Complex semisimple Lie algebras.

Cours-ALSSC.tex

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Part I

Lie Algebras.

I.1 General definitions: Lie algebras.

In this section, $\mathbb{k}$ is an arbitrary field.

Definition I.1.1 – *A Lie algebra over* $\mathbb{k}$ *is a pair* $(\mathfrak{g}, [-,-])$ *where* $\mathfrak{g}$ *is a* $\mathbb{k}$*-vector space and* $[-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ *a map that satisfy the following conditions:*
1. $[-,-]$ *is bilinear,*
2. $[-,-]$ *is alternate:* $\forall x \in \mathfrak{g}, [x,x] = 0$,
3. for all $x, y, z \in \mathfrak{g}$, $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$.
The dimension of a Lie algebra is the dimension of its underlying $\mathbb{k}$*-vector space. A lie algebra* $(\mathfrak{g}, [-,-])$ *is called commutative or abelian whenever* $[-,-]$ *is identically zero.*

Remark I.1.2 – Let $(\mathfrak{g}, [-,-])$ be a Lie algebra.
1. By conditions 1 and 2 of Definition I.1.1, the map $[-,-]$ is antisymetric: $\forall x, y \in \mathfrak{g}$, $[x,y] = -[y,x]$.
2. If the characteristic of $\mathbb{k}$ is different from 2, condition 2 in Definition I.1.1 is equivalent to the antisymmetry of $[-,-]$ (under condition 1).
3. The identity in the third point of the Definition I.1.1 is called the *Jacobi identity.*

Definition I.1.3 – *Let* $(\mathfrak{g}, [-,-])$ *be a Lie algebra.*
1. A Lie subalgebra $\mathfrak{l}$ *of* $(\mathfrak{g}, [-,-])$ *is a vector subspace* $\mathfrak{l}$ *of* $\mathfrak{g}$ *stable under* $[-,-]$*, that is: for all* $x, y \in \mathfrak{l}$, $[x,y] \in \mathfrak{l}$. *(The pair* $(\mathfrak{l}, [-,-]_{|\mathfrak{l}})$ *is then a Lie algebra in its own right.)*
2. A Lie ideal $\mathfrak{i}$ *of* $(\mathfrak{g}, [-,-])$ *is a subspace* $\mathfrak{i}$ *of* $\mathfrak{g}$ *such that, for all* $(x,y) \in \mathfrak{g} \times \mathfrak{i}$, $[x,y] \in \mathfrak{i}$. *(In particular: a Lie ideal of* $\mathfrak{g}$ *is a Lie subalgebra of* $\mathfrak{g}$*.)*

Exercise I.1.4 – Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$ an ideal of $\mathfrak{g}$.
1. There is a unique map $[-,-] : \mathfrak{g}/\mathfrak{i} \times \mathfrak{g}/\mathfrak{i} \longrightarrow \mathfrak{g}/\mathfrak{i}$ such that, for all $x, y \in \mathfrak{g}$,

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x,y] + \mathfrak{i}.$$

2. The pair $(\mathfrak{g}/\mathfrak{i}, [-,-])$ is a Lie algebra over $\mathbb{k}$, called the *quotient Lie algebra* of $\mathfrak{g}$ by $\mathfrak{i}$.

Exercise I.1.5 – Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$, $\mathfrak{j}$ be Lie ideals of $\mathfrak{g}$. The subspaces

$$\mathfrak{i} + \mathfrak{j} = \{x + y \,|\, x \in \mathfrak{i}, \, y \in \mathfrak{j}\} \qquad \text{and} \qquad [\mathfrak{i}, \mathfrak{j}] = \mathrm{Span}\{[x,y] \,|\, x \in \mathfrak{i}, \, y \in \mathfrak{j}\}$$

are Lie ideals of $\mathfrak{g}$.

Definition I.1.6 – *Let* $\mathfrak{g}$ *be a Lie algebra. The ideal* $[\mathfrak{g}, \mathfrak{g}]$*, denoted* $D(\mathfrak{g})$*, of* $\mathfrak{g}$ *is called the* derived ideal *of* $\mathfrak{g}$*.*

Exercise I.1.7 – **Lie subalgebra generated by a subset.** Let $\mathfrak{g}$ be a Lie algebra.
1. The intersection of any family of Lie subalgebras of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$.
2. Let $\mathcal{X}$ be a subset of $\mathfrak{g}$.
2.1. The intersection of all the Lie subalgebras of $\mathfrak{g}$ containing $\mathcal{X}$ is a Lie subalgebra of $\mathfrak{g}$; it is called *the Lie subalgebra of* $\mathfrak{g}$ *generated by* $\mathcal{X}$.
2.2. The set of all Lie subalgebras of $\mathfrak{g}$ containing $\mathcal{X}$, ordered by inclusion, has a minimum element which is the Lie subalgebra generated by $\mathcal{X}$.

Definition I.1.8 – *Let $\mathfrak{g}$ be a Lie algebra, I be a nonempty set and $\mathcal{X} = (x_i)_{i\in I}$ be a family of elements of $\mathfrak{g}$ indexed by I.*
1. The Lie subalgebra generated by $\mathcal{X}$ is the Lie subalgebra generated by the underlying set of $\mathcal{X}$ (that is by the image of the map $I \longrightarrow \mathfrak{g}$, $i \mapsto x_i$).
2. We say that $\mathcal{X}$ generates $\mathfrak{g}$ if the Lie subalgebra of $\mathfrak{g}$ generated by $\mathcal{X}$ is $\mathfrak{g}$.

Exercise I.1.9 – **Lie ideal generated by a subset.** Let $\mathfrak{g}$ be a Lie algebra.
1. The intersection of any family of Lie ideals of $\mathfrak{g}$ is a Lie ideal of $\mathfrak{g}$.
2. Let $\mathcal{X}$ be a subset of $\mathfrak{g}$.
2.1. The intersection of all the Lie ideals of $\mathfrak{g}$ containing $\mathcal{X}$ is a Lie ideal of $\mathfrak{g}$; it is called *the Lie ideal of $\mathfrak{g}$ generated by $\mathcal{X}$*.
2.2. The set of all Lie ideals of $\mathfrak{g}$ containing $\mathcal{X}$, ordered by inclusion, has a minimum element which is the Lie ideal generated by $\mathcal{X}$.

Definition I.1.10 – *Let $\mathfrak{g}$ be a Lie algebra, I be a nonempty set and $\mathcal{X} = (x_i)_{i\in I}$ be a family of elements of $\mathfrak{g}$ indexed by I.*
1. The Lie ideal generated by $\mathcal{X}$ is the Lie ideal generated by the underlying set of $\mathcal{X}$ (that is by the image of the map $I \longrightarrow \mathfrak{g}$, $i \mapsto x_i$).
2. If $\mathfrak{i}$ is a Lie ideal of $\mathfrak{g}$, we say that $\mathcal{X}$ generates $\mathfrak{i}$ if the Lie ideal of $\mathfrak{g}$ generated by $\mathcal{X}$ is $\mathfrak{i}$.

Definition I.1.11 – *Let $(\mathfrak{g}, [-,-])$ be a Lie algebra. The centre of $\mathfrak{g}$ is the set, denoted $Z(\mathfrak{g})$, defined by $Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0, \forall y \in \mathfrak{g}\}$.*

Exercise I.1.12 – The center of a Lie algebra is an ideal.

Exercise I.1.13 – **Center and decomposition as direct sum** – Let $\mathfrak{g}$ be a Lie algebra, I a nonempty set and, for all $i \in I$, $\mathfrak{g}_i$ a Lie subalgebra of $\mathfrak{g}$. Suppose that $\mathfrak{g} = \oplus_{i\in I}\mathfrak{g}_i$ and for all $i, j \in I$, $i \neq j$, $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$.
1. Let $g = \sum_{i\in I} g_i$ and $h = \sum_{i\in I} h_i$ be two elements of $\mathfrak{g}$ (together with their decomposition with respect to that of $\mathfrak{g}$). If $[g, h] = 0$, then, $\forall i, j \in I$, $[g_i, h_j] = 0$.
2. We have $Z(\mathfrak{g}) = \oplus_{i\in I} (Z(\mathfrak{g}) \cap \mathfrak{g}_i) = \oplus_{i\in I} Z(\mathfrak{g}_i)$.

Exercise I.1.14 – **Centraliser and normaliser** Let $\mathfrak{g}$ be a Lie algebra and X be a subspace of $\mathfrak{g}$. Define the *centraliser*, $C_{\mathfrak{g}}(X)$, and the normaliser, $N_{\mathfrak{g}}(X)$, of X as follows:

$$C_{\mathfrak{g}}(X) = \{y \in \mathfrak{g} | [y, x] = 0, \forall x \in X\} \qquad \text{and} \qquad N_{\mathfrak{g}}(X) = \{y \in \mathfrak{g} | [y, x] \in X, \forall x \in X\}.$$

These two sets are Lie subalgebras of $\mathfrak{g}$.

We start with a list of examples that will be central in the sequel.

Example I.1.15 –
1. Let A be an associative algebra over $\Bbbk$ and consider the map $[-,-] : A \times A \longrightarrow A$, $(x, y) \mapsto xy - yx$. Then $(A, [-,-])$ is a Lie algebra over $\Bbbk$.
2. The general linear Lie algebras.
2.1. Let V be a vector space over $\Bbbk$. The set $\mathrm{End}_{\Bbbk}(V)$ of endomorphisms of V is an associative $\Bbbk$-algebra. Point 1 above then shows that it may be endowed with a Lie algebra structure. To stress that $\mathrm{End}_{\Bbbk}(V)$ is considered as a Lie algebra, we denote it $\mathfrak{gl}(V)$.
2.2. Let $n \in \mathbb{N}^*$. The set $M_n(\Bbbk)$ of $n \times n$ matrices with entries in $\Bbbk$ is an associative $\Bbbk$-algebra. Point 1 above then shows that it may be endowed with a Lie algebra structure. To stress that

$M_n(\mathbb{k})$ is considered as a Lie algebra, we denote it $\mathfrak{gl}_n(\mathbb{k})$.
3. The special linear Lie algebras.
3.1. Let V be a finite dimensional vector space over $\mathbb{k}$. The set of trace zero endomorphisms of V is a Lie subalgebra of $\mathfrak{gl}(V)$. It is denoted by $\mathfrak{sl}(V)$.
3.2. Let $n \in \mathbb{N}^*$. The set of trace zero $n \times n$ matrices with entries in $\mathbb{k}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{k})$. It is denoted by $\mathfrak{sl}_n(\mathbb{k})$.
4. Lie algebras associated to flags.
4.1. Let V be a finite dimensional vector space over $\mathbb{k}$ of dimension $n \in \mathbb{N}^*$ and

$$(0) = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

be a full flag of V, which we denote by $\mathcal{F}$. The subset $\mathfrak{n}_{\mathcal{F}}(V)$ of the endomorphisms x of V such that, for all $1 \leq i \leq n$, $x(V_i) \subseteq V_{i-1}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. It is included in $\mathfrak{sl}(V)$. The subset $\mathfrak{b}_{\mathcal{F}}(V)$ of the endomorphisms x of V such that, for all $1 \leq i \leq n$, $x(V_i) \subseteq V_i$ is a Lie subalgebra of $\mathfrak{gl}(V)$. Clearly: $\mathfrak{n}_{\mathcal{F}}(V) \subseteq \mathfrak{b}_{\mathcal{F}}(V) \subseteq \mathfrak{gl}(V)$.
4.2. Let $n \in \mathbb{N}^*$. The set $\mathfrak{n}_n(\mathbb{k})$ of $n \times n$ strict upper triangular matrices with entries in $\mathbb{k}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{k})$. The set $\mathfrak{b}_n(\mathbb{k})$ of $n \times n$ upper triangular matrices with entries in $\mathbb{k}$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{k})$.

Exercise I.1.16 – Let $s \in \mathbb{N}^*$ and let $(\mathfrak{g}_1, [-,-]_1), \ldots, (\mathfrak{g}_s, [-,-]_s)$ be Lie algebras. Put $\mathfrak{g} = \oplus_{1 \leq i \leq s} \mathfrak{g}_i$. The map

$$[-,-] \;:\; \begin{array}{ccc} \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ ((x_1, \ldots, x_s), (y_1, \ldots, y_s)) & \mapsto & ([x_1, y_1]_1, \ldots, [x_s, y_s]_s) \end{array}.$$

Then $(\mathfrak{g}, [-,-])$ is a Lie algebra called the *direct sum* of the family $((\mathfrak{g}_1, [-,-]_1), \ldots, (\mathfrak{g}_s, [-,-]_s))$ of Lie algebras.

We now define morphisms between Lie algebras.

Definition I.1.17 – *Let $(\mathfrak{g}, [-,-])$ and $(\mathfrak{h}, [-,-])$ be Lie algebras. A morphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{l}$ is a morphism of vector spaces $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ such that, for all $x, y \in \mathfrak{g}$, $[f(x), f(y)] = f([x,y])$. An isomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{l}$ is a bijective morphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{l}$.*

Exercise I.1.18 – Let $(\mathfrak{g}, [-,-])$ and $(\mathfrak{h}, [-,-])$ be Lie algebras, let $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a morphism of Lie algebras. Then the image of f is a Lie subalgebra of $\mathfrak{h}$ and its kernel a Lie ideal of $\mathfrak{g}$.

Exercise I.1.19 – Let $(\mathfrak{g}, [-,-])$ be Lie algebras and $\mathfrak{i}$ a Lie ideal of $\mathfrak{g}$. The canonical projection $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{i}$, $x \mapsto x + \mathfrak{i}$ of vector spaces is a morphism of Lie algebras.

Exercise I.1.20 – **Isomorphism Theorems** –
1. Let $(\mathfrak{g}, [-,-])$ and $(\mathfrak{h}, [-,-])$ be Lie algebras, let $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a morphism of Lie algebras and let $\mathfrak{i}$ be a Lie ideal of $\mathfrak{g}$ included in $\ker(f)$.
1.1. There is a unique Lie algebra morphism $\overline{f} : \mathfrak{g}/\mathfrak{i} \longrightarrow \mathfrak{h}$ such that $\pi \circ \overline{f} = f$, where $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{i}$ is the canonical projection.
1.2. If $\mathfrak{i} = \ker(f)$, then $\overline{f}$ is injective and induces an isomorphism of Lie algebras $\mathfrak{g}/\ker(f) \cong \mathrm{im}(f)$.
2. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{i}$ be a Lie ideal of $\mathfrak{g}$ and denote by $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{i}$ the canonical projection. Then, there is a one-to-one correspondence between Lie ideals of $\mathfrak{g}$ containing $\mathfrak{i}$ and

Lie ideals of $\mathfrak{g}/\mathfrak{i}$, given by direct and inverse image under π. Further, if $\mathfrak{j}$ is an ideal of $\mathfrak{g}$ containing $\mathfrak{i}$, then there is an isomorphism of Lie algebras $(\mathfrak{g}/\mathfrak{i})/\pi(\mathfrak{j}) \cong \mathfrak{g}/\mathfrak{j}$.
3. Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{i}$ and $\mathfrak{j}$ are Lie ideals of $\mathfrak{g}$, then there is an isomorphism of Lie algebras $(\mathfrak{i}+\mathfrak{j})/\mathfrak{j} \cong \mathfrak{i}/\mathfrak{i}\cap\mathfrak{j}$.

Definition I.1.21 – **Linear Lie algebra** – *A Lie algebra $\mathfrak{g}$ is called linear if there exists a vector space V over $\mathbb{k}$ such that $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}(V)$.*

Exercise I.1.22 – Let $(\mathfrak{g}, [-,-])$ and $(\mathfrak{h}, [-,-])$ be Lie algebras, let $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a morphism of $\mathbb{k}$-vector spaces, let $(b_i)_{i\in I}$ be a generating set of the $\mathbb{k}$-vector space $\mathfrak{g}$ (where I is any nonempty set). If, for all $i,j \in I$, $[f(b_i), f(b_j)] = f([b_i, b_j])$, then f is a morphism of Lie algebras.

Example I.1.23 – **Classical Lie algebras** –
0. *The General Lie algebra.* Let $n \in \mathbb{N}^*$. We already defined the Lie algebra associated to the associative algebra $M_n(\mathbb{k})$ of $n\times n$ matrices with entries in $\mathbb{k}$; it is denoted $\mathfrak{gl}_n(\mathbb{k})$. For $1 \le i,j \le n$, denote by E_{ij} (or sometimes $E_{i,j}$) the elementary matrix whose only nonzero entry is located in line i and column j and equals 1. Then, the subset $\{E_{ij},\, 1 \le i,j \le n\}$ is a basis of the $\mathbb{k}$-vector space $\mathfrak{gl}_n(\mathbb{k})$ and the bracket is given by the following formula:

$$\forall\, 1 \le i,j,k,l \le n, \quad [E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}. \tag{I.1.1}$$

1. *The special linear Lie algebra.* Let $n \in \mathbb{N}^*$. Denote by $\mathfrak{sl}_{n+1}(\mathbb{k})$ the subspace of $\mathfrak{gl}_{n+1}(\mathbb{k})$ whose elements are the matrices whose trace is zero. It is clear that $\mathfrak{sl}_{n+1}(\mathbb{k})$ is a Lie subalgebra of $\mathfrak{gl}_{n+1}(\mathbb{k})$. It is not difficult to see that the set

$$\{E_{ij},\, 1 \le i \neq j \le n\} \cup \{H_i,\, 1 \le i \le n\}$$

is a basis of the $\mathbb{k}$-vector space $\mathfrak{sl}_{n+1}(\mathbb{k})$, where, for $1 \le i \le n$, $H_i = E_{ii} - E_{i+1,i+1}$. Hence,

$$\dim_{\mathbb{k}}(\mathfrak{sl}_{n+1}(\mathbb{k})) = n^2 + 2n.$$

2. *The symplectic Lie algebra.* Suppose $\mathbb{k}$ has characteristic different from 2 and let $n \in \mathbb{N}^*$. Consider the $2n \times 2n$ matrix with entries in $\mathbb{k}$ (written in $n\times n$ blocs form):

$$B = \begin{pmatrix} 0 & I_{\mathfrak{n}} \\ -I_n & 0 \end{pmatrix}$$

(here, I_n stands for the identity matrix of $M_n(\mathbb{k})$). We put

$$\mathfrak{sp}_{2n}(\mathbb{k}) = \{A \in \mathfrak{gl}_{2n} \,|\, {}^tAB + BA = 0\}.$$

It is easy to check that $\mathfrak{sp}_{2n}(\mathbb{k})$ is a Lie subalgebra of $\mathfrak{gl}_{2n}(\mathbb{k})$. More precisely, let $A \in \mathfrak{gl}_{2n}(\mathbb{k})$ and write A in $n \times n$ blocs form:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$

Then, A belongs to $\mathfrak{sp}_{2n}(\mathbb{k})$ if and only if $A_4 = -{}^tA_1$, ${}^tA_2 = A_2$ and ${}^tA_3 = A_3$. Here is a list of elements in $\mathfrak{sp}_{2n}(\mathbb{k})$:

$$X_{ij} = E_{i,j} - E_{j+n,i+n}, \quad 1 \le i,j \le n;$$

$$U_i = E_{i,i+n}, \quad 1 \le i \le n;$$

$$Y_{ij} = E_{i,j+n} + E_{j,i+n}, \quad 1 \le i < j \le n;$$

$$V_i = E_{i+n,i}, \quad 1 \le i \le n;$$

$$Z_{ij} = E_{i+n,j} + E_{j+n,i}, \quad 1 \le i < j \le n.$$

It is clear from the above that the set

$$\{X_{ij},\, 1 \le i,j \le n\} \cup \{U_i,\, 1 \le i \le n\} \cup \{Y_{ij},\, 1 \le i < j \le n\} \cup \{V_i,\, 1 \le i \le n\} \cup \{Z_{ij},\, 1 \le i < j \le n\}$$

is a basis of the $\mathbb{k}$-vector space $\mathfrak{sp}_{2n}(\mathbb{k})$, so that

$$\dim_{\mathbb{k}}(\mathfrak{sp}_{2n}(\mathbb{k})) = 2n^2 + n.$$

3. *The orthogonal Lie algebra (odd case).* Suppose $\mathbb{k}$ has characteristic different from 2 and let $n \in \mathbb{N}^*$. Consider the $(2n+1) \times (2n+1)$ matrix with entries in $\mathbb{k}$ (written in 1×1, $1 \times n$, $n \times 1$ and $n \times n$ blocs form):

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}.$$

We put

$$\mathfrak{so}_{2n+1}(\mathbb{k}) = \{A \in \mathfrak{gl}_{2n+1}(\mathbb{k}) \,|\, {}^tAB + BA = 0\}.$$

This is a Lie subalgebra of $\mathfrak{gl}_{2n+1}(\mathbb{k})$. It is not difficult to check that the elements of $\mathfrak{so}_{2n+1}(\mathbb{k})$ are the matrices of the form

$$\begin{pmatrix} 0 & L_1 & L_2 \\ -{}^tL_2 & A_1 & A_2 \\ -{}^tL_1 & A_3 & -{}^tA_1 \end{pmatrix},$$

where $L_1, L_2 \in M_{1,n}(\mathbb{k})$, $A_1 \in M_n(\mathbb{k})$ and A_2, A_3 are antisymmetric matrices of $M_n(\mathbb{k})$. It is then clear that

$$\mathfrak{so}_{2n+1}(\mathbb{k}) \subseteq \mathfrak{sl}_{2n+1}(\mathbb{k}) \qquad \text{and} \qquad \dim_{\mathbb{k}}(\mathfrak{so}_{2n+1}(\mathbb{k})) = 2n^2 + n.$$

4. *The orthogonal Lie algebra (even case).* Suppose $\mathbb{k}$ has characteristic different from 2 and let $n \in \mathbb{N}^*$. Consider the $2n \times 2n$ matrix with entries in $\mathbb{k}$ (written in $n \times n$ blocs form):

$$B = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

We put

$$\mathfrak{so}_{2n}(\mathbb{k}) = \{A \in \mathfrak{gl}_{2n}(\mathbb{k}) \,|\, {}^tAB + BA = 0\}.$$

This is a Lie subalgebra of $\mathfrak{gl}_{2n}(\mathbb{k})$. It is not difficult to check that the elements of $\mathfrak{so}_{2n}(\mathbb{k})$ are the matrices of the form

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & -{}^tA_1 \end{pmatrix},$$

where $A_1 \in M_n(\mathbb{k})$ and A_2, A_3 are antisymmetric matrices of $M_n(\mathbb{k})$. It is then clear that

$$\mathfrak{so}_{2n}(\mathbb{k}) \subseteq \mathfrak{sl}_{2n}(\mathbb{k}) \qquad \text{and} \qquad \dim_{\mathbb{k}}(\mathfrak{so}_{2n}(\mathbb{k})) = 2n^2 - n.$$

Remark I.1.24 – The classical Lie algebras may be introduced in a more intrinsic manner.
1. Suppose $\mathbb{k}$ has characteristic different from 2 and let $n \in \mathbb{N}^*$. Let V be a $2n$ dimensional vector space together with a basis $\{e_1, \dots, e_{2n}\}$ and consider the bilinear form $b : V \times V \longrightarrow \mathbb{k}$

whose matrix in the chosen basis is the matrix B of Example I.1.23, Point 2. Then, clearly, b is skew-symmetric and nondegenerate. We may then consider the subspace

$$\mathfrak{sp}(V, b) = \{f \in \mathfrak{gl}(V) \,|\, , \forall v, w \in V,\, b(f(v), w) + b(v, f(w)) = 0\}$$

of $\mathfrak{gl}(V)$. It is easy to check that $\mathfrak{sp}(V, b)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. More precisely, the map

$$\mathfrak{gl}(V) \quad \mapsto \quad \mathfrak{gl}_{2n}(\Bbbk)$$

that sends an element of $\mathfrak{gl}(V)$ to its matrix relative to the above basis is a Lie algebra isomorphism that sends $\mathfrak{sp}(V, b)$ onto $\mathfrak{sp}_{2n}(\Bbbk)$.
2. Suppose $\Bbbk$ has characteristic different from 2 and let $n \in \mathbb{N}^*$. Let V be a $2n + 1$ dimensional vector space together with a basis $\{e_1, \ldots, e_{2n+1}\}$ and consider the bilinear form $b : V \times V \longrightarrow \Bbbk$ whose matrix in the chosen basis is the matrix B of Example I.1.23, Point 3. Then, clearly, b is symmetric and nondegenerate. We may then consider the subspace

$$\mathfrak{so}(V, b) = \{f \in \mathfrak{gl}(V) \,|\, , \forall v, w \in V,\, b(f(v), w) + b(v, f(w)) = 0\}$$

of $\mathfrak{gl}(V)$. It is easy to check that $\mathfrak{so}(V, b)$ is a Lie subalgebra of $\mathfrak{gl}(V)$, isomorphic to $\mathfrak{so}_{2n+1}(\Bbbk)$ via the map sending an endomorphism of V to its matrix relative to the chosen basis. Clearly, the same holds when V is of dimension $2n$ and b is the bilinear form of V whose matrix is the matrix B of Example I.1.23, Point 4 relative to an arbitrary choice of basis.

