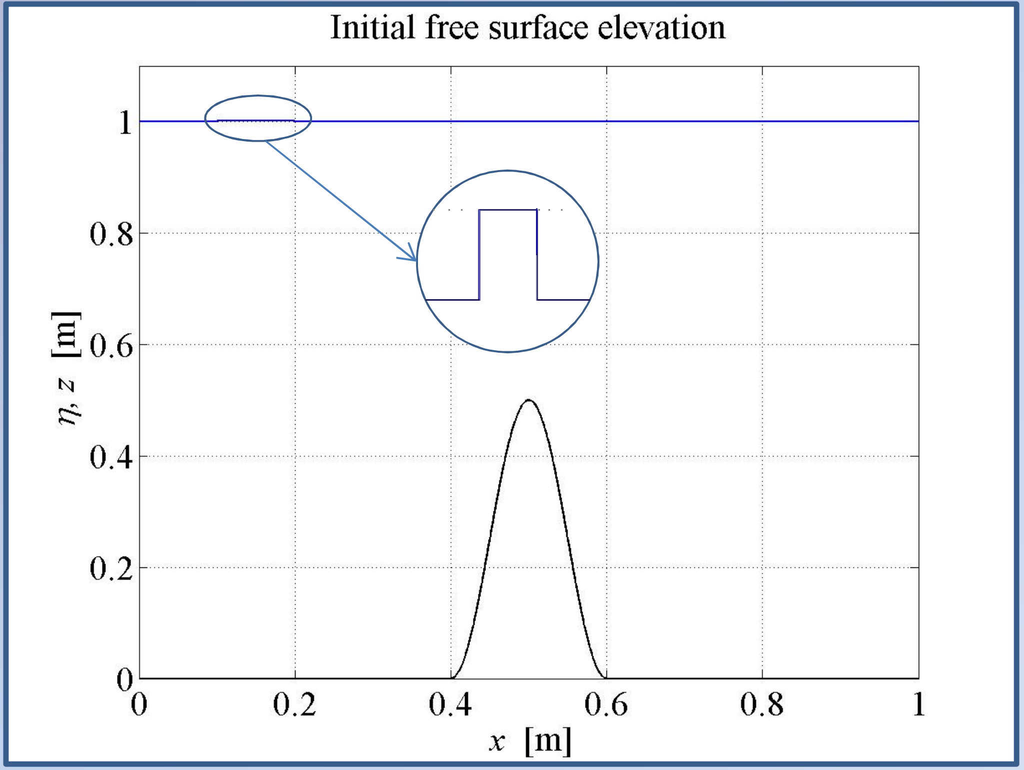
C-property – Quiescent flow over a non-flat bottom (Xing & Shu, 2005)



quantity	L^1	L^2	L^∞
η	4.69E-13	1.86E-13	1.42E-13
Uh	3.49E-12	1.39E-12	8.99E-13

Applications

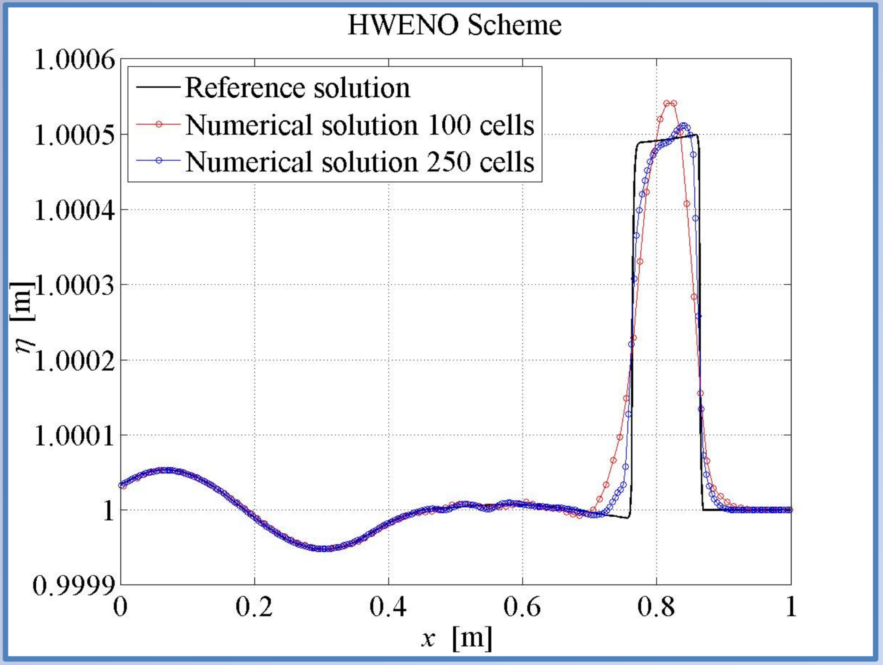
Pulse over a bump (LeVeque, 1998)