I.2 General definitions: representations of a Lie algebra.

In this section, $\Bbbk$ is an arbitrary field.

Definition I.2.1 – **Representation of a Lie algebra** – *Let $\mathfrak{g}$ be a Lie algebra. A representation of $\mathfrak{g}$ is a pair (V, ρ) where V is a vector space over $\Bbbk$ and $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ a morphism of Lie algebras. A finite dimensional representation of $\mathfrak{g}$ is a representation (V, f) with V finite dimensional.*

Example I.2.2 – **The trivial representation** – Let $\mathfrak{g}$ be a Lie algebra and V be a vector space over $\Bbbk$. The map $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ which send any $x \in \mathfrak{g}$ to the zero endomorphism of $\mathfrak{g}$ is a morphism of Lie algebra. This definies the trivial representation of $\mathfrak{g}$ on V.

Definition I.2.3 – *Let $\mathfrak{g}$ be a Lie algebra. A representation (V, ρ) of $\mathfrak{g}$ is said to be faithful if the morphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is injective.*

Definition I.2.4 – **Module over a Lie algebra** – *Let $\mathfrak{g}$ be a Lie algebra. A module over $\mathfrak{g}$ is a pair (V, f) where V is a vector space over $\Bbbk$ and $f : \mathfrak{g} \times V \longrightarrow V$, $(x, v) \mapsto x.v$, a map satisfying the following conditions, for all $\lambda, \mu \in \Bbbk$, for all $x, y \in \mathfrak{g}$ and for all $v, w \in V$:*
(i) $(\lambda x + \mu y).v = \lambda(x.v) + \mu(y.v)$;
(ii) $x.(\lambda v + \mu w) = \lambda(x.v) + \mu(y.w)$;
(iii) $[x, y].v = x.(y.v) - y.(x.v)$.
A finite dimensional module over $\mathfrak{g}$ is a module over $\mathfrak{g}$ whose underlying vector space is finite dimensional.

Exercise I.2.5 – **Modules versus representations** – Let $\mathfrak{g}$ be a Lie algebra.
1. Let (V, f) be a module over $\mathfrak{g}$. Then (V, ρ_f) is a representation of $\mathfrak{g}$, where

$$\begin{array}{rcccl} \rho_f & : & \mathfrak{g} & \longrightarrow & \mathfrak{gl}(V) \\ & & x & \longmapsto & f(x,-) \end{array}$$

and, for all $x \in \mathfrak{g}$, $f(x,-) : v \mapsto x.v$.
2. Let (V, ρ) be a representation of $\mathfrak{g}$. Then (V, f_ρ) is a module over $\mathfrak{g}$, where

$$\begin{array}{rcccl} f_\rho & : & \mathfrak{g} \times V & \longrightarrow & V \\ & & (x,v) & \longmapsto & \rho(x)(v) \end{array}.$$

3. The processes that associate a representation to a module (Point 1 above) and a module to a representation (Point 2 above) are inverse to each other.

Exercise I.2.6 – **Construction of representations** – Let (V, ρ) and (V', ρ') be two representations of the Lie algebra $\mathfrak{g}$.
1. The map

$$\begin{array}{rcccl} \tau & : & \mathfrak{g} & \longrightarrow & \mathfrak{gl}(V \otimes V') \\ & & x & \longmapsto & \rho(x) \otimes \mathrm{id}_{V'} + \mathrm{id}_V \otimes \rho'(x) \end{array}$$

is a morphism of Lie algebras, so that $(V \otimes V', \tau)$ is a representation of $\mathfrak{g}$. This representation is called the tensor product of the representations (V, ρ) and (V', ρ').
2. The map

$$\begin{array}{rcccl} \mu & : & \mathfrak{g} & \longrightarrow & \mathfrak{gl}(\mathrm{Hom}_{\Bbbk}(V, V')) \\ & & x & \longmapsto & \mu(x) \end{array}$$

where, for all $x \in \mathfrak{g}$, $\mu(x) : \mathrm{Hom}_{\Bbbk}(V, V') \longrightarrow \mathrm{Hom}_{\Bbbk}(V, V')$, $\phi \mapsto \rho'(x) \circ \phi - \phi \circ \rho(x)$ is a morphism of Lie algebras, so that $(\mathrm{Hom}_{\Bbbk}(V, V'), \mu)$ is a representation of $\mathfrak{g}$.

This applies in particular to the case where $V' = \Bbbk$ and (ρ', V') is the trivial representation of $\mathfrak{g}$ on V'. Hence, we get a representation of $\mathfrak{g}$ (ρ^*, V^*) defined by

$$\begin{array}{rcccl} \rho^* & : & \mathfrak{g} & \longrightarrow & \mathfrak{gl}(V^*) \\ & & x & \longmapsto & -{}^t\rho(x) \end{array};$$

it is called the dual representation of (ρ, V).
3. The natural isomorphism of vector spaces

$$\begin{array}{ccc} V^* \otimes V' & \longrightarrow & \mathrm{Hom}_{\Bbbk}(V, V') \\ \lambda \otimes v' & \longmapsto & \lambda(-)v' \end{array}$$

is an isomorphism of representations.

The following example of representation of an arbitrary Lie algebra will be fundamental to the theory.

Proposition I.2.7 – *Let $\mathfrak{g}$ be a Lie algebra.*
1. For all $x \in \mathfrak{g}$, the map $\mathrm{ad}_{\mathfrak{g}}(x) : \mathfrak{g} \longrightarrow \mathfrak{g}$, $y \mapsto [x, y]$ is an endomorphism of the vector space $\mathfrak{g}$.
2. The map

$$\begin{array}{rcccl} \mathrm{ad}_{\mathfrak{g}} & : & \mathfrak{g} & \longrightarrow & \mathfrak{gl}(\mathfrak{g}) \\ & & x & \longmapsto & \mathrm{ad}_{\mathfrak{g}}(x) \end{array}$$

is a morphism of Lie algebras, so that $(\mathfrak{g}, \mathrm{ad}_{\mathfrak{g}})$ is a representation of $\mathfrak{g}$.
3. We have, $\ker(\mathrm{ad}_{\mathfrak{g}}) = Z(\mathfrak{g})$.

Proof. Point 1 and 3 are clear. Point 2 follows from the Jacobi identity using the antisymetry. ■

Definition I.2.8 – *Let* $\mathfrak{g}$ *be a Lie algebra. The representation* $(\mathfrak{g}, \mathrm{ad}_{\mathfrak{g}})$ *is called the adjoint representation of* $\mathfrak{g}$.

Exercise I.2.9 – Multibrackets and generating families. Let $\mathfrak{g}$ be a Lie algebra. For all finite sequence $(x_1, \ldots, x_t)$ of elements of $\mathfrak{g}$, $t \in \mathbb{N}^*$, put

$$[x_1] = x_1 \quad \text{if} \quad t = 1 \qquad \text{and} \qquad [x_1, \ldots, x_t] = \mathrm{ad}_{\mathfrak{g}}(x_1) \circ \ldots \circ \mathrm{ad}_{\mathfrak{g}}(x_{t-1})(x_t) \quad \text{if} \quad t \geq 2.$$

The element $[x_1, \ldots, x_t]$ is called the *multibracket* associated to the sequence $(x_1, \ldots, x_t)$.
1. Let $(x_1, \ldots, x_t)$ be a finite sequence of elements of $\mathfrak{g}$, $t \in \mathbb{N}^*$. Then, $\mathrm{ad}_{\mathfrak{g}}([x_1, \ldots, x_t]) = [\mathrm{ad}_{\mathfrak{g}}(x_1), \ldots, \mathrm{ad}_{\mathfrak{g}}(x_t)]$.
2. Let I be a nonempty set and $\mathcal{F} = (x_i)_{i \in I}$ be a family of elements of $\mathfrak{g}$. We denote by $\mathcal{V}_{\mathcal{F}}$ the vector subspace of $\mathfrak{g}$ generated by the elements $[x_{i_1}, \ldots, x_{i_t}]$, where $t \in \mathbb{N}^*$ and $i_1, \ldots, i_t \in I$.
2.1. For all $[x_{i_1}, \ldots, x_{i_t}]$, $t \in \mathbb{N}^*$ and $i_1, \ldots, i_t \in I$, $\mathcal{V}_{\mathcal{F}}$ is stable under $\mathrm{ad}_{\mathfrak{g}}([x_{i_1}, \ldots, x_{i_t}])$. (Hint: induction on t and $\mathrm{ad}_{\mathfrak{g}}$ is a morphism of Lie algebras).
2.2. The vector subspace $\mathcal{V}_{\mathcal{F}}$ is a Lie subalgebra of $\mathfrak{g}$; it is the Lie subalgebra of $\mathfrak{g}$ generated by $\mathcal{F}$. (In particular, if $\mathcal{F}$ generates $\mathfrak{g}$, then $\mathcal{V}_{\mathcal{F}} = \mathfrak{g}$.)
3. Let I be a nonempty set and $\mathcal{F} = (x_i)_{i \in I}$ be a family of elements of $\mathfrak{g}$ which generates $\mathfrak{g}$. Let $\mathfrak{i}$ be a subspace of $\mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{g}}(x_i)(\mathfrak{i}) \subseteq \mathfrak{i}$, for all $i \in I$.
3.1. For all $[x_{i_1}, \ldots, x_{i_t}]$, $t \in \mathbb{N}^*$ and $i_1, \ldots, i_t \in I$, $\mathrm{ad}_{\mathfrak{g}}([x_{i_1}, \ldots, x_{i_t}])(\mathfrak{i}) \subseteq \mathfrak{i}$.
3.2. The subspace $\mathfrak{i}$ is a Lie ideal of $\mathfrak{g}$.
4. Let I be a nonempty set and $\mathcal{F} = (x_i)_{i \in I}$ be a family of elements of $\mathfrak{g}$. We denote by $\mathcal{W}_{\mathcal{F}}$ the vector subspace of $\mathfrak{g}$ generated by the multibrackets whose rightmost term is in the family $\mathcal{F}$, that is elements $[x_1, \ldots, x_t]$, where $t \in \mathbb{N}^*$, $x_1, \ldots, x_{t-1} \in \mathfrak{g}$ and $x_t \in \mathcal{F}$.
4.1. For all $x \in \mathfrak{g}$, $\mathcal{W}_{\mathcal{F}}$ is stable under $\mathrm{ad}_{\mathfrak{g}}(x)$.
4.2. The vector subspace $\mathcal{W}_{\mathcal{F}}$ is a Lie ideal of $\mathfrak{g}$; it is the Lie ideal of $\mathfrak{g}$ generated by $\mathcal{F}$.

Exercise I.2.10 – Generalised eigenspaces – Let $\mathcal{E}, V$ be $\Bbbk$-vector spaces and let $\rho : \mathcal{E} \longrightarrow \mathrm{End}_{\Bbbk}(V)$ be a morphism of vector spaces. For all $\lambda \in \mathcal{E}^*$, put

$$V_\lambda = \{v \in V \mid \forall f \in \mathcal{E},\, \rho(f)(v) = \lambda(f)v\}.$$

Then, $\forall \lambda \in \mathcal{E}^*$, V_λ is a $\Bbbk$-vector subspace of V and further the sum of these subspaces is direct:

$$\sum_{\lambda \in \mathcal{E}^*} V_\lambda = \bigoplus_{\lambda \in \mathcal{E}^*} V_\lambda.$$

This applies in particular when $\mathcal{E}$ is a Lie algebra and (V, ρ) a representation of this Lie algebra.

Definition I.2.11 – *Let* $\mathfrak{g}$ *be a Lie algebra. Let* (V, f) *be a representation of* $\mathfrak{g}$. *A subrepresentation of* V *is a subspace* W *of* V *stable under* $f(x)$ *for all* $x \in \mathfrak{g}$. *(The map* f *then induces a map* $\mathfrak{g} \longrightarrow \mathfrak{gl}(W)$, $x \mapsto f(x)_{|W}$ *which is a representation of* $\mathfrak{g}$.*)*

Exercise I.2.12 – Quotient by a subrepresentation – Let $\mathfrak{g}$ be a Lie algebra, (V, f) be a representation of $\mathfrak{g}$ and W a subrepresentation of (V, f). Let $\pi : V \longrightarrow V/W$ be the canonical projection.
1. For all x in $\mathfrak{g}$, there is a unique element, $\overline{\rho(x)}$, of $\mathrm{End}_{\Bbbk}(V/W)$ such that $\pi \circ \rho(x) = \overline{\rho(x)} \circ \pi$.
2. The map $\overline{\rho} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V/W)$, $x \mapsto \overline{\rho(x)}$ is a morphism of Lie algebras. The representation $(V/W, \overline{\rho})$ is called the quotient representation of (V, ρ) by W.

Definition I.2.13 – *Let $\mathfrak{g}$ be a Lie algebra. Let (V, f) and (W, g) be representations of $\mathfrak{g}$. 1. A morphism of representations from V to W is a morphism of vector spaces $\varphi : V \longrightarrow W$ such that, for all $x \in \mathfrak{g}$, the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "f(x)"] & W \arrow[d, "g(x)"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

The subset of $\mathrm{Hom}_{\Bbbk}(V, W)$ *consisting of morphisms of representations from V to W is denoted* $\mathrm{Hom}_{\mathfrak{g}}(V, W)$*. An isomorphism of representations from V to W is a bijective morphism of representations.*
2. An endomorphism of the representation (V, f) is a morphism of representations from (V, f) to itself. An automorphism of the representation (V, f) is an isomorphism of representations from (V, f) to itself.

Definition I.2.14 – *Let $\mathfrak{g}$ be a Lie algebra. A representation (V, f) of $\mathfrak{g}$ is called irreducible (or simple) if $V \neq \{0\}$ and $\{0\}$ and V are the only subrepresentations of V.*

Definition I.2.15 – *Let $\mathfrak{g}$ be a Lie algebra. A representation (V, f) of $\mathfrak{g}$ is called completely reducible (or semisimple) if there exists a set I and a family of simple subrepresentations $(V_i)_{i \in I}$ of V such that $V = \bigoplus_{i \in Y} V_i$.*

Remark I.2.16 – Let $\mathfrak{g}$ be a Lie algebra. According to Definition I.2.15, the trivial representation is completely reducible, as it is the direct sum of a family of simple subrepresentations indexed by the empty set.

Definition I.2.17 – *Let $\mathfrak{g}$ be a Lie algebra and (U, f) a representation of $\mathfrak{g}$. We say that (U, f) has the direct summand property if, for every subrepresentation V of (U, f), there exists a subrepresentation W of (U, f) such that $U = V \oplus W$.*

Remark I.2.18 – Let $\mathfrak{g}$ be a Lie algebra and (U, f) a representation of $\mathfrak{g}$. If (U, f) has the direct summand property, then the same holds for every subrepresentation V of (U, f). Indeed, given a subrepresentation V of U and a subrepresentation W of V. If X is a subrepresentation of U such that $U = W \oplus X$, then $V = W \oplus (X \cap V)$.

Theorem I.2.19 – *Let $\mathfrak{g}$ be a Lie algebra and (U, f) a representation of $\mathfrak{g}$. The following statements are equivalent:*
(i) (U, f) is completely reducible;
(ii) (U, f) has the direct summand property.

Proof. Suppose U satisfies (i). (If U is trivial, then it clearly satisfies (ii). Hence we may assume $U \neq (0)$.) By hypothesis, there exists a nonempty set I and a family $(S_i)_{i \in I}$ of irreducible subrepresentations of U such that

$$U = \bigoplus_{i \in I} S_i.$$

For any subset J of I, put $S_J = \bigoplus_{i \in J} S_i$. Let now V be a subrepresentation of U. We consider the set

$$\mathcal{E} = \{J,\, J \subseteq I \,|\, S_J \cap V = (0)\},$$

ordered by inclusion. Clearly, $\mathcal{E}$ is not empty since it contains the empty set. We want to prove that the ordered set $(\mathcal{E}, \subseteq)$ is inductively ordered. For this, consider a totally ordered subset $\mathcal{F}$ of $\mathcal{E}$ and put $K = \cup_{J\in\mathcal{F}} J$. It is not difficult to check that, $\mathcal{F}$ being totally ordered, we have that $S_K = \cup_{J\in\mathcal{F}} S_J$, from which we get that $K \in \mathcal{E}$. Hence, indeed, $(\mathcal{E}, \subseteq)$ is inductively ordered. By Zorn's Lemma, it follows that $\mathcal{E}$ has a maximal element. Let M be such an element. We claim that

$$U = V \bigoplus S_M.$$

To show this equality, it is enough to show that

$$\forall\, i \in I, \quad S_i \subseteq V \bigoplus S_M. \tag{I.2.2}$$

Of course, (I.2.2) holds for $i \in M$. Let now $i \in I \setminus M$. By the maximality of M, we have that

$$(S_i \oplus S_M) \cap V = S_{M\cup\{i\}} \cap V \neq (0).$$

Thus, there exists $v \in V$, $s_i \in S_i$ and $s_M \in S_M$ such that $0 \neq v = s_i + s_M$. And, s_i must be nonzero for, otherwise we would have that s_M is a nonzero element in $S_M \cap V$. Therefore, $0 \neq s_i = v - s_M \in S_i \cap (V \oplus S_M)$, which shows that $S_i \cap (V \oplus S_M) \neq (0)$. But, S_i being irreducible, this leads to $S_i \cap (V \oplus S_M) = S_i$, that is $S_i \subseteq V \oplus S_M$, as requierred. We have shown that U satisfies (ii).

Conversally, suppose that U satisfies (ii). (If U is trivial, then it is completely reducible by definition, so we suppose $U \neq (0)$ in the sequel.)

We first show that any nonzero subrepresentation of U contains an irreducible subrepresentation. Let V be a nonzero subrepresentation of U. Take $0 \neq v \in V$. Applying Zorn's Lemma to the set of subrepresentations of V which do not contain v, we get that there exists a maximal such subrepresentation, say Z. By Remark I.2.18, there exists a subrepresentation Y of V such that $V = Z \oplus Y$. Clearly, $Y \neq (0)$. Further, if Y' is a nonzero subrepresentation of Y such that $Y' \subset Y$, by Remark I.2.18 again, there is a subrepresentation Y'' of Y such that $Y = Y' \oplus Y''$, and $(0) \subset Y'' \subset Y$. But, since $V = Z \oplus Y' \oplus Y''$, we have $(Z \oplus Y') \cap (Z \oplus Y'') = Z$. Therefore v cannot be in both $Z \oplus Y'$ and $Z \oplus Y''$. But, this contradicts the maximality property of Z. Hence, such a subrepresentation Y' of Y does not exist. This shows that Y is irreducible.

By the above, the set of irreducible subrepresentations of U is not empty. Therefore, there exists a non empty set I and for all $i \in I$ an irreducible subrepresentation S_i of U such that $\{S_i,\, i \in I\}$ is the set of the irreducible subrepresentations of U. Let now $\mathcal{S}$ be the set of those subsets J of I such that the sum of the S_j, $j \in J$, is direct. Clearly, $\mathcal{S}$ is not empty and we order it by inclusion. It is easy to see that $\mathcal{S}$ is actually inductive. Therefore, by Zorn's Lemma, there is a subset J of I such that J is maximal as an element of $\mathcal{S}$. Consider then

$$V = \oplus_{j\in J} S_j.$$

By the hypothesis on U, there exists a subrepresentation W of U such that $U = V \oplus W$. Now, if W where nonzero, by the above, it would contain an irreducible subrepresentation S and this would contradict the maximality of J. Therefore, $U = \oplus_{j\in J} S_j$. ∎

Corollary I.2.20 – *Let $\mathfrak{g}$ be a Lie algebra. Every subrepresentation and every quotient representation of a completely reducible representation is completely reducible.*

Proof. The case of subrepresentations follows at once from Theorem I.2.19 and Remark I.2.18. Now, if U is a completely reducible representation of $\mathfrak{g}$ and V is a subrepresentation of U, by

Theorem I.2.19, there exists a subrepresentation W of U such that $U = V \oplus W$. Clearly then, U/V and W are isomorphic representations. By the above, W is completely reducible, therefore, so is U/V. ■

Lemma I.2.21 – **(Schur's Lemma)** – *Let $\mathfrak{g}$ be a Lie algebra. Let (V, f) and (W, g) be irreducible representations of $\mathfrak{g}$.*
1. If V and W are not isomorphic representations of $\mathfrak{g}$, then $\mathrm{Hom}_{\mathfrak{g}}(V, W) = 0$.
2. The $\Bbbk$-algebra $\mathrm{End}_{\mathfrak{g}}(V)$ is a division ring.
3. If $\Bbbk$ is algebraically closed and V finite dimensional, then $\mathrm{End}_{\mathfrak{g}}(V) = \Bbbk.\mathrm{id}_V$.

Proof. Let $\varphi : V \longrightarrow W$ be a nonzero morphism of representations. Since the kernel and image of φ are subrepresentations, φ must be surjective and injective, hence an isomorphism. This shows the two first points of the statement.

Suppose in addition that $\Bbbk$ is algebraically closed and V finite dimensional. Let $\varphi \in \mathrm{End}_{\mathfrak{g}}(V)$. Then φ must have an eigenvalue. Let λ be such an eigenvalue. Then, $\ker(\varphi - \lambda \mathrm{id}_V)$ is a subrepresentation of V; as it is nonzero, it must equal V. So $\varphi = \lambda \mathrm{id}_V$. ■

I.3 Lie algebras of derivations.