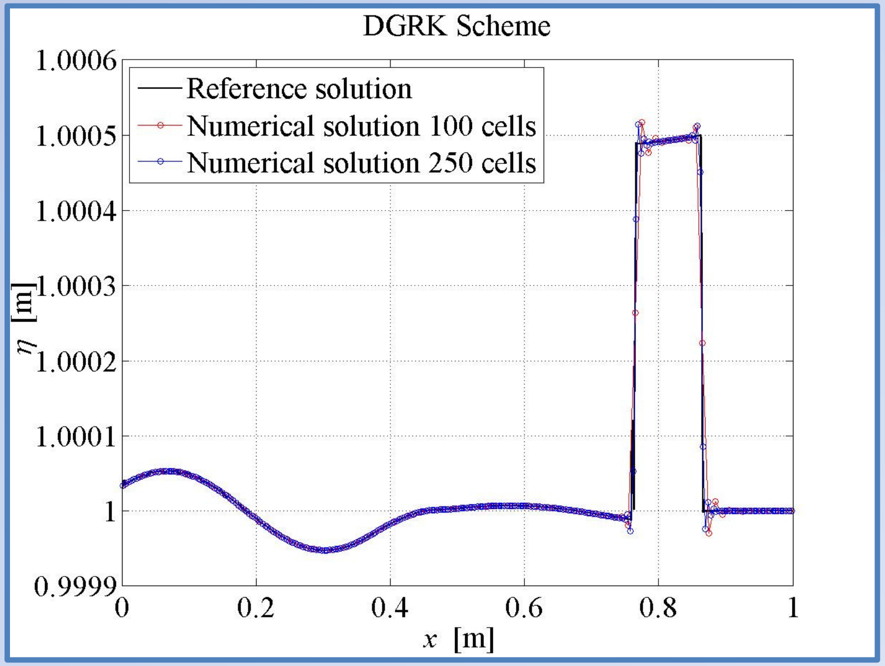


$$z(x)=\begin{cases}0.25\left[\cos\left(10\pi(x-1/2)\right)+1\right]\text{m} & \text{if } |x-1/2|<0.1\text{ m;}\\ 0 & \text{otherwise.}\end{cases}$$