In this section, $\Bbbk$ is an arbitrary field and, by a $\Bbbk$-algebra, we mean a $\Bbbk$-vector space $\mathcal{A}$, equipped with a bilinear map

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} & \longrightarrow & \mathcal{A} \\ (a, b) & \mapsto & ab \end{array}.$$

This notion then includes both Lie algebras and associative algebras (with or without a unit).

Beware: unless otherwise specified, outside the present section, $\Bbbk$-algebra means *associative unital $\Bbbk$-algebra*. The reason here to introduce this more general notion is that it allows to deal with derivations of Lie algebras and associative algebras all together.

Remark I.3.1 – Let $\mathcal{A}$ be a $\Bbbk$-algebra. The map

$$\begin{array}{cccc} [-,-] & : & \mathcal{A} \times \mathcal{A} & \longrightarrow & \mathcal{A} \\ & & (a, b) & \mapsto & ab - ba \end{array}$$

is bilinear and alternate. However, it need not satisfy the Jacobi identity.

Definition I.3.2 – *Let $\mathcal{A}$ be a $\Bbbk$-algebra. A derivation of $\mathcal{A}$ is an element d of $\mathrm{End}_{\Bbbk}(\mathcal{A})$ satisfying the Leibniz rule, namely:*

$$\forall (a, b) \in \mathcal{A} \times \mathcal{A}, \quad d(ab) = a\, d(b) + d(a) b.$$

The set of all derivations of $\mathcal{A}$ will be denoted $\mathrm{Der}_{\Bbbk}(\mathcal{A})$.

Exercise I.3.3 – **Generalised Leibniz formula** – Let $\mathcal{A}$ be a $\Bbbk$-algebra and $d \in \mathrm{Der}_{\Bbbk}(\mathcal{A})$. Then, for all $n \in \mathbb{N}$, and for all $x, y \in \mathcal{A}$,

$$d^n(xy) = \sum_{0 \le i \le n} \binom{n}{i} d^i(x) d^{n-i}(y).$$

(Here, for all $0 \le i \le n$, $\binom{n}{i}$ is defined as the number of subsets with i elements in the set $\{1, \ldots, n\}$.)

Exercise I.3.4 – **Locally nilpotent derivations** – Assume $\mathbb{k}$ has characteristic 0.
1. Let V be a $\mathbb{k}$-vector space. An endomorphism $f : V \longrightarrow V$ is called *locally nilpotent* if it satisfies the following condition: for all $v \in V$, there exists $k \in \mathbb{N}$ such that $f^k(v) = 0$. Let $f : V \longrightarrow V$ be a locally nilpotent endomorphism of f. We define its exponential as the following map:

$$\begin{array}{rccc} \exp(f) & : & V & \longrightarrow & V \\ & & v & \mapsto & \sum_{k\geq 0} \frac{1}{k!} f^k(v) \end{array}.$$

1.1. If $f \in \mathrm{End}_{\mathbb{k}}(V)$ is locally nilpotent, then $\exp(f) \in \mathrm{End}_{\mathbb{k}}(V)$.
1.2. If $f, g \in \mathrm{End}_{\mathbb{k}}(V)$ are locally nilpotent and commute, then so is $f + g$.
1.3. If $f, g \in \mathrm{End}_{\mathbb{k}}(V)$ are locally nilpotent and commute, then $\exp(f + g) = \exp(f) \circ \exp(g)$.
1.4. If $f \in \mathrm{End}_{\mathbb{k}}(V)$ is locally nilpotent, then $\exp(f) \in \mathrm{Aut}_{\mathbb{k}}(V)$.
2. Let $\mathcal{A}$ be a $\mathbb{k}$-algebra and $d \in \mathrm{Der}_{\mathbb{k}}(\mathcal{A})$. Assume that d is a locally nilpotent endomorphism of $\mathcal{A}$. Then, $\exp(d)$ is an automorphism of the algebra $\mathcal{A}$; that is, $\exp(d)$ is an automorphism of the $\mathbb{k}$-vector space $\mathcal{A}$ such that, for all $x, y \in \mathcal{A}$, $\exp(d)(xy) = \exp(d)(x)\exp(d)(y)$.

Exercise I.3.5 – Let $\mathcal{A}$ be a $\mathbb{k}$-algebra. Then, $\mathrm{Der}_{\mathbb{k}}(\mathcal{A})$ is a Lie subalgebra of the Lie algebra $\mathfrak{gl}(\mathcal{A})$. (Since $\mathcal{A}$ is a vector space, $\mathrm{End}_{\mathbb{k}}(\mathcal{A})$ is of course an associative algebra and it is thus a Lie algebra by means of the commutator.)

Remark I.3.6 – Let $\mathfrak{g}$ be a Lie algebra. Recall the $\mathbb{k}$-linear map $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$.
1. The Jacobi identity exactly says that, for all $x \in \mathfrak{g}$, $\mathrm{ad}_{\mathfrak{g}}(x)$ is a derivation of $\mathfrak{g}$, that is,

$$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathrm{Der}_{\mathbb{k}}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g}).$$

2. Exercise I.3.5 shows that $\mathrm{Der}_{\mathbb{k}}(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$; by the above, $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is thus a Lie subalgebra of $\mathrm{Der}_{\mathbb{k}}(\mathfrak{g})$. But, actually, an easy calculation shows that,

$$\forall d \in \mathrm{Der}_{\mathbb{k}}(\mathfrak{g}),\ \forall x \in \mathfrak{g},\ [d, \mathrm{ad}_{\mathfrak{g}}(x)] = \mathrm{ad}_{\mathfrak{g}}(d(x)),$$

so that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is actually an ideal of $\mathrm{Der}_{\mathbb{k}}(\mathfrak{g})$.

Definition I.3.7 – *Let $\mathfrak{g}$ be a Lie algebra. A derivation of $\mathfrak{g}$ is called* inner *if it belongs to* $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ *and* outer *otherwise.*

We finish this section by the following result which shows that, provided $\mathbb{k}$ is algebraically closed, the semisimple and nilpotent part of a derivation are again derivations. It will be at the heart of the proof of the existence of an abstract Jordan-Chevalley decomposition.

Proposition I.3.8 – *Assume $\mathbb{k}$ is algebraically closed. Suppose $\mathcal{A}$ is a finite dimensional $\mathbb{k}$-algebra. If d is a derivation of $\mathcal{A}$ and if s and n are endomorphisms of $\mathcal{A}$ with s diagonalisable, n nilpotent, $s \circ n = n \circ s$ and $d = s + n$ (that is, $d = s + n$ is the Jordan-Chevalley decomposition of d as an endomorphism of $\mathcal{A}$), then s and n are derivations of $\mathcal{A}$.*

Proof. Let $\mathrm{Spec}(d)$ be the set of eigenvalues of d. For all $\lambda \in \mathrm{Spec}(d)$, let $\mathcal{A}_\lambda$ be the caracteristic subspace of d associated to λ. Then, as is well known,

$$\forall \lambda \in \mathrm{Spec}(d),\ \mathcal{A}_\lambda = \left\{ x \in \mathcal{A} \,|\, \exists k \in \mathbb{N}, (d - \lambda \mathrm{id}_{\mathcal{A}})^k(x) = 0 \right\} \qquad \text{and} \qquad \mathcal{A} = \bigoplus_{\lambda \in \mathrm{Spec}(f)} \mathcal{A}_\lambda.$$

In addition, for all $\lambda \in \mathrm{Spec}(d)$, s, n leave $\mathcal{A}_\lambda$ stable and, actually, $\mathcal{A}_\lambda = \ker(s - \lambda \mathrm{id}_{\mathcal{A}})$.

Now, let $\lambda, \mu \in \Bbbk$, then the following identity holds, as an easy induction on n shows:

$$\forall x, y \in \mathcal{A}, \forall n \in \mathbb{N}, \quad (d-(\lambda+\mu)\mathrm{id}_{\mathcal{A}})^n\,(xy) = \sum_{0\le i\le n} \binom{n}{i} (d-\lambda\mathrm{id}_{\mathcal{A}})^{n-i}(x)(d-\mu\mathrm{id}_{\mathcal{A}})^{i}(y).$$

And it follows at once that

$$\forall \lambda, \mu \in \mathrm{Spec}(d), \quad \mathcal{A}_\lambda \mathcal{A}_\mu \subseteq \mathcal{A}_{\lambda+\mu}.$$

As a consequence, for all $\lambda, \mu \in \mathrm{Spec}(d)$, for all $x \in \mathcal{A}_\lambda$ and for all $y \in \mathcal{A}_\mu$,

$$s(xy) = (\lambda+\mu)xy = (\lambda x)y + x(\mu y) = s(x)y + xs(y).$$

And, as $\mathcal{A} = \bigoplus_{\lambda\in\mathrm{Spec}(f)} \mathcal{A}_\lambda$ and the multiplication on $\mathcal{A}$ is bilinear, the above formula extends to any $(x, y) \in \mathcal{A} \times \mathcal{A}$.

We have shown that s is a derivation and, since $n = d - s$, so is n. ∎

I.4 Nilpotent Lie algebras.

In this section, $\Bbbk$ is an arbitrary field.

Definition I.4.1 – *Let $\mathfrak{g}$ be a Lie algebra. The* descending (or lower) central series *of $\mathfrak{g}$, $(C_i(\mathfrak{g}))_{i\in\mathbb{N}}$, is the sequence of ideals of $\mathfrak{g}$ defined recursively by: $C_0(\mathfrak{g}) = \mathfrak{g}$ and, for $i \in \mathbb{N}$, $C_{i+1}(\mathfrak{g}) = [\mathfrak{g}, C_i(\mathfrak{g})]$. Then, $\mathfrak{g}$ is called nilpotent if there exists $n \in \mathbb{N}$ such that $C_n(\mathfrak{g}) = (0)$.*

Exercise I.4.2 –
1. Any abelian Lie algebra is nilpotent.
2. Any subalgebra or homomorphic image of a nilpotent Lie algebra is nilpotent.
3. Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then so is $\mathfrak{g}$.

Exercise I.4.3 – Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$ be a Lie ideal of $\mathfrak{g}$. Then all the terms in the descending central series of the Lie algebra $\mathfrak{i}$ are Lie ideals of $\mathfrak{g}$.

Example I.4.4 – We use the notation of example I.1.15.
1. If V is a finite dimensional vector space and $\mathcal{F}$ is a full flag of V, then $\mathfrak{n}_{\mathcal{F}}(V)$ is nilpotent.
2. For all $n \in \mathbb{N}^*$, $\mathfrak{n}_n(\Bbbk)$ is nilpotent.

The following exercise is easy. It connects nilpotency in the sense of Lie algebras and the nilpotency of endomorphisms. The necessary condition it states will turn out to be sufficient (this is the content of Engel's Theorem below).

Definition I.4.5 – *Let $\mathfrak{g}$ be any finite dimensional Lie algebra. An element $x \in \mathfrak{g}$ is said to be* ad*-nilpotent if the endomorphism* $\mathrm{ad}(x) : \mathfrak{g} \longrightarrow \mathfrak{g}$ *is nilpotent.*

Exercise I.4.6 – Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{g}$ is nilpotent, then any elements of $\mathfrak{g}$ is ad-nilpotent.

Theorem I.4.8 below will be of central importance in the sequel. It shows that the Lie algebras $\mathfrak{n}_{\mathcal{F}}(V)$ (cf. Example I.1.15) are prototypes of nilpotent Lie algebras. Its proof needs some preparatory statements.

Lemma I.4.7 – *Let V be a vector space over $\Bbbk$. If $x \in \mathfrak{gl}(V)$ is a nilpotent endomorphism, then $\mathrm{ad}_{\mathfrak{gl}(V)}(x) \in \mathfrak{gl}(\mathfrak{gl}(V))$ is a nilpotent endomorphism.*

Proof. To any $x \in \mathfrak{gl}(V)$, we may associate the two endomorphisms of $\mathfrak{gl}(V)$ given by left and right composition with x:

$$\begin{array}{cccc} \lambda_x : & \mathfrak{gl}(V) & \longrightarrow & \mathfrak{gl}(V) \\ & y & \mapsto & x \circ y \end{array} \quad \text{and} \quad \begin{array}{cccc} \rho_x : & \mathfrak{gl}(V) & \longrightarrow & \mathfrak{gl}(V) \\ & y & \mapsto & y \circ x \end{array}$$

Of course, these two endomorphisms commute. Clearly, if x is nilpotent, the elements λ_x and ρ_x of the algebra $\mathfrak{gl}(\mathfrak{gl}(V))$ are also nilpotent (with the same nilpotency index as x). But then, by standard arguments, $\mathrm{ad}_{\mathfrak{gl}(V)}(x) = \lambda_x - \rho_x$ is nilpotent (of index bounded above by twice that of x). ∎

Theorem I.4.8 – **(Preparatory to Engel's Theorem.)** *Let V be a nonzero finite dimensional vector space and $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ consisting of nilpotent endomorphisms.*
1. There exists a nonzero vector in V which is in the kernel of all the endomorphisms lying in $\mathfrak{g}$.
2. There exists a full flag $\mathcal{F}$ of V such that $\mathfrak{g} \subseteq \mathfrak{n}_{\mathcal{F}}(V)$.

Proof. Notice that the hypotheses imply that $\mathfrak{g}$ is finite dimensional.
1. We proceed by induction on the dimension of $\mathfrak{g}$. The result is obvious when $\dim_{\Bbbk}(\mathfrak{g}) = 0$.
Suppose now that $\dim_{\Bbbk}(\mathfrak{g}) > 0$.
Let $\mathfrak{h}$ be any Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}$. Consider the adjoint action of $\mathfrak{gl}(V)$ on itself: $\mathrm{ad} : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(\mathfrak{gl}(V))$. It induces an action of $\mathfrak{g}$ on $\mathfrak{g}$ and further an action of $\mathfrak{h}$ on $\mathfrak{g}$:

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathfrak{gl}(\mathfrak{g}) \\ y & \mapsto & (x \mapsto [y,x]) \end{array}$$

which stabilises $\mathfrak{h}$. We get that way an action as follows

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}) \\ y & \mapsto & (x + \mathfrak{h} \mapsto [y,x] + \mathfrak{h}) \end{array}. \tag{I.4.3}$$

Now, by Lemma I.4.7, for all $x \in \mathfrak{h}$, $\mathrm{ad}(x)$ is a nilpotent endomorphism of the vector space $\mathfrak{gl}(V)$. Hence the image of the map (I.4.3) consists in nilpotent endomorphisms of the vector space $\mathfrak{g}/\mathfrak{h}$. Therefore, we are in position to apply the induction hypothesis to this image:

$$\exists x \in \mathfrak{g} \setminus \mathfrak{h} \text{ such that, } \forall\, y \in \mathfrak{h},\ [y,x] \in \mathfrak{h}.$$

In other words, $\mathfrak{h}$ is properly included in its normaliser in $\mathfrak{g}$: $\mathfrak{h} \subset N_{\mathfrak{g}}(\mathfrak{h})$.
We may choose $\mathfrak{h}$ to be maximal among proper subalgebras of $\mathfrak{g}$. The above then shows that $\mathfrak{g}$ is the normaliser of $\mathfrak{h}$ in $\mathfrak{g}$. That is, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. Now, the assumption that $\mathfrak{h}$ is maximal among proper subalgebras of $\mathfrak{g}$ forces it to be of codimension 1. Indeed, otherwise, the inverse image of a one dimensional Lie subalgebra of $\mathfrak{g}/\mathfrak{h}$ under the canonical projection would be a Lie subalgebra strictly included between $\mathfrak{h}$ and $\mathfrak{g}$. As a consequence, we get that

$$\forall z \in \mathfrak{g} \setminus \mathfrak{h},\ \mathfrak{g} = \mathfrak{h} \oplus \Bbbk z.$$

By the induction hypothesis, the vector space

$$W = \{v \in V | \forall h \in \mathfrak{h},\ h(v) = 0\}$$

is nonzero and, as $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, this subspace is stable under any element of $\mathfrak{g}$. But, any z as above is a nilpotent endomorphism of V, hence induces a nilpotent element of W. This forces z to have a non zero element w of W in its kernel. Such a w is now an element in the kernel of any element of $\mathfrak{g}$. The proof of point 1 is now complete.
2. To prove point 2, we use induction on the dimension of V. The result is true by Point 1 in case V is one dimensional. Take now any finite dimensional vector space V of dimension $n \geq 2$. By point 1, there exists a nonzero element $v \in V$ that is in the kernel of any element of $\mathfrak{g}$. Clearly, the line $\Bbbk v$ is a subrepresentation of V for the obvious action of $\mathfrak{g}$ and we get a morphism of Lie algebras

$$r : \mathfrak{g} \longrightarrow \mathfrak{gl}(V/\Bbbk v).$$

By the induction hypothesis, There is a full flag $(0) = W_0 \subset \ldots \subset W_{n-1} = V/\Bbbk v$ of $V/\Bbbk v$ such that, $\forall x \in \mathfrak{g}$, $r(x)(W_i) \subseteq W_{i-1}$, $1 \leq i \leq n-1$. Let now V_{i+1}, $0 \leq i \leq n-1$, be the inverse image of W_i under the canonical projection $V \longrightarrow V/\Bbbk v$ and put $V_0 = (0)$. It is then clear that, for all $x \in \mathfrak{g}$, and for all $1 \leq i \leq n$, $x(V_i) \subseteq V_{i-1}$. Therefore, the full flag $V_0 \subset \ldots \subset V_n$ of V has the required property and the proof is complete. ∎

Here is a first consequence of Theorem I.4.8.

Exercise I.4.9 – Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra.
1. If $\mathfrak{i}$ is a nonzero ideal of $\mathfrak{g}$, then $\mathfrak{i}$ intersect $Z(\mathfrak{g})$ nontrivialy. (Hint: Consider the representation $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{i})$ and apply Theorem I.4.8 to its image.)
2. If $\mathfrak{g}$ is nonzero, then $Z(\mathfrak{g})$ is nonzero.

We are now in position to establish Engel's Theorem, which charactrises the nilpotency of a Lie algebra by means of its image under the adjoint representation (see Exercise I.4.6).

Theorem I.4.10 – **(Engel's Theorem.)** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if every $x \in \mathfrak{g}$ is* ad*-nilpotent.*

Proof. Exercice I.4.6 proves the necessity. Conversaly, consider the adjoint representation and suppose that its image consist in nilpotent endomorphisms. Then, by Theorem I.4.8, the image of $\mathfrak{g}$ under the adjoint representation is a nilpotent Lie algebra. But the latter is isomorphic to $\mathfrak{g}/Z(\mathfrak{g})$. Hence, $\mathfrak{g}$ is nilpotent (see Exercise I.5.3). ∎

Remark I.4.11 – At this stage, a comment may be useful which underlines a certain lack of symetry between Theorem I.4.8 and Theorem I.5.7 below.

Let V be a finite dimensional nonzero vector space and $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. By Theorem I.4.8, the nilpotency of the endomorphisms contained in $\mathfrak{g}$ forces the nilpotency of $\mathfrak{g}$. But, the converse is trivialy false. Indeed, for all $n \in \mathbb{N}^*$, the set of diagonal matrices in $\mathfrak{gl}_n(\mathbb{C})$ is an abelian, hence nilpotent, Lie algebra. However, none of its elements is nilpotent except zero.

I.5 Solvable Lie algebras.

In this section, unless otherwise specified, $\Bbbk$ is an arbitrary field.

Definition I.5.1 – *Let $\mathfrak{g}$ be a Lie algebra. Define inductively the decreasing sequence $(D_i(\mathfrak{g}))_{i\in\mathbb{N}}$ of ideals of $\mathfrak{g}$, called the* derived series *of $\mathfrak{g}$ by: $D_0(\mathfrak{g}) = \mathfrak{g}$ and, for $i \in \mathbb{N}$, $D_{i+1}(\mathfrak{g}) = [D_i(\mathfrak{g}), D_i(\mathfrak{g})]$. Then $\mathfrak{g}$ is called solvable if there exists $n \in \mathbb{N}$ such that $D_n(\mathfrak{g}) = (0)$.*

Example I.5.2 – Let $\mathfrak{g}$ be a Lie algebra. It is clear that, for all $i \in \mathbb{N}$, $C_i(\mathfrak{g}) \supseteq D_i(\mathfrak{g})$. Hence, any nilpotent Lie algebra is solvable.

Exercise I.5.3 –
1. Any Lie subalgebra or homorphic image of a solvable Lie algebra is solvable.
2. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$ a Lie ideal of $\mathfrak{g}$. If $\mathfrak{i}$ and $\mathfrak{g}/\mathfrak{i}$ are solvable, then so is $\mathfrak{g}$.
3. Let $\mathfrak{g}$ be a Lie algebra. The sum of two solvable ideals is a solvable ideal.
4. Let $\mathfrak{g}$ be a Lie algebra. If the derived ideal of $\mathfrak{g}$ is solvable, then so is $\mathfrak{g}$.

Exercise I.5.4 – Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{i}$ be a Lie ideal of $\mathfrak{g}$. Then all the terms in the derived series of the Lie algebra $\mathfrak{i}$ are Lie ideals of $\mathfrak{g}$.

Example I.5.5 – (For the notation, see Example I.1.15.)
1. For any finite dimensional vector space V and any full flag $\mathcal{F}$ of V, $\mathfrak{b}_{\mathcal{F}}(V)$ is solvable.
2. For all $n \in \mathbb{N}^*$, $\mathfrak{b}_n(\mathbb{k})$ is solvable.

The following Theorems I.5.6 and I.5.7 will be of central importance in the sequel. They show that the Lie algebras $\mathfrak{b}_{\mathcal{F}}(V)$ (cf. Example I.1.15) are prototypes of solvable Lie algebras.

Theorem I.5.6 – **(Preparatory to Lie's Theorem.)** *Assume that $\mathbb{k}$ is algebraically closed and of characteristic 0. Let V be a nonzero vector space of finite dimension. If $\mathfrak{g}$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$, there exists a nonzero common eigenvector for all the elements of $\mathfrak{g}$.*

Proof. We proceed by induction on the dimension of $\mathfrak{g}$. The case where $\dim_{\mathbb{k}}(\mathfrak{g}) = 0$ is trivial. The case where $\dim_{\mathbb{k}}(\mathfrak{g}) = 1$ is easy. Indeed, any nonzero element x of $\mathfrak{g}$ must have a nonzero eigenvector by the hypothesis that $\mathbb{k}$ is algebraicaly close and, since $\mathfrak{g} = \mathbb{k}x$, it is an eigenvector for all the elements of $\mathfrak{g}$.

Fix $n \geq 1$ and suppose now that the result is true whenever $\dim_{\mathbb{k}}(\mathfrak{g}) \leq n$.

Suppose now that $\dim_{\mathbb{k}}(\mathfrak{g}) = n+1$. Since $\mathfrak{g}$ is solvable, we have that $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$ (strict inclusion). Thus, the Lie algebra $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian and nonzero, so that it contains a Lie ideal of codimension 1. Now, the inverse image of such an ideal under the canonical projection is a codimension 1 ideal of $\mathfrak{g}$. Let $\mathfrak{i}$ be such a codimension one ideal of $\mathfrak{g}$.

By the induction hypothesis, there exists a nonzero element $v \in V$ which is a common eigenvector for all the elements of $\mathfrak{i}$. Hence, there exists a linear form λ on $\mathfrak{i}$ such that, for all $x \in \mathfrak{i}$, $x(v) = \lambda(x)v$. Put now

$$(0) \subset W = \{w \in V | x(w) = \lambda(x)w,\ \forall x \in \mathfrak{i}\} \subseteq V.$$

We now proceed to show that all the elements of $\mathfrak{g}$ leave W invariant. An easy calculation shows that this is equivalent to showing that, for all $x \in \mathfrak{g}$ and $y \in \mathfrak{i}$, $\lambda([x,y]) = 0$.

To this aim, fix $x \in \mathfrak{g}$ and $w \in W \setminus \{0\}$. Put $W_0 = (0)$ and, for all $i \in \mathbb{N}^*$, put

$$W_i = \mathrm{Span}\{w, x(w), \dots, x^{i-1}(w)\}.$$

Then $(W_i)_{i \in \mathbb{N}}$ is an increasing sequence of subspaces of V. Let then $d \in \mathbb{N}^*$ denote the least integer such that

$$W_0 \subset W_1 \subset \dots \subset W_d = W_{d+1} = W_{d+2} = \dots.$$

It is clear by definition that, for all $i \in \mathbb{N}$, $x(W_i) \subseteq W_{i+1}$.

Consider now $z \in \mathfrak{i}$. An easy induction on i shows that,

$$\forall i \in \mathbb{N}, \quad z(x^i(w)) - \lambda(z)x^i(w) \in W_i.$$

From this, it follows that z stabilises W_d and that the trace of its restriction to W_d is $d\lambda(z)$. Take now any element $y \in \mathfrak{i}$ and apply this to the element $[x,y] \in \mathfrak{i}$. We get that $d\lambda([x,y]) = 0$; indeed, since x and y stabilize W_d, the endomorphism induced by $[x,y]$ on W_d must be a commutator and hence have trace 0. As the characteristic of $\Bbbk$ is assumed to be 0, we end up with the desired equality: $\lambda([x,y]) = 0$.

At this stage, summing up the above, $\mathfrak{i}$ is a codimension 1 Lie ideal of $\mathfrak{g}$ and $\mathfrak{g}$ leaves the subspace

$$(0) \subset W = \{w \in V | x(w) = \lambda(x)w, \, \forall x \in \mathfrak{i}\} \subseteq V$$

invariant. Take any $z \in \mathfrak{g} \setminus \mathfrak{i}$, take a nonzero eigenvector of the restriction of z to W. This nonzero eigenvector is clearly a common eigenvector of all the elements of $\mathfrak{g}$, since $\mathfrak{g} = \mathfrak{i} + \Bbbk z$. Hence, we have proved that the result is true for $\mathfrak{g}$, which finishes the induction. ∎

Theorem I.5.7 – **(Lie's Theorem.)** *Assume that* $\Bbbk$ *is algebraically closed and of characteristic* 0*. Let* V *be a nonzero vector space of finite dimension. If* $\mathfrak{g}$ *is a solvable Lie subalgebra of* $\mathfrak{gl}(V)$*, then there exists a full flag* $\mathcal{F}$ *of* V *such that* $\mathfrak{g} \subseteq \mathfrak{b}_{\mathcal{F}}(V)$*.*

Proof. The result follows easily from Theorem I.5.6 using an induction on the dimension of V based on an argument similar to that of the proof of the second Point in Theorem I.4.8. ∎

We finish this section by two corollaries derived from Lie's Theorem.

Corollary I.5.8 – *Assume that* $\Bbbk$ *is algebraically closed and of characteristic* 0*. Let* $\mathfrak{g}$ *be a finite dimensional solvable Lie algebra. Then, there exists an increasing sequence* $(\mathfrak{g}_i)_{0 \leq i \leq \dim_{\Bbbk}(\mathfrak{g})}$ *of ideals of* $\mathfrak{g}$ *such that, for all* $0 \leq i \leq \dim_{\Bbbk}(\mathfrak{g})$, $\dim_{\Bbbk}(\mathfrak{g}_i) = i$*.*

Proof. Consider the adjoint representation of $\mathfrak{g}$, $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$. Since $\mathfrak{g}$ is finite dimensional, we may apply Lie's Theorem to the solvable Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$. It asserts that there exists a full flag $(\mathfrak{g}_i)_{0 \leq i \leq \dim_{\Bbbk}(\mathfrak{g})}$ of $\mathfrak{g}$ whose subspaces are left stable by $\mathrm{ad}_{\mathfrak{g}}(x)$, for all $x \in \mathfrak{g}$. But this last property just means that, for $0 \leq i \leq \dim_{\Bbbk}(\mathfrak{g})$, $\mathfrak{g}_i$ is a Lie ideal of $\mathfrak{g}$. ∎

Corollary I.5.9 – *Assume that* $\Bbbk$ *is algebraically closed and of characteristic* 0*. Let* $\mathfrak{g}$ *be a finite dimensional solvable Lie algebra.*

1. For all $x \in [\mathfrak{g}, \mathfrak{g}]$, $\mathrm{ad}_{\mathfrak{g}}(x)$ *is a nilpotent endomorphism of* $\mathfrak{gl}(\mathfrak{g})$*.*

2. The Lie subalgebra $[\mathfrak{g}, \mathfrak{g}]$ *of* $\mathfrak{g}$ *is nilpotent*

Proof. Since $\mathfrak{g}$ is finite dimensional, we may apply Lie's Theorem to the solvable Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$. There exists a full flag $\mathcal{F} = (\mathfrak{g}_i)_{0 \leq i \leq \dim_{\Bbbk}(\mathfrak{g})}$ of $\mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathfrak{b}_{\mathcal{F}}(\mathfrak{g})$.

But then, if $x \in [\mathfrak{g}, \mathfrak{g}]$, $\mathrm{ad}_{\mathfrak{g}}(x) \in [\mathfrak{b}_{\mathcal{F}}(\mathfrak{g}), \mathfrak{b}_{\mathcal{F}}(\mathfrak{g})] \subseteq \mathfrak{n}_{\mathcal{F}}(\mathfrak{g})$, so that $\mathrm{ad}_{\mathfrak{g}}(x)$ is a nilpotent endomorphism of $\mathfrak{gl}(\mathfrak{g})$.

This shows that the image of the adjoint action

$$\mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} : [\mathfrak{g}, \mathfrak{g}] \longrightarrow \mathfrak{gl}([\mathfrak{g}, \mathfrak{g}])$$

of $[\mathfrak{g}, \mathfrak{g}]$ consists in nilpotent endomorphisms. By Engel's Theorem, it follows that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. ∎

I.6 Cartan's criterion for solvability.

In this section, unless otherwise specified, $\Bbbk$ is an arbitrary field.

Definition I.6.1 – *An endomorphism f of a vector space U over $\Bbbk$ is called semisimple if, for all subspace V of U stable under f, there exists a subspace W of U stable under f and such that $U = V \oplus W$.*

Remark I.6.2 – Suppose $\Bbbk$ is algebraically closed, and let V be a finite dimensional vector space over $\Bbbk$. Then, as is well known, an endomorphism of V is semisimple if and only if it is diagonalisable.

We begin with a Lemma which relates the Jordan-Chevalley decomposition of an endomorphism and that of its image under the adjoint representation.

Lemma I.6.3 – *Let V be a finite dimensional vector space.*
1. If x is a nilpotent element of $\mathfrak{gl}(V)$, then $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$ is a nilpotent element of $\mathfrak{gl}(\mathfrak{gl}(V))$.
2. If x is a diagonalisable element of $\mathfrak{gl}(V)$, then $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$ is a diagonalisable element of $\mathfrak{gl}(\mathfrak{gl}(V))$.
3. Assume $\Bbbk$ is algebraicaly closed. Let $x \in \mathfrak{gl}(V)$ and suppose d, n are elements of $\mathfrak{gl}(V)$ such that d is semisimple, n nilpotent, $[d, n] = 0$ and $x = d + n$ (that is, $x = d + n$ is the Jordan-Chevalley decomposition of x as an endomorphism of V). Then, $\mathrm{ad}_{\mathfrak{gl}(V)}(x) = \mathrm{ad}_{\mathfrak{gl}(V)}(d) + \mathrm{ad}_{\mathfrak{gl}(V)}(n)$ is the Jordan-Chevalley decomposition of $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$ as an endomorphism of $\mathfrak{gl}(V)$.

Proof. 1. This statement is the content of Lemma I.4.7.
2. Let $\mathcal{B} = (v_1, \ldots, v_m)$ be a basis of V consisting of eigenvectors of x whose respective eigenvalues are $\lambda_1, \ldots, \lambda_m$. For $1 \leq i, j \leq m$, let $e_{i,j}$ be the endomorphism of V that sends v_j to v_i and any other vector in $\mathcal{B}$ to zero. Then $(e_{i,j})_{1 \leq i,j \leq m}$ is a basis of $\mathfrak{gl}(V)$. A straightforward calculation shows that:

$$\begin{array}{rcccl} \mathrm{ad}_{\mathfrak{gl}(V)}(x) & : & \mathfrak{gl}(V) & \longrightarrow & \mathfrak{gl}(V) \\ & & e_{i,j} & \mapsto & (\lambda_i - \lambda_j) e_{i,j} \end{array},$$

so that $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$ is diagonalisable.
3. We have that $\mathrm{ad}_{\mathfrak{gl}(V)}(x) = \mathrm{ad}_{\mathfrak{gl}(V)}(d) + \mathrm{ad}_{\mathfrak{gl}(V)}(n)$ and by Points 1 and 2 above, $\mathrm{ad}_{\mathfrak{gl}(V)}(d)$ is semisimple and $\mathrm{ad}_{\mathfrak{gl}(V)}(n)$ is nilpotent. In addition, $[\mathrm{ad}_{\mathfrak{gl}(V)}(d), \mathrm{ad}_{\mathfrak{gl}(V)}(n)] = \mathrm{ad}_{\mathfrak{gl}(V)}([d, n]) = 0$. Hence, indeed, $\mathrm{ad}_{\mathfrak{gl}(V)}(x) = \mathrm{ad}_{\mathfrak{gl}(V)}(d) + \mathrm{ad}_{\mathfrak{gl}(V)}(n)$ is the Jordan-Chevalley decomposition of $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$ as an endomorphism of $\mathfrak{gl}(V)$. ■

Lemma I.6.4 – *Assume that $\Bbbk$ is algebraically closed and of characteristic 0. Let V be a finite dimensional vector space and consider subspaces A and B of $\mathfrak{gl}(V)$ such that $A \subseteq B \subseteq \mathfrak{gl}(V)$. Put*

$$M = \{x \in \mathfrak{gl}(V) | \forall y \in B, [x, y] \in A\}.$$

Then, if x is an element of M such that, for all $y \in M$, $\mathrm{Tr}(xy) = 0$, then x is nilpotent.

Proof. Fix an element x of M such that, for all $y \in M$, $\mathrm{Tr}(xy) = 0$.

Let $x = x_s + x_n$ be the Jordan-Chevalley decomposition of x. That is, x_s is a diagonalisable endomorphism of V, x_n is a nilpotent endomorphism of V and these two endomorphisms commute. Let $\mathcal{B} = (v_1, \ldots, v_m)$ be a basis of V consisting of eigenvectors of x_s whose respective eigenvalues we denote $\lambda_1, \ldots, \lambda_m$.

We consider the following vector subspace of the $\mathbb{Q}$-vector space $\Bbbk$:

$$E = \sum_{1 \le i \le m} \mathbb{Q}\lambda_i.$$

Let now $f : E \longrightarrow \mathbb{Q}$ be any linear form on the $\mathbb{Q}$-vector space E. We consider the endomorphism y of V defined by

$$\begin{array}{rcccl} y & : & V & \longrightarrow & V \\ & & v_i & \mapsto & f(\lambda_i)v_i, \quad 1 \le i \le m \end{array}.$$

We equip the vector space $\mathfrak{gl}(V)$ with the basis associated to $\mathcal{B}$: this is the family $(e_{i,j})_{1\le i,j\le m}$ of endomorphisms such that, for all $1 \le i,j \le m$, $e_{i,j}$ sends v_j to v_i and any other vector in $\mathcal{B}$ to zero. It is straightforward to verify (see also the proof of Lemma I.6.3) that:

$$\begin{array}{rcccl} \mathrm{ad}_{\mathfrak{g}}(x_s) & : & \mathfrak{gl}(V) & \longrightarrow & \mathfrak{gl}(V) \\ & & e_{i,j} & \mapsto & (\lambda_i - \lambda_j)e_{i,j} \end{array} \quad \text{and} \quad \begin{array}{rcccl} \mathrm{ad}_{\mathfrak{g}}(y) & : & \mathfrak{gl}(V) & \longrightarrow & \mathfrak{gl}(V) \\ & & e_{i,j} & \mapsto & (f(\lambda_i) - f(\lambda_j))e_{i,j} \end{array}.$$

Now, by Lagrange interpolation, there exists a polynomial $P \in \Bbbk[T]$; without constant term and such that $P(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$, for all $1 \le i,j \le m$. But then, clearly,

$$\mathrm{ad}_{\mathfrak{g}}(y) = P(\mathrm{ad}_{\mathfrak{g}}(x_s)).$$

On the other hand, by Lemma I.6.3, $\mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{g}}(x_s) + \mathrm{ad}_{\mathfrak{g}}(x_n)$ is the Jordan-Chevalley decomposition of $\mathrm{ad}_{\mathfrak{g}}(x) \in \mathfrak{gl}(\mathfrak{gl}(V))$. Hence, there exists a polynomial Q in $\Bbbk[T]$, without constant term, such that

$$\mathrm{ad}_{\mathfrak{g}}(x_s) = Q(\mathrm{ad}_{\mathfrak{g}}(x)).$$

Therefore, there exists a polynomial R in $\Bbbk[T]$, without constant term, such that

$$\mathrm{ad}_{\mathfrak{g}}(y) = R(\mathrm{ad}_{\mathfrak{g}}(x)).$$

From this, since $x \in M$, it follows that $y \in M$. By hypothesis on x, we thus get that

$$0 = \mathrm{Tr}(xy) = \sum_{1\le i\le m} \lambda_i f(\lambda_i).$$

The right hand side is an element of E, so that we may apply f to it and get $0 = \sum_{1\le i\le m} f(\lambda_i)^2$, which entails that $f(\lambda_i) = 0$, for all $1 \le i \le m$, since this sum is a sum of positive rational numbers.

At this stage, we have proved that f is zero. Hence, the dual of E, and therefore E, is zero. It follows that all the eigenvalues of x_s are zero and that x_s is thus zero, which proves that x is nilpotent. ∎

Theorem I.6.5 – **Cartan's criterion for solvability** – *Assume that $\Bbbk$ is algebraically closed and of characteristic* 0. *Let V be a finite dimensional vector space and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V)$. Suppose that, for all $x \in [\mathfrak{g},\mathfrak{g}]$ and all $y \in \mathfrak{g}$, $\mathrm{Tr}(x \circ y) = 0$, then $\mathfrak{g}$ is solvable.*

Proof. We are in position to apply Lemma I.6.4 with $A = [\mathfrak{g},\mathfrak{g}]$, $B = \mathfrak{g}$. Put $M = \{x \in \mathfrak{gl}(V) | \forall y \in B, [x,y] \in A\}$, as in this Lemma. It is clear that $\mathfrak{g} \subseteq M$. Thus

$$[\mathfrak{g},\mathfrak{g}] \subseteq \mathfrak{g} \subseteq M \subseteq \mathfrak{gl}(V).$$

Take $x \in [\mathfrak{g}, \mathfrak{g}]$, $y \in M$. As is easily verified,

$$\forall a, b, c \in \mathfrak{gl}(V),\ \mathrm{Tr}([a, b] \circ c]) = \mathrm{Tr}(a \circ [b, c]).$$

From this equality, we easily get that $\mathrm{Tr}(x \circ y) = 0$. Hence, Lemma I.6.4 proves that x is nilpotent. Applying now Engel's Theorem (in the form of Theorem I.4.8), we get that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, hence solvable. Thus $\mathfrak{g}$ is solvable by Exercise I.5.3. ∎

Corollary I.6.6 – *Assume that* $\Bbbk$ *is algebraically closed and of characteristic* 0. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra. Suppose that, for all* $x \in [\mathfrak{g}, \mathfrak{g}]$ *and all* $y \in \mathfrak{g}$, $\mathrm{Tr}(\mathrm{ad}_{\mathfrak{g}}(x) \circ \mathrm{ad}_{\mathfrak{g}}(y)) = 0$, *then* $\mathfrak{g}$ *is solvable.*

Proof. Consider the adjoint representation of $\mathfrak{g}$, $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$. By the hypothesis on $\mathfrak{g}$, we are in position to apply Cartan's criterion to the Lie subalgebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ of $\mathfrak{gl}(\mathfrak{g})$. Hence, $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is solvable. But $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is isomorphic to $\mathfrak{g}/Z(\mathfrak{g})$. Hence, $\mathfrak{g}$ is solvable. ∎

I.7 Simple and semisimple Lie algebras.

In this section, $\Bbbk$ is an arbitrary field and Lie algebras are assumed to be finite dimensional.

Definition I.7.1 – *A Lie algebra* $\mathfrak{g}$ *is called* simple *if it is nonabelian and if it has no other ideals than* $\{0\}$ *and* $\mathfrak{g}$.

Exercise I.7.2 – A simple Lie algebra is not solvable.

Exercise I.7.3 – Let $n \in \mathbb{N}^*$. If the characteristic of $\Bbbk$ is different from 2 and does not divide n, then $\mathfrak{sl}_n(\Bbbk)$ is a simple Lie algebra.

We now introduce semisimple Lie algebras.

We have already mentionned (cf. Exercise I.5.3) that the sum of two solvable ideals of a Lie algebra is a solvable ideal. It follows that, for any Lie algebra, the set of solvable ideals, ordered by inclusion, as a greatest element. This justifies the following definition.

Definition I.7.4 – *Let* $\mathfrak{g}$ *be a Lie algebra. The* radical *of* $\mathfrak{g}$, *denoted* $\mathrm{Rad}(\mathfrak{g})$, *is the greatest solvable ideal of* $\mathfrak{g}$.

Definition I.7.5 – *A Lie algebra* $\mathfrak{g}$ *is* semisimple *if* $\mathrm{Rad}(\mathfrak{g}) = (0)$.

Example I.7.6 – Let $\mathfrak{g}$ be a nonzero Lie algebra. Then, using the second point in Exercise I.5.3, we get that $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is semisimple.

Exercise I.7.7 – A simple Lie algebra is semisimple (see Exercise I.7.2). The Lie algebra (0) is semisimple (though not simple).

Exercise I.7.8 – Let $\mathfrak{g}$ be a Lie algebra.
1. We have $Z(\mathfrak{g}) \subseteq \mathrm{Rad}(\mathfrak{g})$.
2. If $\mathfrak{g}$ is semisimple, its adjoint representation is faithful, *i.e.*, $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is injective.

Actually, abelian ideals detect semisimplicity.

Lemma I.7.9 – *Let $\mathfrak{g}$ be a nonzero Lie algebra. Then, $\mathfrak{g}$ is semisimple if and only if it has no nonzero abelian ideal.*

Proof. Clearly, any abelian ideal of $\mathfrak{g}$ is contained in $\mathrm{Rad}(\mathfrak{g})$. So, the condition is necessary. Conversally, if $\mathrm{Rad}(\mathfrak{g})$ is nonzero, the last nonzero term in the derived series of $\mathrm{Rad}(\mathfrak{g})$ is an abelian ideal of $\mathrm{Rad}(\mathfrak{g})$. But, by Exercise I.5.4, this last nonzero term is actually an ideal of $\mathfrak{g}$. ■

Remark I.7.10 – **Levi decomposition** – Exercise I.7.6 shows that any Lie algebra is the extension of a semisimple Lie algebra by a solvable Lie algebra. It turns out that, actually, under mild hypotheses on the base field, such an extension may be choosen to be split. In other words, any Lie algebra $\mathfrak{g}$ is the semi-direct product of its radical and a Lie subalgebra isomorphic to $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$. This is the *Levi* decomposition. It clearly emphasises the importance of solvable and semisimple Lie algebras.

We now introduce a major tool to characterise semisimplicity: the Killing form.

Definition I.7.11 – *Let $\mathfrak{g}$ be a Lie algebra. The Killing form of $\mathfrak{g}$ is the map:*

$$\begin{array}{rcccl} \kappa_{\mathfrak{g}} & : & \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathbb{k} \\ & & (x,y) & \mapsto & \mathrm{Tr}(\mathrm{ad}(x)\circ\mathrm{ad}(y)) \end{array},$$

where Tr *stands for the usual trace map on $\mathfrak{gl}(\mathfrak{g})$.*

Exercise I.7.12 – Let $\mathfrak{g}$ be a Lie algebra.
1. Show that the Killing form of $\mathfrak{g}$ is is a symmetric bilinear form, and that, for all $x, y, z \in \mathfrak{g}$:

$$\kappa_{\mathfrak{g}}(x,[y,z]) = \kappa_{\mathfrak{g}}([x,y],z).$$

(A symmetric bilinear form over a Lie algebra satisfying the above compatibility condition with the Lie bracket is called *invariant*.)
2. Show that the radical of the Killing form:

$$\mathrm{Rad}(\kappa_{\mathfrak{g}}) := \mathfrak{g}^{\perp} = \{x\in\mathfrak{g} \mid \kappa_{\mathfrak{g}}(x,y)=0,\, \forall y\in\mathfrak{g}\}$$

is an ideal of $\mathfrak{g}$.
3. Show that, more generaly, the orthogonal for $\kappa_{\mathfrak{g}}$ of an ideal of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$.
4. Let $\mathfrak{i}$ be an ideal of $\mathfrak{g}$. Then, $\kappa_{\mathfrak{i}} = (\kappa_{\mathfrak{g}})_{|\mathfrak{i}\times\mathfrak{i}}$. (The same result does not hold if $\mathfrak{i}$ is just a Lie subalgebra of $\mathfrak{g}$.)

Remark I.7.13 – **(Cartan's criterion for solvability revisited)** – Assume $\mathbb{k}$ is algebraically closed and of characteristic 0. Let $\mathfrak{g}$ be a Lie algebra. Corollary I.6.6 actually states the following. If $[\mathfrak{g},\mathfrak{g}] \subseteq \mathrm{Rad}(\kappa_{\mathfrak{g}})$, then $\mathfrak{g}$ is solvable.

Lemma I.7.14 – *Assume $\mathbb{k}$ is algebraically closed and of characteristic 0. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathrm{Rad}(\kappa_{\mathfrak{g}}) \subseteq \mathrm{Rad}(\mathfrak{g})$.*

Proof. Put $\mathfrak{i} = \mathrm{Rad}(\kappa_{\mathfrak{g}})$ and recall that $\mathfrak{i}$ is an ideal of $\mathfrak{g}$ (see Exercise, I.7.12).

Let $x \in \mathfrak{i}$, by definition, we have $\kappa_{\mathfrak{g}}(x,y) = 0$, for all $y \in \mathfrak{i}$. By Point 4 of Exercise, I.7.12, this entails that $\kappa_{\mathfrak{i}} = 0$. In particular, $[\mathfrak{i},\mathfrak{i}] \subseteq \mathrm{Rad}(\kappa_{\mathfrak{i}})$. By Cartan's criterion for solvability (under the form of Remark I.7.13); this means that $\mathfrak{i}$ is solvable. Hence, being an ideal, $\mathfrak{i} \subseteq \mathrm{Rad}(\mathfrak{g})$ by definition of the radical of a Lie algebra. ■

Exercise I.7.15 – Let $\mathfrak{g}$ be a Lie algebra.
1. Let $\mathfrak{i}$ be an abelian ideal of $\mathfrak{g}$. For all $(x,y) \in \mathfrak{g} \times \mathfrak{i}$, the endomorphism $\mathrm{ad}(x) \circ \mathrm{ad}(y)$ is nilpotent (of index bounded above by 2), hence has zero trace.
2. Thus $\mathrm{Rad}(\kappa_{\mathfrak{g}})$ contains any abelian ideal of $\mathfrak{g}$.