Applications

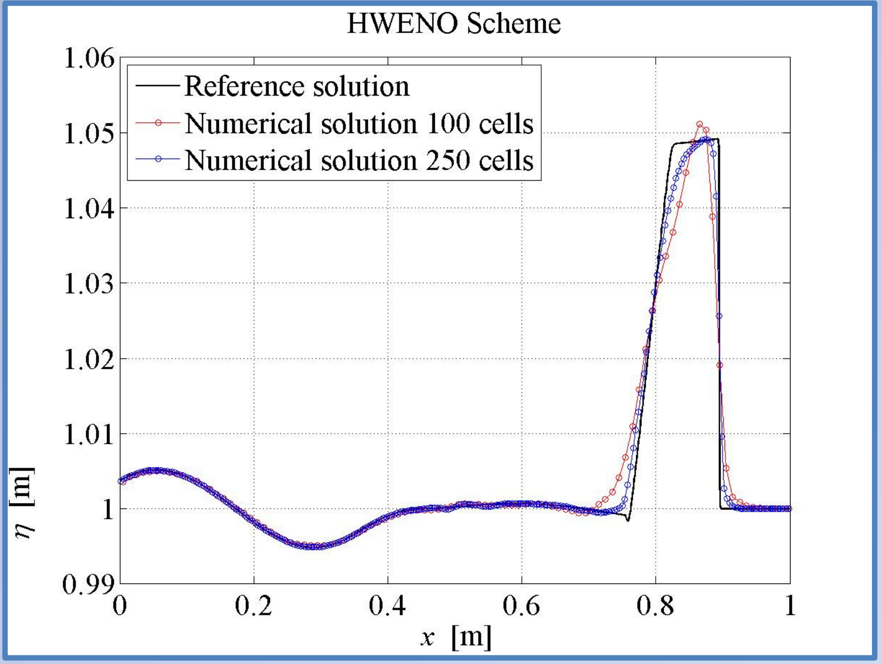
Pulse over a bump – small amplitude – (10^{-3})

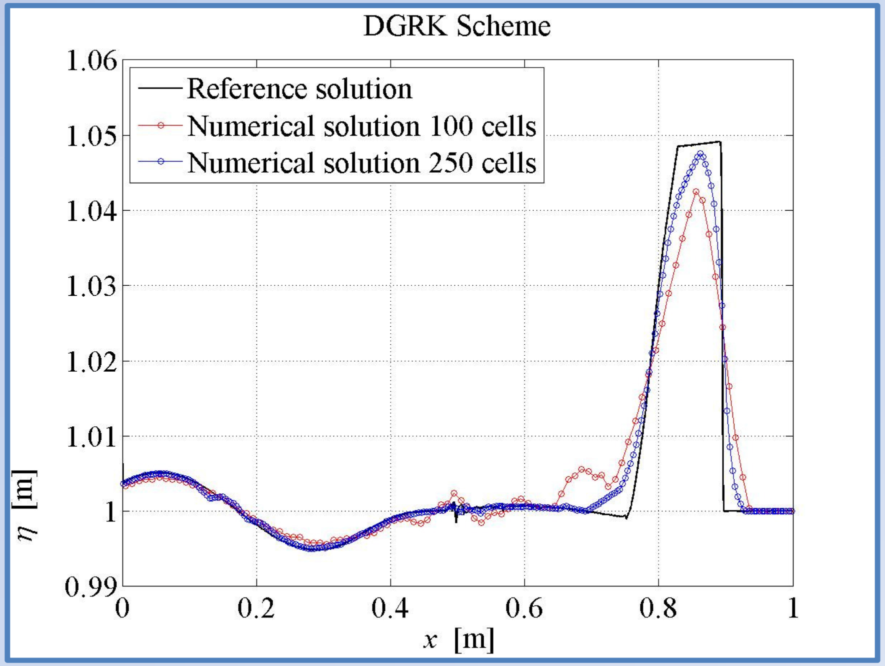




Applications

Pulse over a bump – large amplitude– (10^{-1})





Conclusions on HWENO scheme

This work regards the development of a well-balanced, 4^{th} order accurate, compact HWENO scheme for the integration of the SWE. In particular, several aspects of the scheme, originally proposed in the context of the gas dynamics, are modified and extended to allow the application to SWE.

We have obtained:

- *a very good compactness of the scheme;*
- *the C-property fulfillment;*
- *An high-order accuracy;*
- *a good computational efficiency;*
- *a good stability.*

Conversely, further improvements are necessary to obtain good, problem independent, non-oscillatory properties of the scheme. A new index of smoothness may be introduced?

RKDG

Balancing RKDG methods on domains with curved boundaries

Work in progress…

General conclusion

Any engineering approach cannot do without mathematics and physics,

so …

suggestions are welcome

Acknowledgments

- Many Thanks to the Organizing Committee for this invitation

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Thank you for your attention!