The following Theorem is fundamental; it connects the Killing form and semisimplicity.

Theorem I.7.16 – **(Cartan-Killing's criterion.)** – *Assume $\Bbbk$ is algebraically closed and of characteristic 0. Let $\mathfrak{g}$ be a Lie algebra. Then, $\mathfrak{g}$ is semisimple if and only if $\kappa_{\mathfrak{g}}$ is nondegenerate.*

Proof. It follows immediately from Lemma I.7.14 that, if $\mathfrak{g}$ is semisimple, then its Killing form must be nondegenerate.

Conversaly, suppose that $\kappa_{\mathfrak{g}}$ is nondegenerate. By Exercise I.7.15, then $\mathfrak{g}$ has no nontrivial abelian ideal. But then, Lemma I.7.9 shows that $\mathfrak{g}$ is semisimple. ∎

Exercise I.7.17 – Assume $\Bbbk$ is algebraically closed and of characteristic 0. The direct sum of two semisimple Lie algebras is again semisimple. (*Hint.* use the Cartan-Killing criterion.) In particular, the (finite) direct sum of simple Lie algebras is semisimple.

We then get a structure Theorem for semisimple Lie algebras which reduces their study to that of simple Lie algebras.

Lemma I.7.18 – *Assume $\Bbbk$ is algebraically closed and of characteristic 0. Let $\mathfrak{g}$ be a semisimple Lie algebra. If $\mathfrak{i}$ is a Lie ideal of $\mathfrak{g}$, then*
1. $\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{i}^{\perp}$ and $[\mathfrak{i}, \mathfrak{i}^{\perp}] = 0$;
2. $\mathfrak{i}$ and $\mathfrak{i}^{\perp}$ are semisimple as Lie algebras.

Proof. Let $\mathfrak{i}$ be a Lie ideal of $\mathfrak{g}$. By Exercise I.7.12, we know that $\mathfrak{i}^{\perp}$ is a Lie ideal of $\mathfrak{g}$. Consider the ideal $\mathfrak{j} = \mathfrak{i} \cap \mathfrak{i}^{\perp}$. We have that
$$\kappa_{\mathfrak{j}} = (\kappa_{\mathfrak{g}})_{|\mathfrak{j} \times \mathfrak{j}} = 0,$$
the first equality is Exercise I.7.12, Point 4, the second is trivial. Hence, $[\mathfrak{j}, \mathfrak{j}] \subseteq \mathrm{Rad}(\kappa_{\mathfrak{j}})$ and Cartan's criterion gives that $\mathfrak{j}$ is solvable. But, being an ideal of the semisimple Lie algebra $\mathfrak{g}$, this forces $\mathfrak{j}$ to be trivial. That is, $\mathfrak{i} \cap \mathfrak{i}^{\perp} = 0$, and thus
$$\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{i}^{\perp}$$
since $\kappa_{\mathfrak{g}}$ is nondegenerate. Moreover, $[\mathfrak{i}, \mathfrak{i}^{\perp}] \subseteq \mathfrak{i} \cap \mathfrak{i}^{\perp}$, so that $[\mathfrak{i}, \mathfrak{i}^{\perp}] = 0$.

But, this last equality shows that any ideal of the Lie algebra $\mathfrak{i}$ (resp. $\mathfrak{i}^{\perp}$) is actually an ideal of $\mathfrak{g}$. Hence, the existence of a non trivial solvable ideal of the Lie algebra $\mathfrak{i}$ (resp. $\mathfrak{i}^{\perp}$) would imply the existence of a non trivial solvable ideal of $\mathfrak{g}$. Hence $\mathfrak{i}$ and $\mathfrak{i}^{\perp}$ are semisimple as Lie algebras. ∎

Theorem I.7.19 – **Structure of semisimple Lie algebras** – *Assume $\Bbbk$ is algebraically closed and of characteristic 0. Let $\mathfrak{g}$ be a nonzero semisimple Lie algebra.*
1. There exist $t \in \mathbb{N}^*$, Lie ideals $\mathfrak{s}_1, \dots, \mathfrak{s}_t$ of $\mathfrak{g}$ which are simple as Lie algebras, such that
$$\mathfrak{g} = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_t$$
and, for $1 \leq i < j \leq t$, $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$.
2. If $\mathfrak{s}$ is an ideal of $\mathfrak{g}$ which is a simple Lie algebra, then there exists $1 \leq i \leq t$ such that $\mathfrak{s} = \mathfrak{s}_i$.

Proof. Suppose that $\mathfrak{g}$ is a semisimple Lie algebra which is not simple. By definition, there exists an ideal $\mathfrak{i}$ of $\mathfrak{g}$ such that $(0) \subset \mathfrak{i} \subset \mathfrak{g}$. By Lemma I.7.18, we know that $\mathfrak{i}$ and $\mathfrak{i}^\perp$ are semisimple Lie algebras and that

$$\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{i}^\perp \qquad \text{and} \qquad [\mathfrak{i}, \mathfrak{i}^\perp] = 0.$$

Choose now $\mathfrak{i}$ minimal among proper nontrivial ideals of $\mathfrak{g}$. We even have that $\mathfrak{i}$ is a simple Lie algebra (and $\mathfrak{i}^\perp$ a semisimple one).

Let now $d \in \mathbb{N}^*$ be the least integer for which there exists a semisimple Lie algebra of dimension d. We are now ready to prove the theorem by induction on $\dim_{\Bbbk}(\mathfrak{g})$. If $\mathfrak{g}$ has dimension d, then the above reasoning shows that $\mathfrak{g}$ is actually simple. Hence the result is true. Suppose now that $\dim_{\Bbbk}(\mathfrak{g}) > d$. If $\mathfrak{g}$ is simple, then the result holds for it. Otherwise, choose a minimal proper non trivial ideal of $\mathfrak{g}$. Then by the above, $\mathfrak{i}^\perp$ is a semisimple Lie algebra of strictly lower dimension and we may apply the induction hypothesis to it. By the above argument, we deduce that the result holds for $\mathfrak{g}$.
2. Let $\mathfrak{s}$ be an ideal of $\mathfrak{g}$ which is a simple Lie algebra. Then, $[\mathfrak{s}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$ and it is nonzero since $Z(\mathfrak{g})$ is zero. But then,

$$\mathfrak{s} = [\mathfrak{s}, \mathfrak{g}] = \oplus_{1 \le i \le t} [\mathfrak{s}, \mathfrak{s}_i],$$

(the first equality follows from the simplicity of $\mathfrak{s}$, the second is obvious). By the simplicity of $\mathfrak{s}$, there must exist a unique $1 \le i \le t$ such that $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}_i]$. In particular, $\mathfrak{s} \subseteq \mathfrak{s}_i$ and thus $\mathfrak{s} = \mathfrak{s}_i$ by the simplicity of $\mathfrak{s}_i$. ■

The following Corollary describes Lie ideals of semisimple Lie algebras.

Corollary I.7.20 – *Assume $\Bbbk$ is algebraically closed and of characteristic 0. Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $t \in \mathbb{N}^*$ and $\mathfrak{s}_1, \dots, \mathfrak{s}_t$ be the Lie ideals of $\mathfrak{g}$ which are simple (see Theorem I.7.19). Then the following holds:*
1. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$;
2. any nonzero Lie ideal is a sum $\bigoplus_{j \in J} \mathfrak{s}_j$ for some subset $J \subseteq \{1, \dots, t\}$;
3. any Lie ideal or homomorphic image of $\mathfrak{g}$ is semisimple.

Proof. By Theorem I.7.19, we have that $\mathfrak{g} = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_t$ and, $\forall\, 1 \le i < j \le t$, $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$.
1. Clearly, $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{s}_1, \mathfrak{s}_1] \oplus \ldots \oplus [\mathfrak{s}_t, \mathfrak{s}_t]$. But, for all $1 \le i \le t$, $[\mathfrak{s}_i, \mathfrak{s}_i] = \mathfrak{s}_i$ by the simplicity of $\mathfrak{s}_i$. Hence Point 1.
2. Let $\mathfrak{i}$ be a nonzero ideal of $\mathfrak{g}$. By Lemma I.7.18, $\mathfrak{i}$ is semisimple, so that $[\mathfrak{i}, \mathfrak{i}] = \mathfrak{i}$, by Point 1. Thus,

$$\mathfrak{i} = [\mathfrak{i}, \mathfrak{i}] \subseteq [\mathfrak{i}, \mathfrak{g}] \subseteq \bigoplus_{1 \le i \le t} [\mathfrak{i}, \mathfrak{s}_i] \subseteq \bigoplus_{1 \le i \le t} \mathfrak{i} \cap \mathfrak{s}_i \subseteq \mathfrak{i}.$$

Hence,

$$\mathfrak{i} = \bigoplus_{1 \le i \le t} \mathfrak{i} \cap \mathfrak{s}_i.$$

On the other hand, for $1 \le i \le t$, $\mathfrak{s}_i$ is simple, so that $\mathfrak{i} \cap \mathfrak{s}_i$ equals $\mathfrak{s}_i$ or (0). Hence Point 2.
3. This is now clear using Exercise I.7.17. ■

Part II

Semisimple Lie algebras.

II.1 Complete reducibility of finite dimensional representations.

Assume $\Bbbk$ is algebraically closed of characteristic 0.

Lemma II.1.1 – *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and (V, f) a faithful finite dimensional representation of $\mathfrak{g}$. Consider*

$$\begin{array}{rccl} \beta_f & : & \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \Bbbk \\ & & (x, y) & \mapsto & \mathrm{Tr}(f(x) \circ f(y)) \end{array}.$$

Then, β_f is an invariant, nondegenerate and symmetric bilinear form on $\mathfrak{g}$.

Proof. Bilinearity and symmetry of the form β_f are clear. The invariance of β_f is easy. Let now $\mathrm{Rad}(\beta_f)$ be the radical of β_f. By the invariance of β_f, $\mathrm{Rad}(\beta_f)$ is a Lie ideal of $\mathfrak{g}$. Now, $f(\mathrm{Rad}(\beta_f))$ is a Lie subalgebra of $\mathfrak{gl}(V)$, isomorphic to $\mathrm{Rad}(\beta_f)$, since f is faithful. On the other hand, Cartan's criterion for solvability clearly applies to $f(\mathrm{Rad}(\beta_f))$ and shows that it is solvable. It follows that $f(\mathrm{Rad}(\beta_f))$ is solvable and hence that so is $\mathrm{Rad}(\beta_f)$. But $\mathfrak{g}$ is semisimple, so $\mathrm{Rad}(\beta_f) = (0)$. ■

Lemma II.1.2 – *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $\beta : \mathfrak{g} \times \mathfrak{g} \longrightarrow \Bbbk$ be an invariant, nondegenerate, symmetric bilinear form on $\mathfrak{g}$ and (V, f) a representation of $\mathfrak{g}$. Let $(x_1, \ldots, x_n)$ be a basis of $\mathfrak{g}$ and $(y_1, \ldots, y_n)$ be its dual basis with respect to β. Then, the element*

$$c = \sum_{1 \le i \le n} f(x_i) \circ f(y_i) \in \mathrm{End}_\Bbbk(V)$$

is a morphism of representation of (V, f).

Proof. Let z be an element of $\mathfrak{g}$. Consider elements $a_{i,j}, b_{i,j} \in \Bbbk$, $1 \le i, j \le n$ such that

$$[z, x_i] = \sum_{1 \le j \le n} a_{i,j} x_j \qquad \text{and} \qquad [z, y_i] = \sum_{1 \le j \le n} b_{i,j} y_j.$$

By the invariance of β, we have that, for all $1 \le i, j \le n$,

$$a_{i,j} = \beta([z, x_i], y_j) = -\beta([x_i, z], y_j) = -\beta(x_i, [z, y_j]) = -b_{j,i}.$$

Recall now that $[f(z), -] : \mathrm{End}_\Bbbk(V) \longrightarrow \mathrm{End}_\Bbbk(V))$ is a derivation of the associative algebra $\mathrm{End}_\Bbbk(V)$. So,

$$\begin{array}{rcl} [f(z), c] & = & \sum_{1 \le i \le n} [f(z), f(x_i) \circ f(y_i)] \\ & = & \sum_{1 \le i \le n} [f(z), f(x_i)] \circ f(y_i) + \sum_{1 \le i \le n} f(x_i) \circ [f(z), f(y_i)] \\ & = & \sum_{1 \le i \le n} f([z, x_i]) \circ f(y_i) + \sum_{1 \le i \le n} f(x_i) \circ f([z, y_i]) \\ & = & \sum_{1 \le i, j \le n} a_{i,j} f(x_j) \circ f(y_i) + \sum_{1 \le i, j \le n} b_{i,j} f(x_i) \circ f(y_j) \\ & = & \sum_{1 \le i, j \le n} a_{i,j} f(x_j) \circ f(y_i) + \sum_{1 \le i, j \le n} b_{j,i} f(x_j) \circ f(y_i) \\ & = & \sum_{1 \le i, j \le n} (a_{i,j} + b_{j,i}) f(x_j) \circ f(y_i) \\ & = & 0. \end{array}$$

Which proves that c is a morphism of representation of (V, f). ■

Remark II.1.3 – Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and (V, f) a faithful finite dimensional representation of $\mathfrak{g}$. Lemma II.1.1 proves that

$$\begin{array}{cccc} \beta_f & : & \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \mathbb{k} \\ & & (x, y) & \mapsto & \mathrm{Tr}(f(x) \circ f(y)) \end{array}$$

is an invariant, nondegenerate, symmetric bilinear form on $\mathfrak{g}$. Let $(x_1, \ldots, x_n)$ be a basis of $\mathfrak{g}$ and $(y_1, \ldots, y_n)$ be its dual basis with respect to β_f. Then the element,

$$c = \sum_{1 \leq i \leq n} f(x_i) \circ f(y_i) \in \mathrm{End}_{\mathbb{k}}(V) \tag{II.1.4}$$

is a morphism of representation of (V, f), by Lemma II.1.2. In addition, we have

$$\mathrm{Tr}(c) = \sum_{1 \leq i \leq n} \mathrm{Tr}(f(x_i) \circ f(y_i)) = \sum_{1 \leq i \leq n} \beta_f(x_i, y_i) = \dim_{\mathbb{k}}(\mathfrak{g}). \tag{II.1.5}$$

Suppose in addition that (V, f) is simple. Then, by Schur's Lemma, $c \in \mathbb{k}\mathrm{id}_V$. It follows that

$$c = \frac{\dim_{\mathbb{k}}(\mathfrak{g})}{\dim_{\mathbb{k}}(V)} \mathrm{id}_V.$$

In particular, the above element is independant of the choice of the basis $(x_1, \ldots, x_n)$. For this reason, we denote it c_f and call it *the Casimir element* associated to f.

We are now ready to prove Weyl's Theorem of complete reducibility of finite dimensional representations of finite dimensional semisimple Lie algebras. According to Theorem I.2.19, the complete reducibility of a representation is equivalent to the fact that it verifies the direct summand property. Hence, we prove that any finite dimensional representation of a finite dimensional semisimple Lie algebra does verify the direct summand property.

Lemma II.1.4 – *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and let (V, ρ) be a finite dimensional representation of $\mathfrak{g}$. If W is a codimension 1 subrepresentation of (V, ρ), then there exists a subrepresentation X of (V, ρ) such that $V = W \oplus X$.*

Proof. We proceed by induction on the dimension of V. The result is trivial whenever V has dimension less than or equal to 1. We now consider a finite dimensional representation (V, ρ) of $\mathfrak{g}$ with $\dim_{\mathbb{k}}(V) > 1$ and a codimension 1 subrepresentation W of V.

Suppose W is not simple. Then, there exists a subrepresentation W' of V such that

$$(0) \subset W' \subset W \subset V.$$

We consider the quotient representation $(V/W', \overline{\rho})$ of $\mathfrak{g}$ and denote by $\pi : V \longrightarrow V/W'$ the canonical projection. Clearly, $\pi(W)$ is a codimension 1 subrepresentation of V/W' and, since $\dim_{\mathbb{k}}(V/W') < \dim_{\mathbb{k}}(V)$, the induction hypothesis yields a subrepresentation $\widetilde{W}$ of V such that

$$W' \subseteq \widetilde{W}, \quad \dim_{\mathbb{k}}(\widetilde{W}) = \dim_{\mathbb{k}}(W') + 1 \quad \text{and} \quad V/W' = \pi(W) \oplus \pi(\widetilde{W}).$$

Now, $\dim_{\mathbb{k}}(\widetilde{W}) = \dim_{\mathbb{k}}(W') + 1 \leq \dim_{\mathbb{k}}(V) - 1$. So again, we may apply the induction hypothesis to $\widetilde{W}$ and its subrepresentation W'. It provides a subrepresentation X of $\widetilde{W}$, of dimension 1, such that

$$\widetilde{W} = W' \oplus X.$$

But, on the one hand, we have that $W \cap X \subseteq W \cap \widetilde{W} = W'$, so that $W \cap X \subseteq W' \cap X = (0)$. And, on the other hand, $\dim_{\Bbbk}(W) + \dim_{\Bbbk}(X) = (\dim_{\Bbbk}(V) - 1) + 1 = \dim_{\Bbbk}(V)$. Therefore, we get that

$$V = W \oplus X.$$

So, in case W is not simple, it has the desired complement subrepresentation.

Suppose now that W is simple. Observe first that we may suppose, without loss of generality, that ρ is faithful. Hence, to the arbitrary choice of a basis of $\mathfrak{g}$, we may associate an element c as in (II.1.4); the element c is an endomorphism of the representation V (Lemma II.1.2) which, by construction, stabilises all the subrepresentations of V.

Observe that, $\mathfrak{g}$ being semisimple, we have that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, from which it follows that $\mathfrak{g}$ acts trivially on any one dimensional representation. Therefore, $\mathfrak{g}$ acts trivialy, on V/W. In other words, $\rho(\mathfrak{g})(V) \subseteq W$. By construction, we therefore have $c(V) \subseteq W$. Further, by Schurs's Lemma, c acts on W by scalar multiplication: there exists $\lambda \in \Bbbk$ such that $c_{|W} = \lambda \mathrm{id}_W$. All in all, in any basis of V obtained completing one of W, the matrix of c is as follows:

$$\begin{pmatrix} \lambda & 0 & \dots & 0 & * \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \lambda & * \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}.$$

But, by (II.1.5), we know that the trace of c is nonzero. Hence, $\lambda \neq 0$. From which it follows that $\ker(c)$ is a one dimensional subspace of V and $W \cap \ker(c) = (0)$, so that:

$$V = W \oplus \ker(c).$$

It remains to notice that, c being an endomorphism of representation, $\ker(c)$ must be a subrepresentation to conclude that $\ker(c)$ is the desired complement to W.

The proof is now complete. ∎

Theorem II.1.5 – **(Weyl's Theorem.)** *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Any finite dimensional representation of $\mathfrak{g}$ is completely reducible (equivalently, has the direct summand property).*

Proof. Let (V, ρ) be a finite dimensional representation of $\mathfrak{g}$. By Theorem I.2.19, the result amounts to showing that (V, ρ) has the direct summand property. Of course, the result holds if (V, ρ) is simple.

Suppose (V, ρ) is not simple and let $(0) \subset W \subset V$ be a subrepresentation of V. We consider the representation $(\mathrm{Hom}_{\Bbbk}(V, W), \mu)$ of $\mathfrak{g}$ as defined in Exercise I.2.6. We consider the two following subspaces of $\mathrm{Hom}_{\Bbbk}(V, W)$: let $\mathcal{V}$ (resp. $\mathcal{W}$) be the subspace of linear maps $f : V \longrightarrow W$ which act by scalar multiplication (resp. by 0) on W. Let $f \in \mathcal{V}$ and let $\lambda \in \Bbbk$ such that $f_{|W} = \lambda \mathrm{id}_W$. Then, by definition of μ, we have that,

$$\forall\, x \in \mathfrak{g},\ \forall\, w \in W, \quad (\mu(x)(f))(w) = \rho(x) \circ f(w) - f \circ \rho(x)(w) = \rho(x)(\lambda w) - \lambda \rho(x)(w) = 0.$$

Therefore, $\mathcal{V}$ and $\mathcal{W}$ are subrepresentations of $(\mathrm{Hom}_{\Bbbk}(V, W), \mu)$ and, further, the action of $\mathfrak{g}$ on this representation sends $\mathcal{V}$ into $\mathcal{W}$. In addition, it is clear that $\mathcal{W}$ is a subspace of $\mathcal{V}$ of codimension 1.

Therefore, we are in position to apply the result of Lemma II.1.4: there exists a one dimensional subrepresentation of $\mathcal{V}$ which is a complement of $\mathcal{W}$. Let $f \in \mathcal{V}$ be a basis for such a complement. Multiplying f by a nonzero scalar, if necessary, we may suppose that $f_{|W} = \mathrm{id}_W$. As we already noticed in the proof of Lemma II.1.4, $\mathfrak{g}$ must act trivially on the subrepresentation $\Bbbk f$, since it is semisimple. This means that f is, actually, a morphism of representations from V to W. As such, its kernel must be a subrepresentation of V. Further, as $f_W = \mathrm{id}_W$, we have that $\ker(f) \cap W = (0)$. Now, an obvious dimension argument gives that

$$V = W \oplus \ker(f).$$

Therefore, $\ker(f)$ is the desired complement to W in V. ∎

Corollary II.1.6 – *Let $\mathfrak{h}$ be a finite dimensional Lie algebra. If $\mathfrak{g}$ is a semisimple ideal of $\mathfrak{g}$, then there exists a unique Lie ideal $\mathfrak{i}$ of $\mathfrak{h}$ such that*

$$\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{i}.$$

Proof. The proof uses two representations attached to $\mathrm{ad}_{\mathfrak{h}} : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{h})$.

Consider first the finite dimensional representation of $\mathfrak{g}$: $(\mathrm{ad}_{\mathfrak{h}})_{|\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{h})$. Clearly, $\mathfrak{g}$ is a subrepresentation of $\mathfrak{h}$ and thus, by Weyl's Theorem, there exists a subrepresentation $\mathfrak{i}$ of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{i}$. We have that $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$ because $\mathfrak{i}$ is a subrepresentation of $\mathfrak{h}$ and $[\mathfrak{g}, \mathfrak{i}] \subseteq [\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{g}$ because $\mathfrak{g}$ is an ideal of $\mathfrak{h}$. Therefore, we have

$$[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i} \cap \mathfrak{g} = (0).$$

Consider now the subrepresentation $f : \mathfrak{h} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ of $(\mathfrak{h}, \mathrm{ad}_{\mathfrak{h}})$. We claim that $\mathfrak{i} = \ker(f)$. The inclusion $\mathfrak{i} \subseteq \ker(f)$ has been proved before. Conversaly, let $x \in \ker(f)$. Write $x = x_{\mathfrak{g}} + x_{\mathfrak{i}}$, $x_{\mathfrak{g}} \in \mathfrak{g}$, $x_{\mathfrak{i}} \in \mathfrak{i}$. For all $y \in \mathfrak{g}$, we have

$$0 = [x, y] = [x_{\mathfrak{g}}, y] + [x_{\mathfrak{i}}, y] = [x_{\mathfrak{g}}, y].$$

It follows that $x_{\mathfrak{g}} \in Z(\mathfrak{g})$. But, $\mathfrak{g}$ being semisimple, $Z(\mathfrak{g}) = (0)$ and we get that $x \in \mathfrak{i}$. As $\mathfrak{i}$ is the kernel of f, it must be an ideal of $\mathfrak{h}$. The existence is established.

Unicity is easy as, using the same argument as above, any $\mathfrak{i}$ as in the statement must equal $\ker(f)$. ∎

II.2 Derivations.

Assume $\Bbbk$ is algebraically closed of characteristic 0.

Recall from Remark I.3.6 that, given any Lie algebra $\mathfrak{g}$, we have $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathrm{Der}_{\Bbbk}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$ and that $\mathrm{Der}_{\Bbbk}(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ a Lie ideal of the Lie algebra $\mathrm{Der}_{\Bbbk}(\mathfrak{g})$.

We now proceed to show that, if $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra, then $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) = \mathrm{Der}_{\Bbbk}(\mathfrak{g})$.

Proposition II.2.1 – *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Then* $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) = \mathrm{Der}_{\Bbbk}(\mathfrak{g})$*; that is, all the derivations of $\mathfrak{g}$ are inner derivations.*

Proof. We work in the finite dimensional Lie algebra $\mathrm{Der}_{\Bbbk}(\mathfrak{g})$ and consider its Lie ideal $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ (see Remark I.3.6). By the semisimplicity of $\mathfrak{g}$, the map $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathrm{Der}_{\Bbbk}(\mathfrak{g})$ is injective. Hence, $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is a semisimple Lie ideal of $\mathrm{Der}_{\Bbbk}(\mathfrak{g})$. Thus, we are in position to apply Corollary II.1.6 which asserts that there exists an ideal $\mathfrak{i}$ of $\mathrm{Der}_{\Bbbk}(\mathfrak{g})$ such that

$$\mathrm{Der}_{\Bbbk}(\mathfrak{g}) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \oplus \mathfrak{i}.$$

Let $d \in \mathfrak{i}$. For all $x \in \mathfrak{g}$, we have that

$$\mathrm{ad}_{\mathfrak{g}}(d(x)) = [d, \mathrm{ad}_{\mathfrak{g}}(x)] = 0;$$

indeed: the first equality above comes from Point 2 of Remark I.3.6, and the second follows from the fact that $\mathfrak{i}$ and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ are ideals of $\mathrm{Der}_{\Bbbk}(\mathfrak{g})$ with trivial intersection. Hence, $d(x)$ is in the center of the semisimple Lie algebra $\mathfrak{g}$, so that it is zero. We have shown that $d = 0$. Hence $\mathfrak{i} = (0)$, which proves the statement. ■

II.3 Abstract Jordan-Chevalley decomposition.

Assume $\Bbbk$ is algebraically closed of characteristic 0.

The results in this section show that any element of a finite dimensional semisimple Lie algebra may be written uniquely as the sum of a *nilpotent* and a *semisimple* elements (in a sense that needs to be defined) which commute. This amounts to an abstract form of the classical Jordan-Chevalley decomposition of endomorphisms. The point with this decomposition is that it is universal in some sense, since this decomposition recovers the classical one on any finite dimensional representation.

The first notion defined below has already been introduced; we recall it for the sake of symmetry.

Definition II.3.1 – *Let $\mathfrak{g}$ be any finite dimensional Lie algebra and let $x \in \mathfrak{g}$.*
1. We say that x is ad*-nilpotent if the endomorphism* $\mathrm{ad}(x) : \mathfrak{g} \longrightarrow \mathfrak{g}$ *is nilpotent.*
2. We say that x is ad*-semisimple is the endomorphism* $\mathrm{ad}(x) : \mathfrak{g} \longrightarrow \mathfrak{g}$ *is diagonalisable.*

The next Theorem establishes the existence of an *abstract Jordan-Chevalley decomposition* for any finite dimensional semisimple Lie algebra.

Theorem II.3.2 – **Abstract Jordan-Chevalley decomposition** – *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. For all $x \in \mathfrak{g}$, there exists a unique pair (x_n, x_s) of elements of $\mathfrak{g}$ with x_n* ad*-nilpotent, x_s* ad*-semisimple, $[x_n, x_s] = 0$ and $x = x_n + x_s$. Further, any element of $\mathfrak{g}$ which commute with x also commutes with x_s and x_n.*

Proof. Since $\mathfrak{g}$ is semisimple, the adjoint representation is faithful. In addition, by Proposition II.2.1, its image is $\mathrm{Der}_{\Bbbk}(\mathfrak{g})$. Hence, the adjoint representation induces an isomorphism of Lie algebras as follows:

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathrm{Der}_{\Bbbk}(\mathfrak{g}) \\ x & \mapsto & \mathrm{ad}_{\mathfrak{g}}(x) \end{array}.$$

Now, let $x \in \mathfrak{g}$. The endomorphism $\mathrm{ad}_{\mathfrak{g}}(x) \in \mathrm{End}_{\Bbbk}(\mathfrak{g})$ as a (usual) Jordan-Chevalley decomposition. That is, there exist endomorphisms s and n in $\mathrm{End}_{\Bbbk}(\mathfrak{g})$ with s diagonalisable, n nilpotent and such that $\mathrm{ad}_{\mathfrak{g}}(x) = s + n$ and s and n commute. But, by Proposition I.3.8, s and n must belong to $\mathrm{Der}_{\mathfrak{g}}(\mathbb{k})$. Hence, there exists $x_s, x_n \in \mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{g}}(x_s) = s$ and $\mathrm{ad}_{\mathfrak{g}}(x_n) = n$. It then follows from the properties of s an n that x_n is ad-nilpotent, x_s is ad-semisimple, $[x_n, x_s] = 0$, $x = x_n + x_s$ and that, any element which commutes with x must also commute with x_s and x_n. In addition, the unicity in the Jordan-Chevalley decomposition for endomorphisms implies the unicity of such a pair (x_s, x_n). ∎

Definition II.3.3 – *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Let $x \in \mathfrak{g}$. If (x_n, x_s) is the pair of elements of $\mathfrak{g}$ whose existence and unicity is given by Theorem II.3.8, x_n is called the nilpotent part of x, x_s the semisimple part of x and the decomposition $x = x_n + x_s$ is called the abstract Jordan-Chevalley decomposition of x.*

It will be useful latter to know how the abstract Jordan-Chevalley decomposition behaves with respect to the decomposition of a semisimple Lie algebra as the sum of its simple ideals (see Theorem I.7.19). This is the aim of the following exercise.

Exercise II.3.4 – Let $\mathfrak{g}$ be a nonzero semisimple Lie algebra, $t \in \mathbb{N}^*$, $\mathfrak{g}_1, \dots, \mathfrak{g}_t$, $t \in \mathbb{N}^*$, semisimple Lie subalgebras of $\mathfrak{g}$ such that, for all $1 \leq i \neq j \leq t$, $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$. Consider $x \in \mathfrak{g}$ and write $x = \sum_{1 \leq i \leq t} x_i$, $x_i \in \mathfrak{g}_i$ for all $1 \leq i \leq t$.
1. If x is ad-semisimple then, for all $1 \leq i \leq t$, x_i is ad-semisimple both as an element of $\mathfrak{g}$ and as an element of $\mathfrak{g}_i$.
2. If x is ad-nilpotent then, for all $1 \leq i \leq t$, x_i is ad-nilpotent both as an element of $\mathfrak{g}$ and as an element of $\mathfrak{g}_i$.
3. Let x_s be the semisimple part of x and x_n its nilpotent part. For $1 \leq i \leq t$, let $(x_i)_s$ be the semisimple part of x_i and $(x_i)_n$ its nilpotent part, as an element of $\mathfrak{g}_i$. Then, for all $1 \leq i \leq t$, $x_s = \sum_{1 \leq i \leq t} (x_i)_s$ and $x_n = \sum_{1 \leq i \leq t} (x_i)_n$.

At this stage, a first problem arises concerning the abstract Jordan-Chevalley decomposition, that we have to fix in order to avoid ambiguity. Indeed, let V be a finite dimensional vector space and let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. Given any element of $\mathfrak{g}$ we have at our disposal the usual Jordan-Chevalley decomposition of that element (that is, as an endomorphism of the vector space V) and its abstract Jordan-Chevalley decomposition (as an element of the semisimple Lie algebra $\mathfrak{g}$). This leads to the obvious question of the relationship between these decompositions. This problem is fixed by Lemma II.3.6, via the preparatory Lemma II.3.5.

Lemma II.3.5 – *Let V be a finite dimensional vector space and let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. For all $x \in \mathfrak{g}$, $\mathfrak{g}$ contains the semisimple and nilpotent part of the Jordan-Chevalley decomposition of x (as an endomorphism of $\mathrm{End}_{\mathbb{k}}(V)$).*

Proof. Let $\mathcal{V}$ be the set of subrepresentations of V (seen as a representation of $\mathfrak{g}$ by means of $\mathfrak{g} \xrightarrow{\subseteq} \mathfrak{gl}(V)$). For all $W \in \mathcal{V}$, let

$$\mathfrak{g}_W = \left\{ y \in \mathfrak{gl}(V), \mid y(W) \subseteq W),\ \mathrm{Tr}(y_{|W}) = 0 \right\}.$$

In addition, let

$$\mathfrak{g}_* = N_{\mathfrak{gl}(V)}(\mathfrak{g}) \bigcap \left(\bigcap_{W \in \mathcal{V}} \mathfrak{g}_W \right)$$

and observe that $\mathfrak{g}_*$ is a Lie subalgebra of $\mathfrak{gl}(V)$. We wish to show that $\mathfrak{g} = \mathfrak{g}_*$.

First, it is clear that, for all $W \in \mathcal{V}$, $\mathfrak{g} \subseteq \mathfrak{g}_W$, indeed, W is a subrepresentation of V (for $\mathfrak{g}$) and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ since $\mathfrak{g}$ is semisimple (see Corollary I.7.20), so that the restriction of any element of $\mathfrak{g}$ to a subrepresentation W is a commutator and, hence, has zero trace. We have shown that $\mathfrak{g} \subseteq \mathfrak{g}_*$.

We now show the converse inclusion. To start with, observe that $\mathfrak{g}$ is actually an ideal of the Lie algebra $\mathfrak{g}_*$, since $\mathfrak{g}_* \subseteq N_{\mathfrak{gl}(V)}(\mathfrak{g})$. So, by Corollary II.1.6, there exists a unique Lie ideal $\mathfrak{i}$ of $\mathfrak{g}_*$ such that

$$\mathfrak{g}_* = \mathfrak{g} \oplus \mathfrak{i}.$$

Consider $y \in \mathfrak{i}$. If W is a simple subrepresentation of V for $\mathfrak{g}$, then $y(W) \subset W$, so that y induces an endomorphism of W and, since $[y, \mathfrak{g}] \subseteq [\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i} \cap \mathfrak{g} = (0)$, $y_{|W}$ is actually an endomorphism of the representation W of $\mathfrak{g}$. By Schur's Lemma this implies that there exists $\lambda \in \mathbb{k}$ such that $y_{|W} = \lambda \mathrm{id}_W$ and, since in addition $\mathrm{Tr}(y_{|W})$ is zero, then $y_{|W} = 0$. But, by Weyl's Theorem, V is the sum of such simple representations, so $y = 0$. We have shown that $\mathfrak{i} = (0)$ and therefore that $\mathfrak{g} = \mathfrak{g}_*$.

It remains to show that $\mathfrak{g}^*$ contains the semisimple part and nilpotent part of each of its element. Let $x \in \mathfrak{g}_*$ and let d and n be its semisimple and nilpotent part, respectively (as an element of $\mathrm{End}_{\mathbb{k}}(V)$). We know that there exist polynomials D and N in $\mathbb{k}[T]$ (without constant terms) such that $d = D(x)$ and $n = N(x)$. Thus, d and n stabilise any $W \in \mathcal{V}$. In addition, since $n_{|W}$ is nilpotent, its trace is zero, and so is that of $d_{|W}$ since $d = x - n$. Hence, $n, d \in \bigcap_{W \in \mathcal{V}} \mathfrak{g}_W$. On the other hand, $x \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$; that is $\mathrm{ad}_{\mathfrak{gl}(V)}(x)(\mathfrak{g}) \subseteq \mathfrak{g}$. But, by Lemma I.6.3, $\mathrm{ad}_{\mathfrak{gl}(V)}(x) = \mathrm{ad}_{\mathfrak{gl}(V)}(d) + \mathrm{ad}_{\mathfrak{gl}(V)}(n)$ is the Jordan-Chevalley decomposition of $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$. Hence, both $\mathrm{ad}_{\mathfrak{gl}(V)}(d)$ and $\mathrm{ad}_{\mathfrak{gl}(V)}(n)$ are polynomials in $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$. Thus, $\mathrm{ad}_{\mathfrak{gl}(V)}(d)(\mathfrak{g}) \subseteq \mathfrak{g}$ and $\mathrm{ad}_{\mathfrak{gl}(V)}(n)(\mathfrak{g}) \subseteq \mathfrak{g}$. In other terms, $d, n \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$. We have proved that $d, n \in \mathfrak{g}_*$. This finishes the proof. ∎

Lemma II.3.6 – *Let V be a finite dimensional vector space and let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. For all $x \in \mathfrak{g}$, the abstract Jordan-Chevalley decomposition of x (that is, as an element of the semisimple Lie algebra $\mathfrak{g}$) coincides with its usual Jordan-Chevalley decomposition (that is, as an endomorphism of* $\mathrm{End}_{\mathbb{k}}(V)$*).*

Proof. Write $x = d + n$ the Jordan-Chevalley decomposition of x as an element of $\mathrm{End}_{\mathbb{k}}(V)$, with d semisimple and n nilpotent. We know (by Lemma I.6.3) that $\mathrm{ad}_{\mathfrak{gl}(V)}(x) = \mathrm{ad}_{\mathfrak{gl}(V)}(d) + \mathrm{ad}_{\mathfrak{gl}(V)}(n)$ is the Jordan-Chevalley decomposition of $\mathrm{ad}_{\mathfrak{gl}(V)}(x)$. We also know (Lemma II.3.5) that $d, n \in \mathfrak{g}$. Hence, $\mathrm{ad}_{\mathfrak{gl}(V)}(x), \mathrm{ad}_{\mathfrak{gl}(V)}(d), \mathrm{ad}_{\mathfrak{gl}(V)}(n) \in \mathfrak{gl}(V)$ actually leave $\mathfrak{g}$ invariant and, clearly,

$$\mathrm{ad}_{\mathfrak{gl}(V)}(x)_{|\mathfrak{g}} = \mathrm{ad}_{\mathfrak{g}}(x), \quad \mathrm{ad}_{\mathfrak{gl}(V)}(d)_{|\mathfrak{g}} = \mathrm{ad}_{\mathfrak{g}}(d) \quad \text{and} \quad \mathrm{ad}_{\mathfrak{gl}(V)}(n)_{|\mathfrak{g}} = \mathrm{ad}_{\mathfrak{g}}(n).$$

In addition, $\mathrm{ad}_{\mathfrak{g}}(d)$ is semisimple, as the restriction to $\mathfrak{g}$ of the semisimple endomorphism $\mathrm{ad}_{\mathfrak{gl}(V)}(d)$ of $\mathfrak{gl}(V)$, and similarly, $\mathrm{ad}_{\mathfrak{g}}(n)$ is nilpotent, as the restriction to $\mathfrak{g}$ of the nilpotent endomorphism $\mathrm{ad}_{\mathfrak{gl}(V)}(n)$ of $\mathfrak{gl}(V)$. So, the identity

$$\mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{g}}(d) + \mathrm{ad}_{\mathfrak{g}}(n),$$

which we deduce from the above, exactly says that $x = d + n$ is the abstract Jordan-Chevalley decomposition of x. ∎

The following Theorem shows that, actually, the abstract Jordan-Chevalley decomposition of a semisimple Lie algebra provides a universal procedure to recover the usual one on any finite dimensional representation.

We need a lemma first. Recall (see Corollary I.7.20) that any homomorphic image of a semisimple Lie algebra is semisimple.

Lemma II.3.7 – *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, $\mathfrak{i}$ be an ideal of $\mathfrak{g}$ and let $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{i}$ be the canonical projection. For all $x \in \mathfrak{g}$, if $x = x_s + x_n$ is the abstract Jordan-Chevalley decomposition of x, then $\pi(x) = \pi(x_s) + \pi(x_n)$ is the abstract Jordan-Chevalley decomposition of $\pi(x)$ in the (semisimple) Lie algebra $\mathfrak{g}/\mathfrak{i}$.*

Proof. Since $\mathfrak{i}$ is an ideal of $\mathfrak{g}$, it is a subrepresentation of the adjoint representation $(\mathfrak{g}, \mathrm{ad}_{\mathfrak{g}})$ and we have a commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}_{\mathfrak{g}}"] \arrow[d, "\pi"] & \{\varphi \in \mathfrak{gl}(\mathfrak{g}) | \varphi(\mathfrak{i}) \subseteq \mathfrak{i}\} \arrow[d, "\Pi"] & \subseteq \mathfrak{gl}(\mathfrak{g}) \\ \mathfrak{g}/\mathfrak{i} \arrow[r, "\mathrm{ad}_{\mathfrak{g}/\mathfrak{i}}"] & \mathfrak{gl}(\mathfrak{g}/\mathfrak{i}) & \end{tikzcd}$$

where Π sends an element of $\{\varphi \in \mathfrak{gl}(\mathfrak{g}) | \varphi(\mathfrak{i}) \subseteq \mathfrak{i}\}$ to the endomorphism it induces on $\mathfrak{g}/\mathfrak{i}$.

Let $x \in \mathfrak{g}$ and let $x = x_s + x_n$ be its abstract Jordan-Chevalley decomposition in $\mathfrak{g}$. By definition, $\mathrm{ad}_{\mathfrak{g}}(x_s)$ is semisimple and $\mathrm{ad}_{\mathfrak{g}}(x_n)$ is nilpotent, from which it follows that $\Pi \circ \mathrm{ad}_{\mathfrak{g}}(x_s)$ is semisimple and $\Pi \circ \mathrm{ad}_{\mathfrak{g}}(x_n)$ is nilpotent, that is $\mathrm{ad}_{\mathfrak{g}/\mathfrak{i}}(\pi(x_s))$ is semisimple and $\mathrm{ad}_{\mathfrak{g}/\mathfrak{i}}(\pi(x_n))$ is nilpotent. Of course, $[\pi(x_s), \pi(x_n)] = 0$. So, $\pi(x) = \pi(x_s) + \pi(x_n)$ is the abstract Jordan-Chevalley decomposition of $\pi(x)$ in the semisimple Lie algebra $\mathfrak{g}/\mathfrak{i}$. ∎

Theorem II.3.8 – *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and $f : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ a finite dimensional representation of $\mathfrak{g}$. For all $x \in \mathfrak{g}$, if $x = x_n + x_s$ is the abstract Jordan-Chevalley decomposition of x, $f(x_n)$ is a nilpotent endomorphism of V and $f(x_s)$ a semisimple endomorphism of V. That is, $f(x) = f(x_n) + f(x_s)$ is the usual Jordan-Chevalley decomposition of the endomorphism $f(x) : V \longrightarrow V$.*

Proof. We have the following commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f"] \arrow[d, "\pi"] & \mathfrak{gl}(V) \\ \mathfrak{g}/\ker(f) \arrow[ur, "\overline{f}"] & \end{tikzcd}$$

where π is the canonical projection and $\overline{f}$ the faithful representation of $\mathfrak{g}/\ker(f)$ induced by f. Let $x \in \mathfrak{g}$ and let $x = x_s + x_n$ be its abstract Jordan-Chevalley decomposition. By Lemma II.3.7, $\pi(x) = \pi(x_s) + \pi(x_n)$ is the abstract Jordan-Chevalley decomposition of $\pi(x)$ in the semisimple Lie algebra $\mathfrak{g}/\ker(f)$. It follows, by Lemma II.3.6, that $\overline{f} \circ \pi(x) = \overline{f} \circ \pi(x_s) + \overline{f} \circ \pi(x_n)$ is the usual Jordan-Chevalley decomposition of $\overline{f} \circ \pi(x)$ in $\mathfrak{gl}(V)$. In other words, $f(x) = f(x_s) + f(x_n)$ is the usual Jordan-Chevalley decomposition of $f(x)$ in $\mathfrak{gl}(V)$. ∎

II.4 Finite dimensional representations of $\mathfrak{sl}_2(\Bbbk)$.

Assume $\Bbbk$ is algebraically closed of characteristic 0.

Of course, the Lie algebra $\mathfrak{sl}_2(\mathbb{k})$ is of interest on its own as it is the first example of a simple Lie algebra over $\mathbb{k}$.

However, there is a much better reason to be interrested in $\mathfrak{sl}_2(\mathbb{k})$: its representation theory is the key to many subtle properties of the structure of *any* semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{k}$. This is due to the ubiquity of $\mathfrak{sl}_2(\mathbb{k})$ in any semisimple Lie algebra, as we will see via the Cartan-Chevalley decomposition. For this reason, $\mathfrak{g}$, or even any of its representation, may be considered (actually in many different ways) as a representation of $\mathfrak{sl}_2(\mathbb{k})$, which provides a lot of information.

We now describe finite dimensional representations of $\mathfrak{sl}_2(\mathbb{k})$. According to Weyl's theorem, we may concentrate on simple representations.

Recall that $\mathfrak{sl}_2(\mathbb{k})$ is the Lie subalgebra of $\mathfrak{gl}_2(\mathbb{k})$ consisting of those matrices whose trace is zero. The elements x, y and h form a basis of $\mathfrak{sl}_2(\mathbb{k})$, where:

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{et} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In addition, the following relations hold:

$$[x, y] = h, \qquad [h, x] = 2x \quad \text{and} \quad [h, y] = -2y. \tag{II.4.6}$$

Lemma II.4.1 – *Let (V, ρ) be a finite dimensional representation of $\mathfrak{sl}_2(\mathbb{k})$. The following holds:*
1. $\rho(h)$ is a semisimple endomorphism;
2. $\rho(x)$ and $\rho(y)$ are nilpotent endomorphisms.

Proof. It follows easily from relations (II.4.6) that h is ad-semisimple and x and y are ad-nilpotent. Thus, the result follows from Theorem II.3.8. ∎

Motivated by Lemma II.4.1, we make the following definition.

Definition II.4.2 *Let (V, ρ) be a representation of $\mathfrak{sl}_2(\mathbb{k})$. For all $\lambda \in \mathbb{k}$ put*

$$V_\lambda = \{v \in V \mid h.v = \lambda v\}.$$

A weight of (V, ρ) is an element $\lambda \in \mathbb{k}$ such that $V_\lambda \neq \{0\}$. If $\lambda \in \mathbb{k}$ is a weight of (V, ρ), V_λ is called the weight space of V associated to the weight λ and any element of V_λ is called a weight vector of weight λ.

Remark II.4.3 Let (V, ρ) be a representation of $\mathfrak{sl}_2(\mathbb{k})$.
1. For all weight $\lambda \in \mathbb{k}$ of (V, ρ), V_λ is just the eigenspace of $\rho(h)$ of eigenvalue λ.
2. Suppose that V is finite dimensional. By Lemma II.4.1, $V = \oplus_{\lambda \in \mathbb{k}} V_\lambda$ and the set of weights of (V, ρ) is finite.

Lemma II.4.4 – *Let (V, ρ) be a nonzero representation of $\mathfrak{sl}_2(\mathbb{k})$. Then, the following holds.*
1. For all $\lambda \in \mathbb{k}$ $\rho(x)(V_\lambda) \subseteq V_{\lambda+2}$ and $\rho(y)(V_\lambda) \in V_{\lambda-2}$.
2. Let $\lambda \in \mathbb{k}$. Suppose $v \in V_\lambda$ satisfies $\rho(x)(v) = 0$, then,

$$\forall\, t \in \mathbb{N}^*, \qquad \rho(x)(\rho(y)^t)(v) = t(\lambda - t + 1)\rho(y)^{t-1}(v).$$

3. Suppose V is finite dimensional. Then, there exists a weight $\lambda \in \mathbb{k}$ of (V, ρ) such that $V_{\lambda+2} = 0$.

Proof. Point 1 follows easily from relations (II.4.6). Point 2 is proved by an easy induction on t using relations (II.4.6) and Point 1. Point 3 is clear as V is finite dimensional. ∎

Lemma II.4.5 – *Let (V,ρ) be a nonzero finite dimensional representation of $\mathfrak{sl}_2(\Bbbk)$.*
1. There exists a weight vector $v \in V \setminus \{0\}$ in the kernel of $\rho(x)$.
2. Let v be a nonzero weight vector of weight λ in the kernel of $\rho(x)$. Put

$$v_{-1} = 0, \qquad v_0 = v, \quad \textit{and} \quad v_i = \frac{1}{i!} y^i.v, \ \forall i \in \mathbb{N}.$$

Then, for all $i \in \mathbb{N}$:
2.1. $h.v_i = (\lambda - 2i)v_i$;
2.2. $y.v_i = (i+1)v_{i+1}$;
2.3. $x.v_i = (\lambda - i + 1)v_{i-1}$.

Proof. Point 1 and 2 follow easily from Lemma II.4.4. ∎

Proposition II.4.6 – *Let $m \in \mathbb{N}$. Let (ρ, V) be a simple representation of dimension $m+1$ of $\mathfrak{sl}_2(\Bbbk)$. Let v be a nonzero weight vector of weight $\lambda \in \Bbbk$ in the kernel of $\rho(x)$ (which exists by Lemma II.4.5). For $i \in \mathbb{N}$, put $v_i = (1/i!)y^i.v$. Then, the following hold:*
1. $\{v_0, \dots, v_m\}$ is a basis of V;
2. $\lambda = m$ and $V = \bigoplus_{0 \leq i \leq m} V_{m-2i}$;
3. for $0 \leq i \leq m$, $V_{m-2i} = \Bbbk v_i$.

Proof. By Lemma II.4.5, the family $(v_i)_{i\in\mathbb{N}}$ spans a nonzero subrepresentation of V. Hence, it spans V since (V,ρ) is simple.

Point 2.1 of lemma II.4.5 shows that the v_i, $i \in \mathbb{N}$, are eigenvectors of $\rho(h)$ with eigenvalue $\lambda - 2i$. In particular, the nonzero vectors in $(v_i)_{i\in\mathbb{N}}$ have pairwise distinct eigenvalues and, hence, form a linearly independant family which, by the above is a basis of V. So, since V is finite dimensional, there exists $i \in \mathbb{N}$ such that $v_i = 0$. And, as Point 2.2 of the same Lemma shows, if v_i is zero for some $i \in \mathbb{N}$, then v_{i+1} is also zero. Therefore, the nonzero elements in $(v_i)_{i\in\mathbb{N}}$ must be $(v_0, \dots, v_m)$.

Point 2.3 of lemma II.4.5 gives $0 = x.v_{m+1} = (\lambda - m)v_m$. Hence $\lambda = m$ since $v_m \neq 0$. The rest easily follows. ∎

Proposition II.4.6 put drastic limits on finite dimensional representations of $\mathfrak{sl}_2(\Bbbk)$. It also motivates the following definition.

Let $m \in \mathbb{N}$. Let $\mathcal{B}$ be the canonical basis of $\Bbbk^{m+1}$. Motivated by Proposition II.4.6, we introduce the morphism of $\Bbbk$-vector spaces:

$$\rho_m : \mathfrak{sl}_2(\Bbbk) \longrightarrow \mathfrak{gl}\left(\Bbbk^{m+1}\right)$$

such that

$$\mathrm{Mat}_{\mathcal{B}}(\rho_m(x)) = \begin{pmatrix} 0 & m & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & m-1 & \ddots & & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & & & & & 0 & 1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix}, \ \mathrm{Mat}_{\mathcal{B}}(\rho_m(y)) = \begin{pmatrix} 0 & 0 & \dots & \dots & \dots & 0 & 0 \\ 1 & 0 & & & & & 0 \\ 0 & 2 & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & 0 \\ 0 & & & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \dots & \dots & 0 & m & 0 \end{pmatrix}$$

$$\mathrm{Mat}_{\mathcal{B}}(\rho_m(h)) = \begin{pmatrix} m & 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & m-2 & 0 & \dots & \dots & 0 & 0 \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ 0 & 0 & \dots & \dots & 0 & -m+2 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 & -m \end{pmatrix}.$$

A straightforward calculation gives the following relations in the Lie algebra $\mathfrak{gl}\left(\Bbbk^{m+1}\right)$:

$$[\rho_m(x), \rho_m(y)] = \rho_m([x,y]), \quad [\rho_m(h), \rho_m(x)] = \rho_m([h,x]), \quad \text{and} \quad [\rho_m(h), \rho_m(y)] = \rho_m([h,y]).$$

Hence, ρ_m defines a representation of the Lie algebra $\mathfrak{sl}_2(\Bbbk)$ in $\Bbbk^{m+1}$ (see Exercise I.1.22).

Lemma II.4.7 – *For all $m \in \mathbb{N}$, $(\Bbbk^{m+1}, \rho_m)$ is a simple representation of $\mathfrak{sl}_2(\Bbbk)$.*

Proof. We have to show that any nonzero subspace of $\Bbbk^{m+1}$ which is left stable by $\rho_m(h)$, $\rho_m(x)$ and $\rho_m(y)$ must equal $\Bbbk^{m+1}$.

Let V be such a vector subspace and let v be a nonzero element of v. Write $\mathcal{B} = (e_i)_{0 \le i \le m}$ for the canonical basis of $\Bbbk_{m+1}$ and $v = \sum_{0 \le i \le m} \alpha_i e_i$, $\alpha_i \in \Bbbk$, for all $0 \le i \le m$. Denote by S the support of v, that is, the subset of $\{0, \dots, m\}$ containing those i's for which α_i is nonzero. Letting h act on v, it is easy to see that, if the support of v has at least two elements, then V must contain a nonzero element whose support as cardinality equal to that of S minus 1. (This is because the eigenvalues of $\rho_m(h)$ are pairwise distinct.) From this, we deduce that V must contain an eigenvector of $\rho_m(h)$, that is: V must contain an element of $\mathcal{B}$. But then, letting $\rho_m(x)$ and $\rho_m(y)$ act on this element of $\mathcal{B}$, we see that V actually contains $\mathcal{B}$. Hence $V = \Bbbk^{m+1}$. ■

The following Theorem gives an explicit classification of simple finite dimensional representations of $\mathfrak{sl}_2(\Bbbk)$.

Theorem II.4.8 –
1. For all $m \in \mathbb{N}$, $(\Bbbk^{m+1}, \rho_m)$ is a simple representation of $\mathfrak{sl}_2(\Bbbk)$.
2. Let $m \in \mathbb{N}$. If (V, ρ) is a finite dimensional simple representation of $\mathfrak{sl}_2(\Bbbk)$ of dimension $m+1$, then (V, ρ) is isomorphic to $(\Bbbk^{m+1}, \rho(m))$.

Proof. The first point is Lemma II.4.7. Use the notation of Proposition II.4.6. It is easy to check, using Lemma II.4.5 and Proposition II.4.6, that the linear map $\Bbbk^{m+1} \mapsto V$, $e_i \mapsto v_i$ is an isomorphism of representations between $(\Bbbk^{m+1}, \rho_m)$ and (V, ρ). ■

II.5 Toral subalgebras and Cartan-Chevalley decomposition.

Assume $\Bbbk$ is algebraically closed of characteristic 0.

Let $\mathfrak{g}$ be a nonzero finite dimensional semisimple Lie algebra. Recall the adjoint representation

$$\begin{array}{cccc} \mathrm{ad}_{\mathfrak{g}} & : & \mathfrak{g} & \longrightarrow & \mathfrak{gl}(\mathfrak{g}) \\ & & x & \mapsto & \mathrm{ad}(x) \end{array},$$

which we know is faithful.

If any element of $\mathfrak{g}$ was ad-nilpotent, then Engel's Theorem would entail that $\mathfrak{g}$ is nilpotente, which is absurde since it is semisimple and nonzero. Hence, by the existence of the abstract Jordan-Chevalley decomposition, $\mathfrak{g}$ must contain nonzero ad-semisimple elements. The line generated by such an element is a (nonzero abelian) Lie subalgebra of $\mathfrak{g}$ whose elements are all semisimple.

Definition II.5.1 – *Let $\mathfrak{g}$ be a nonzero finite dimensional semisimple Lie algebra. A toral Lie subalgebra of $\mathfrak{g}$ is a nonzero Lie subalgebra of $\mathfrak{g}$ whose elements are all* ad*-semisimple.*

Lemma II.5.2 – *Let $\mathfrak{g}$ be a nonzero finite dimensional semisimple Lie algebra. Any toral Lie subalgebra of $\mathfrak{g}$ is abelian.*

Proof. Let $\mathfrak{t}$ be a toral subalgebra of $\mathfrak{g}$. By definition, for any element x of $\mathfrak{t}$, $\mathrm{ad}_{\mathfrak{g}}(x)$ is semisimple and thus so is $\mathrm{ad}_{\mathfrak{t}}(x)$ since it is its restriction to the fixed subspace $\mathfrak{t}$.

We need to show that, for all $x \in \mathfrak{t}$, $\mathrm{ad}_{\mathfrak{t}}(x)$ is zero or, equivalently, that its eigenvalues all are zero. Suppose, to the contrary, that there exists an element x of $\mathfrak{t}$ such that $\mathrm{ad}_{\mathfrak{t}}(x)$ has a nonzero eigenvalue: there exists $\lambda \in \Bbbk \setminus \{0\}$ and $y \in \mathfrak{t} \setminus \{0\}$ such that $[x, y] = \lambda y$. Now, $\mathrm{ad}_{\mathfrak{t}}(y)$ is also semisimple, so that there exists a basis $(y_1, \ldots, y_m)$ of $\mathfrak{t}$ and scalars $\lambda_1, \ldots, \lambda_m$ such that, for $1 \leq i \leq m$, $[y, y_i] = \lambda_i y_i$. Now, write $x = x_1 y_1 + \ldots + x_m y_m$, $x_1, \ldots, x_m \in \Bbbk$. Then,

$$-\lambda y = [y, x] = [y, x_1 y_1 + \ldots + x_m y_m] = x_1 [y, y_1] + \ldots + x_m [y, y_m] = x_1 \lambda_1 y_1 + \ldots + x_m \lambda_m y_m.$$

On the right hand side of the above equation, we have 0 or a linear combination of eigenvectors of $\mathrm{ad}_{\mathfrak{t}}(y)$ with nonzero eigenvalue. In contrast, on the left hand side, we have a nonzero eigenvector $\mathrm{ad}_{\mathfrak{t}}(y)$, with eigenvalue 0. This is absurd. We conclude that an element x as above does not exist. That is, $\mathfrak{t}$ is abelian. ∎

Exercise II.5.3 – Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, $t \in \mathbb{N}^*$, and, for $1 \leq i \leq t$, semisimple Lie subalgebras $\mathfrak{g}_i$ of $\mathfrak{g}$ such that $\mathfrak{g} = \oplus_{1 \leq i \leq t} \mathfrak{g}_i$ and, for all $1 \leq i \neq j \leq t$, $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$.
1. Let $g = \sum_{1 \leq i \leq t} g_i$ be an element of $\mathfrak{h}$, $g_i \in \mathfrak{g}_i$, $1 \leq i \leq t$. Then, for $1 \leq i \leq t$, $\mathfrak{h} + \Bbbk g_i$ is an abelian Lie subalgebra of $\mathfrak{g}$.
2. We have $\mathfrak{h} = \oplus_{1 \leq i \leq t} \mathfrak{h} \cap \mathfrak{g}_i$.
3. For all $1 \leq i \leq t$, $\mathfrak{h} \cap \mathfrak{g}_i$ is a maximal toral subalgebra of $\mathfrak{g}_i$.

Let $\mathfrak{g}$ be a nonzero finite dimensional semisimple Lie algebra and $\mathfrak{h}$ a maximal toral subalgebra of $\mathfrak{g}$. Put $r = \dim_{\Bbbk}(\mathfrak{h})$.

By lemma II.5.2, the elements of the finite dimensional vector space $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ pairwise commute. In addition, they all are semisimple since $\mathfrak{h}$ is toral. It follows that they are simultaneously diagonalisable: there exists a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of $\mathfrak{g}$ such that any vector in $\mathcal{B}$ is an eigenvector for all $\mathrm{ad}_{\mathfrak{g}}(h)$, $h \in \mathfrak{h}$. Now, for $1 \leq i \leq n$ and $h \in \mathfrak{h}$, put

$$[h, b_i] = \alpha_i(h) b_i.$$

It is then obvious that, for all $1 \leq i \leq n$, $\alpha_i : \mathfrak{h} \longrightarrow \Bbbk$, $h \mapsto \alpha_i(h)$ is a linear form on $\mathfrak{h}$.

Exercise II.5.4 – Retain the above notation and recall Exercise I.2.10.
For all $\alpha \in \mathfrak{h}^*$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{h}, [h, x] = \alpha(h) x\}$. Then, $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$.

Now, let $\Phi \subseteq \mathfrak{h}^*$ be the (finite) subset of $\mathfrak{h}^*$ consisting of the nonzero linear forms α such that $\mathfrak{g}_\alpha \neq 0$. Then we get that

$$\mathfrak{g} = \mathfrak{g}_0 \bigoplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right) \tag{II.5.7}$$

and $\mathfrak{h} \subseteq \mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$.

Definition II.5.5 – *In the above notation, the elements of Φ are called the roots of $(\mathfrak{g}, \mathfrak{h})$, Φ is called the set of roots of $(\mathfrak{g}, \mathfrak{h})$ and the decomposition (II.5.7) is called the Cartan-Chevalley decomposition of $(\mathfrak{g}, \mathfrak{h})$.*

Much more can be said about the Cartan-Chevalley decomposition of $\mathfrak{g}$.

We begin with an easy Lemma underlining the compatibility between this decomposition and the Lie bracket as well as the Killing form.

Lemma II.5.6 – *Retain the above notation.*
1. For $\alpha, \beta \in \mathfrak{h}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
2. For $\alpha \in \mathfrak{h}^* \setminus \{0\}$ and $x \in \mathfrak{g}_\alpha$, the endomorphism $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent.
3. If $\alpha, \beta \in \mathfrak{h}^*$ satisfy $\alpha + \beta \neq 0$, then $\kappa_{\mathfrak{g}}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

Proof. Point 1 follows at once from the Jacobi identity.

Fix $\alpha \in \mathfrak{h}^* \setminus \{0\}$. Given any $\beta \in \mathfrak{h}^*$, the set $\{\beta + i\alpha,\ i \in \mathbb{N}\}$ is infinite. Hence, it cannot lie in $\Phi \sqcup \{0\}$. This means that, for all $\beta \in \Phi \sqcup \{0\}$, there exists an integer i_β satisfying $\mathrm{ad}_{\mathfrak{g}}(x)^{i_\beta}(\mathfrak{g}_\beta) = 0$, for all $x \in \mathfrak{g}_\alpha$. Now, let $i = \max\{i_\beta,\ \beta \in \Phi \sqcup \{0\}\}$. By the Cartan-Chevalley decomposition, we have that $\mathrm{ad}_{\mathfrak{g}}(x)^i(\mathfrak{g}) = 0$, which proves point 2.

Let now $\alpha, \beta \in \mathfrak{h}^*$ and consider $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $h \in \mathfrak{h}$. By the invariance of the Killing form (see Exercise I.7.12),

$$\alpha(h)\kappa_{\mathfrak{g}}(x, y) = \kappa_{\mathfrak{g}}([h, x], y) = -\kappa_{\mathfrak{g}}(x, [h, y]) = -\beta(h)\kappa_{\mathfrak{g}}(x, y).$$

If $\alpha + \beta \neq 0$, then we may choose h such that $(\alpha + \beta)(h) \neq 0$. The above identity then gives $\kappa_{\mathfrak{g}}(x, y) = 0$. This proves point 3. ∎

Remark II.5.7 – Point 3 of Lemma II.5.6 shows that, for all $\alpha \in \mathfrak{h}^* \setminus \{0\}$, $\mathfrak{g}_\alpha$ is a totally isotropic subspace of $\mathfrak{g}$ with respect to the Killing form.

Corollary II.5.8 – *Retain the above notation.*
The restriction of the Killing form to $\mathfrak{g}_0$ is nondegenerate.

Proof. The Lie algebra $\mathfrak{g}$ is semisimple, so that $\kappa_{\mathfrak{g}}$ is nondegenerate. Suppose $x \in \mathfrak{h}$ is orthogonal to any element $y \in \mathfrak{h}$. Then, by point 3 of Lemma II.5.6, it is orthogonal to any element of $\mathfrak{g}$. Hence it is 0. ∎

The next Proposition is very important.

Proposition II.5.9 – *Retain the above notation. The following equality hold:*

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} \,|\, \forall h \in \mathfrak{h},\ [h, x] = 0\} = \mathfrak{h}.$$

Proof. Notice that, being a centraliser, $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$. Notice also the two following alternative descriptions of $\mathfrak{g}_0$ (the second one being due to the faithfulness of the adjoint representation):

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} | \mathrm{ad}_{\mathfrak{g}}(x)(\mathfrak{h}) \subseteq (0)\}; \tag{II.5.8}$$

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} | \forall h \in \mathfrak{h},\, [\mathrm{ad}_{\mathfrak{g}}(x), \mathrm{ad}_{\mathfrak{g}}(h)] = 0\}. \tag{II.5.9}$$

(a) *The Lie subalgebra $\mathfrak{g}_0$ contains the semisimple and nilpotent parts of each of its elements.*
Let $x \in \mathfrak{g}_0$. Write $x = x_s + x_n$ its abstract Jordan-Chevalley decomposition. By definition, the usual Jordan decomposition of $\mathrm{ad}_{\mathfrak{g}}(x)$ is $\mathrm{ad}_{\mathfrak{g}}(x) = \mathrm{ad}_{\mathfrak{g}}(x_s) + \mathrm{ad}_{\mathfrak{g}}(x_n)$ with $\mathrm{ad}_{\mathfrak{g}}(x_s)$ semisimple and $\mathrm{ad}_{\mathfrak{g}}(x_n)$ nilpotent. By the properties of the usual Jordan-Chevalley decomposition, we know that $\mathrm{ad}_{\mathfrak{g}}(x_s)$ and $\mathrm{ad}_{\mathfrak{g}}(x_n)$ are polynomials without constant terms of $\mathrm{ad}_{\mathfrak{g}}(x)$. So, the result follows from (II.5.8).
(b) *All the* ad*-semisimple elements of $\mathfrak{g}_0$ are in $\mathfrak{h}$.*
Let $x \in \mathfrak{g}_0$. Then, by definition of $\mathfrak{g}_0$, $\mathfrak{h} + \Bbbk x$ is a Lie subalgebra of $\mathfrak{g}$. If we suppose that x is ad-semisimple, then so are all the elements of $\mathfrak{h} + \Bbbk x$ and $\mathfrak{h} + \Bbbk x$ is toral. By maximality of $\mathfrak{h}$ among toral subalgebras of $\mathfrak{g}$, we are done.
(c) *The restriction to $\mathfrak{h}$ of the Killing form is nondegenerate.*
Let x be an element of $\mathfrak{h}$ such that $\kappa_{\mathfrak{g}}(x, \mathfrak{h}) = 0$. Consider $y \in \mathfrak{g}_0$ and write $y = y_s + y_n$ its abstract Jordan-Chevalley decomposition. By (a) and (b) above, we know that $y_s \in \mathfrak{h}$ and $y_n \in \mathfrak{g}_0$. Hence,

$$\kappa_{\mathfrak{g}}(x, y) = \kappa_{\mathfrak{g}}(x, y_n) = \mathrm{Tr}\left(\mathrm{ad}_{\mathfrak{g}}(x) \circ \mathrm{ad}_{\mathfrak{g}}(y_n)\right).$$

But, since $y_n \in \mathfrak{g}_0$, $\mathrm{ad}_{\mathfrak{g}}(y_n)$ and $\mathrm{ad}_{\mathfrak{g}}(x)$ commute and, by definition, $\mathrm{ad}_{\mathfrak{g}}(y_n)$ is a nilpotent endomorphism. It follows at once that $\mathrm{ad}_{\mathfrak{g}}(x) \circ \mathrm{ad}_{\mathfrak{g}}(y_n)$ is nilpotent and thus that its trace is 0. So, $\kappa_{\mathfrak{g}}(x, y) = 0$. We have shown that $\kappa_{\mathfrak{g}}(x, \mathfrak{g}_0) = 0$. But, by Corollary II.5.8, the restriction of the Killing form to $\mathfrak{g}_0$ is nondegenerate. So we must have $x = 0$.
(d) *The Lie algebra $\mathfrak{g}_0$ is nilpotent.*
Consider $x \in \mathfrak{g}_0$ and write $x = x_s + x_n$ its abstract Jordan-Chevalley decomposition. By (a), (b) above, $x_s \in \mathfrak{h}$ and $x_n \in \mathfrak{g}_0$. It follows that $\mathrm{ad}_{\mathfrak{g}_0}(x_s) = 0$, by definition of $\mathfrak{g}_0$. In addition, $\mathrm{ad}_{\mathfrak{g}_0}(x_n)$ being the restriction of $\mathrm{ad}_{\mathfrak{g}}(x_n)$ to $\mathfrak{g}_0$, it is nilpotent. Hence, $\mathrm{ad}_{\mathfrak{g}_0}(x) = \mathrm{ad}_{\mathfrak{g}_0}(x_n)$ is nilpotent. So, for all $x \in \mathfrak{g}_0$, $\mathrm{ad}_{\mathfrak{g}_0}(x)$ is nilpotent. By Engel's Theorem, $\mathfrak{g}_0$ is nilpotent.
(e) *One has $\mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0] = 0$.*
By the invariance of the Killing form, $\kappa_{\mathfrak{g}}(\mathfrak{h}, [\mathfrak{g}_0, \mathfrak{g}_0]) \subseteq \kappa_{\mathfrak{g}}([\mathfrak{h}, \mathfrak{g}_0], \mathfrak{g}_0) = \kappa_{\mathfrak{g}}(0, \mathfrak{g}_0) = 0$. Hence, an element of $\mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ must be an element of $\mathfrak{h}$ orthogonal to any element of $\mathfrak{h}$. By (c) it must then be zero.
(f) *The Lie algebra $\mathfrak{g}_0$ is abelian.*
By (d), $\mathfrak{g}_0$ is nilpotent. Suppose it is not abelian, then, by Exercice I.4.9, $Z(\mathfrak{g}_0) \cap [\mathfrak{g}_0, \mathfrak{g}_0] \neq (0)$. Take $x \in Z(\mathfrak{g}_0) \cap [\mathfrak{g}_0, \mathfrak{g}_0]$, $x \neq 0$. Write $x = x_s + x_n$ its abstract Jordan-Chevalley decomposition. By (a), (b) and (e) above, $x_s \in \mathfrak{h}$ and $x_n \in \mathfrak{g}_0 \setminus \{0\}$. Further, $\mathrm{ad}_{\mathfrak{g}}(x_n)$ is the nilpotent part of the usual Jordan-Chevalley decomposition of $\mathrm{ad}_{\mathfrak{g}}(x)$ and, as such, it is a polynomial without constant term of $\mathrm{ad}_{\mathfrak{g}}(x)$. So, $x_n \in Z(\mathfrak{g}_0)$. Now, for all $y \in \mathfrak{g}_0$,

$$\kappa_{\mathfrak{g}}(x_n, y) = \mathrm{Tr}\left(\mathrm{ad}_{\mathfrak{g}}(x_n) \circ \mathrm{ad}_{\mathfrak{g}}(y)\right).$$

But, x_n being nilpotent and in the center of $\mathfrak{g}_0$, $\mathrm{ad}_{\mathfrak{g}}(x_n) \circ \mathrm{ad}_{\mathfrak{g}}(y)$ must be nilpotent and thus have trace 0. This shows that x_n is a nonzero element of $\mathfrak{g}_0$, orthogonal to any element in $\mathfrak{g}_0$. By Corollary II.5.8, this is a contradiction. Hence, $\mathfrak{g}_0$ is abelian.
(g) *One has $\mathfrak{g}_0 = \mathfrak{h}$.*
Suppose, on the contrary, that $\mathfrak{h} \subset \mathfrak{g}_0$ and consider $x \in \mathfrak{g}_0 \setminus \mathfrak{h}$. Write $x = x_s + x_n$ its abstract

Jordan-Chevalley decomposition. By (a) and (b), $x_n \in \mathfrak{g}_0 \setminus \{0\}$. Take any element $y \in \mathfrak{g}_0$. By (f),

$$\kappa_{\mathfrak{g}}(x_n, y) = \mathrm{Tr}\left(\mathrm{ad}_{\mathfrak{g}}(x_n) \circ \mathrm{ad}_{\mathfrak{g}}(y)\right) = 0,$$

since $\mathrm{ad}_{\mathfrak{g}}(x_n)$ is nilpotent and commutes with $\mathrm{ad}_{\mathfrak{g}}(y)$ (same argument as used several times above). This contradicts Corollary II.5.8. ∎

Corollary II.5.10 – *Retain the above notation. The restriction to $\mathfrak{h}$ of the Killing form is nondegenerate.*

Proof. The result follows from Corollary II.5.8 and Proposition II.5.9. ∎

Notation II.5.11 – Since the restriction to $\mathfrak{h}$ of the Killing form is nondegenerate (Corollary II.5.10), we have an isomorphism of vector spaces as above:

$$\begin{array}{cccccc} \iota & : & \mathfrak{h} & \longrightarrow & \mathfrak{h}^* \\ & & h & \mapsto & \kappa_{\mathfrak{g}}(h, -) \end{array}$$

which allows to identify canonically $\mathfrak{h}$ and $\mathfrak{h}^*$. For $\alpha \in \mathfrak{h}^*$, we put $t_\alpha = \iota^{-1}(\alpha)$.

We pointed out, in the introduction to Section II.4, the ubiquity of $\mathfrak{sl}_2(\mathbb{k})$ in any semisimple Lie algebra. We are now ready to give precise statements to justify this claim.

Proposition II.5.12 – *Retain the above notation.*
1. The set Φ spans $\mathfrak{h}^*$.
2. Let $\alpha \in \mathfrak{h}^*$. If $\alpha \in \Phi$, then $-\alpha \in \Phi$.
3. Let $\alpha \in \Phi$. If $(x, y) \in \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$, then $[x, y] = \kappa_{\mathfrak{g}}(x, y)t_\alpha$.
4. Let $\alpha \in \Phi$; $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is the one dimensional subspace of $\mathfrak{h}$ spaned by t_α.
5. For $\alpha \in \Phi$, $\alpha(t_\alpha) = \kappa_{\mathfrak{g}}(t_\alpha, t_\alpha) \neq 0$.

Proof. 1. Suppose that Φ does not span $\mathfrak{h}^*$. Then, there exists $h \in \mathfrak{h} \setminus \{0\}$ on which any element of Φ vanishes. Then, for all $\alpha \in \Phi$, $[\mathfrak{g}_\alpha, h] = 0$. But, $\mathfrak{h}$ is abelian, so the existence of the Cartan-Chevalley decomposition of $\mathfrak{g}$ proves that h is in the centre of $\mathfrak{g}$. Hence a nonzero element in the center of the semisimple Lie algebra $\mathfrak{g}$; a contradiction.
2. Let $\alpha \in \Phi$. If we suppose that $-\alpha \notin \Phi$, then Point 3 of Lemma II.5.6 contradicts the nondegeneracy of the Killing form on $\mathfrak{g}$.
3. Lemma II.5.6 shows that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$. More precisely, let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, $h \in \mathfrak{h}$, then

$$\kappa_{\mathfrak{g}}([x, y], h) = \kappa_{\mathfrak{g}}(x, [y, h]) = \alpha(h)\kappa_{\mathfrak{g}}(x, y) = \kappa_{\mathfrak{g}}(t_\alpha, h)\kappa_{\mathfrak{g}}(x, y).$$

Hence, $[x, y] - \kappa_{\mathfrak{g}}(x, y)t_\alpha$ is an element of $\mathfrak{h}$, orthogonal to any element of $\mathfrak{h}$ and thus equal to zero.
4. By Point 3, it is enough to show that $\kappa_{\mathfrak{g}}(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$. But, by Point 3 of Lemma II.5.6, this must be true since otherwise $\kappa_{\mathfrak{g}}$ would be degenerate.
5. By the definition of t_α, $\alpha(t_\alpha) = \kappa_{\mathfrak{g}}(t_\alpha, t_\alpha)$. By Point 4, there exists $(x, y) \in \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ such that $[x, y] = t_\alpha$. Put

$$S = \mathrm{Span}\{x, y, t_\alpha\} \subseteq \mathfrak{g}.$$

Then S is a Lie subalgebra of $\mathfrak{g}$ of dimension 3. More precisely, we have

$$[x, y] = t_\alpha, \quad [t_\alpha, x] = \kappa_{\mathfrak{g}}(t_\alpha, t_\alpha)x \quad \text{and} \quad [t_\alpha, y] = -\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha)y.$$

If we suppose that $\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha) = 0$, then S must be solvable, as is easily verified.

Moreover, $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is injective. Hence, $\mathrm{ad}_{\mathfrak{g}}(S)$ is a solvable Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. By Lie's Theorem, it follows that there exists a full flag $\mathcal{F}$ of $\mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{g}}(S) \subseteq \mathfrak{b}_{\mathcal{F}}(\mathfrak{g})$. This forces $\mathrm{ad}_{\mathfrak{g}}(t_\alpha)$ to be a nilpotent endomorphism since it is the bracket of two elements of $\mathfrak{b}_{\mathcal{F}}(\mathfrak{g})$. But, being an element of the toral algebra $\mathfrak{h}$, t_α is ad-semisimple. Hence, $\mathrm{ad}_{\mathfrak{g}}(t_\alpha)$ is nilpotent and semisimple, hence zero, and thus $t_\alpha = 0$. But this is absurd since $t_\alpha = \iota^{-1}(\alpha)$. Hence, $\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha) \neq 0$. ■

Theorem II.5.13 – *Retain the above notation. Let $\alpha \in \Phi$.*
1. For all nonzero x_α in $\mathfrak{g}_\alpha$, there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that, if we put $h_\alpha = [x_\alpha, y_\alpha]$, the subspace $\mathrm{Span}\{x_\alpha, y_\alpha, h_\alpha\}$ *is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\Bbbk)$ via $x_\alpha \mapsto x$, $y_\alpha \mapsto y$, $h_\alpha \mapsto h$.*
2. Moreover, for any pair $(x_\alpha, y_\alpha) \in \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ such that, putting $h_\alpha = [x_\alpha, y_\alpha]$, the subspace $\mathrm{Span}\{x_\alpha, y_\alpha, h_\alpha\}$ *is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\Bbbk)$ via $x_\alpha \mapsto x$, $y_\alpha \mapsto y$, $h_\alpha \mapsto h$, then* $h_\alpha = \dfrac{2}{\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha)} t_\alpha$.

Proof. 1. Let x_α in $\mathfrak{g}_\alpha \setminus \{0\}$. By Point 3 of Lemma II.5.6 and the nondegenerescy of $\kappa_{\mathfrak{g}}$, $\kappa_{\mathfrak{g}}(x_\alpha, \mathfrak{g}_{-\alpha}) \neq 0$. By Point 3 of Proposition II.5.12, $\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha) \neq 0$. Hence, we may chose $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\kappa_{\mathfrak{g}}(x_\alpha, y_\alpha) = \dfrac{2}{\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha)}$. Put $h_\alpha = [x_\alpha, y_\alpha]$. We then have $[x_\alpha, y_\alpha] = \dfrac{2}{\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha)} t_\alpha$ (by Proposition II.5.12). It is easy to verify that $\mathrm{Span}\{x_\alpha, y_\alpha, h_\alpha\}$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\Bbbk)$ via $x_\alpha \mapsto x$, $y_\alpha \mapsto y$, $h_\alpha \mapsto h$.
2. Let $(x_\alpha, y_\alpha) \in \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ be as in the statement. Then, putting $h_\alpha = [x_\alpha, y_\alpha]$, we must have

$$[x_\alpha, y_\alpha] = h_\alpha, \quad [h_\alpha, x_\alpha] = 2x_\alpha \quad \text{and} \quad [h_\alpha, y_\alpha] = -2y_\alpha.$$

By Point 3 of Proposition II.5.12, we have $h_\alpha = [x_\alpha, y_\alpha] = \kappa_{\mathfrak{g}}(x_\alpha, y_\alpha) t_\alpha$. On the other hand, $2x_\alpha = [h_\alpha, x_\alpha] = \alpha(h_\alpha) x_\alpha$. Hence, $2 = \alpha(h_\alpha) = \alpha(\kappa_{\mathfrak{g}}(x_\alpha, y_\alpha) t_\alpha) = \kappa_{\mathfrak{g}}(x_\alpha, y_\alpha)\alpha(t_\alpha) = \kappa_{\mathfrak{g}}(x_\alpha, y_\alpha)\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha)$. So, indeed, $h_\alpha = \dfrac{2}{\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha)} t_\alpha$. ■

Theorem II.5.13 stresses the ubiquity of $\mathfrak{sl}_2(\Bbbk)$ in the semisimple Lie algebra $\mathfrak{g}$. However, it is not completely satisfactory as it establishes an existence without any kind of uniqueness.

Actually, we can go further using this existence and the representation theory of $\mathfrak{sl}_2(\Bbbk)$. This leads to the following statement which is a crucial step to the study of the representations of $\mathfrak{g}$.

Remark II.5.14 – Fix an element $\alpha \in \Phi$.

As Theorem II.5.13 allows us to do, consider a pair $(x_\alpha, y_\alpha) \in \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ such that, putting $h_\alpha = [x_\alpha, y_\alpha]$ and $S_\alpha = \mathrm{Span}\{x_\alpha, y_\alpha, h_\alpha\}$, we get a Lie algebra isomorphism

$$\begin{array}{rccc} i_\alpha : & S_\alpha & \xrightarrow{\sim} & \mathfrak{sl}_2 \\ & x_\alpha & \mapsto & x \\ & y_\alpha & \mapsto & y \\ & h_\alpha & \mapsto & h \end{array}.$$

Recall further that $h_\alpha = \dfrac{2}{\kappa_{\mathfrak{g}}(t_\alpha, t_\alpha)} t_\alpha$. The adjoint representation $(\mathfrak{g}, \mathrm{ad}_{\mathfrak{g}})$ then induces a representation

$$r_\alpha : S_\alpha \longrightarrow \mathfrak{gl}(\mathfrak{g}).$$

The analysis of the representation $(\mathfrak{g}, r_\alpha)$ (which is a representation of $\mathfrak{sl}_2(\Bbbk)$, up to the isomorphism i_α), on the basis of the results in Section II.4, provides substancial information on the Cartan-Chevalley decomposition of $\mathfrak{g}$, as we now proceed to show.

Proposition II.5.15 – *Retain the above notation. Let $\alpha \in \Phi$.*
1. The only scalar multiples of α in Φ are $\pm\alpha$.
2. We have $\dim_\Bbbk(\mathfrak{g}_\alpha) = 1$.
3. In particular, the Lie subalgebra S_α of Remark II.5.14 is the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. Moreover, in the notation of Theorem II.5.13, Point 1, the element y_α is uniquely determined by the choice of x_α.

Proof. Let $\alpha \in \Phi$. Fix a pair $(x_\alpha, y_\alpha) \in \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ as in Remark II.5.14.

Let us consider the subspace $\mathfrak{g}(\alpha) = \mathfrak{h} \bigoplus \left(\bigoplus_{c \in \Bbbk,\, c\alpha \in \Phi} \mathfrak{g}_{c\alpha} \right)$ of $\mathfrak{g}$. It follows from Lemma II.5.6 that $\mathfrak{g}(\alpha)$ is a subrepresentation of $(\mathfrak{g}, r_\alpha)$.

The kernel $\ker(\alpha)$ of the linear form α is an hyperplane of $\mathfrak{h}$ on which S_α acts trivially, so that any choice of basis of $\ker(\alpha)$ gives a decomposition of this subrepresentation as the direct sum of $\dim(\mathfrak{h}) - 1$ lines which are all subrepresentations of $\mathfrak{g}(\alpha)$, each of which is isomorphic to the one dimensional representation of S_α. In addition, the subspace S_α of $\mathfrak{g}(\alpha)$ is also a subrepresentation of $(\mathfrak{g}(\alpha), r_\alpha)$, isomorphic to the 3-dimensional simple representation of S_α. Since $\mathfrak{h} = \ker(\alpha) \oplus \Bbbk h_\alpha$, we get that

$$\mathfrak{h} \oplus \Bbbk x_\alpha \oplus \Bbbk y_\alpha$$

is a subrepresentation of $(\mathfrak{g}(\alpha), r_\alpha)$, which decomposes as the direct sum of $\dim(\mathfrak{h}) - 1$ copies of the one dimensional and one copy of the 3-dimensional representations of S_α. From this, it follows that the eigenvalues of $r_\alpha(h_\alpha)$ on the subrepresentation $\mathfrak{h} \oplus \Bbbk x_\alpha \oplus \Bbbk y_\alpha$ are 0 (with multiplicity $\dim_\Bbbk(\mathfrak{h})$) and 2 and -2 (with multiplicity 1).

By Weyl's Theorem, there exists a subrepresentation W of $\mathfrak{g}(\alpha)$ such that

$$\mathfrak{g}(\alpha) = (\mathfrak{h} \oplus \Bbbk x_\alpha \oplus \Bbbk y_\alpha) \bigoplus W,$$

and W decomposes as a direct sum of simple subrepresentations.

Suppose now that $2\alpha \in \Phi$. Then, any nonzero element x of $\mathfrak{g}_{2\alpha}$ is an eigenvector of eigenvalue 4 of $r_\alpha(h_\alpha)$, indeed: $r_\alpha(h_\alpha)(x) = [h_\alpha, x] = 2\alpha(h_\alpha)x = 4x$. But, as the restriction of r_α to $\mathfrak{h} \oplus \Bbbk x_\alpha \oplus \Bbbk y_\alpha$ has eigenvalues $-2, 0, 2$, then we must have $x \in W$. This forces the existence of a copy of an odd dimensional simple representation of S_α in the above decomposition of W and, has a consequence, the existence in W of an eigenvector of $r_\alpha(h_\alpha)$ of eigenvalue 0. Now, clearly, the eigenspace of eigenvalue zero of the restriction of $r_\alpha(h_\alpha)$ to $\mathfrak{h} \oplus \Bbbk x_\alpha \oplus \Bbbk y_\alpha$ is $\mathfrak{h}$. So, all in all, the eigenspace of eigenvalue zero of the restriction of $r_\alpha(h_\alpha)$ to $\mathfrak{g}(\alpha)$ must be of dimension at least equal to $\dim_\Bbbk(\mathfrak{h}) + 1$. On the other hand, $\mathfrak{g}(\alpha) = \mathfrak{h} \bigoplus \left(\bigoplus_{c \in \Bbbk,\, c\alpha \in \Phi} \mathfrak{g}_{c\alpha} \right)$, from which it follows at once that the latter eigenspace must be $\mathfrak{h}$ since

$$\mathfrak{g}_{c\alpha} \subseteq \ker\left(r_\alpha(h_\alpha) - 2c\mathrm{id}\right). \tag{II.5.10}$$

This is a contradiction. At this stage, we have proved that, if $\alpha \in \Phi$, then $2\alpha \notin \Phi$. Notice that the above argument also shows that W cannot contain a copy of an odd dimensional simple representation of S_α.

Of course, it follows from the above intermediate conclusion that, if $\alpha \in \Phi$, then $(1/2)\alpha \notin \Phi$. This excludes 1 from the list of eigenvalues of the restriction of $r_\alpha(h_\alpha)$ to $\mathfrak{g}(\alpha)$, by (II.5.10). This shows that a decomposition of W as a direct sum of simple representation of S_α contains no copy of an even dimensional representation.

Putting all this together, we conclude that $W = (0)$, that is $\mathfrak{g}(\alpha) = \mathfrak{h} \oplus \Bbbk x_\alpha \oplus \Bbbk y_\alpha$, which proves Points 1 and 2 of the statement.

Point 3 follows easily. ■

Proposition II.5.16 – *Retain the above notation. Let $\alpha, \beta \in \Phi$.*
1. We have $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$.
2. Suppose $\beta \neq \pm\alpha$. Put $q = \max\{i \in \mathbb{Z} \,|\, \beta + i\alpha \in \Phi\}$ and $r = \max\{i \in \mathbb{Z} \,|\, \beta - i\alpha \in \Phi\}$ (hence $q, r \geq 0$). Then, for all $-r \leq i \leq q$, $\beta + i\alpha \in \Phi$ and $\beta(h_\alpha) = r - q$.
3. If $\alpha + \beta \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. Let $\alpha, \beta \in \Phi$.

If $\beta = \pm\alpha$, the statements of Point 1 have already been proved. Hence, from now on, we assume $\beta \neq \pm\alpha$.

Put $\mathfrak{g}(\alpha, \beta) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha} \subseteq \mathfrak{g}$ and recall from Proposition II.5.15 that this is a decomposition of $\mathfrak{g}(\alpha, \beta)$ as a direct sum of lines. By Lemma II.5.6 and in the notation of Remark II.5.14, $\mathfrak{g}(\alpha, \beta)$ is a subrepresentation of $(\mathfrak{g}, r_\alpha)$. Now, we have that

$$\forall i \in \mathbb{Z}, \quad \mathfrak{g}_{\beta + i\alpha} \subseteq \ker\left(r_\alpha(h_\alpha) - (\beta(h_\alpha) + 2i)\mathrm{id}\right)$$

and, since the elements $\beta(h_\alpha) + 2i$, $i \in \mathbb{Z}$, are clearly pairwise distinct, the above decomposition is the decompositon of $\mathfrak{g}(\alpha, \beta)$ into eigenspaces of the restriction of $r_\alpha(h_\alpha)$ to $\mathfrak{g}(\alpha, \beta)$. Now, by Weyl's Theorem and the classification of simple representations of $\mathfrak{sl}_2(\Bbbk)$, the scalars $\beta(h_\alpha) + 2i$, $i \in \mathbb{Z}$ must be integers and, since these scalars must be all odd or all even, we have that $\mathfrak{g}(\alpha, \beta)$ is an irreducible representation of S_α.

Put

$$q = \max\{i \in \mathbb{Z} \,|\, \beta + i\alpha \in \Phi\} \qquad \text{and} \qquad r = \max\{i \in \mathbb{Z} \,|\, \beta - i\alpha \in \Phi\}.$$

Then $q, r \geq 0$. The form of the simple representations of $\mathfrak{sl}_2(\Bbbk)$ then provides the following. First, the set of eigenvalues of the restriction of $r_\alpha(h_\alpha)$ to $\mathfrak{g}(\alpha, \beta)$ must be $\{\beta(h_\alpha) + 2i, -r \leq i \leq q\}$, which implies that, for all $-r \leq i \leq q$, $\beta + i\alpha \in \Phi$. Second, $-(\beta(h_\alpha) - 2r) = \beta(h_\alpha) + 2q$, which implies $\beta(h_\alpha) = r - q$.

Now, since $q, r \geq 0$, we have $-r \leq q - r \leq q$, so that $\beta - \beta(h_\alpha)\alpha = \beta + (q - r)\alpha \in \Phi$.

Suppose in addition that $\alpha + \beta \in \Phi$. Then, $\beta(h_\alpha)$ is not maximal among eigenvalues of the restriction of $r_\alpha(h_\alpha)$ to $\mathfrak{g}(\alpha, \beta)$, since $\mathfrak{g}_{\alpha+\beta}$ is a nonzero eigenspace of eigenvalue $\beta(h_\alpha) + 2$. Now, using the explicit description of the irreducible representations of $\mathfrak{sl}_2(\Bbbk)$, it follows that the action of x_α on $\mathfrak{g}_\beta$ is nonzero. That is: $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$. Since, on the other hand, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ (cf. Lemma II.5.6) and $\dim_\Bbbk(\mathfrak{g}_{\alpha+\beta}) = 1$ (cf. Proposition II.5.15), we get that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. ■

Proposition II.5.17 – *The set $\sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$ generates $\mathfrak{g}$ as a Lie algebra.*

Proof. Since Φ spans $\mathfrak{h}^*$, it is enough to show that, for all $\alpha \in \Phi$, h_α is in the Lie subalgebra of $\mathfrak{g}$ generated by $\sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$. But this is clear since $h_\alpha = [x_\alpha, y_\alpha]$. ■

The following Exercise gives an explicit expression for the restriction to $\mathfrak{h}$ of the Killing form. It will be useful latter.

Exercise II.5.18 – If $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra and $\mathfrak{h}$ a maximal toral subalgebra. Then the following holds:
1. for $\alpha \in \Phi$, $h, k \in \mathfrak{h}$ and $x \in \mathfrak{g}_\alpha$, $\mathrm{ad}(h) \circ \mathrm{ad}(k)(x) = \alpha(h)\alpha(k)x$.
2. for $h, k \in \mathfrak{h}$, $\kappa_\mathfrak{g}(h, k) = \sum_{\alpha \in \Phi} \alpha(h)\alpha(k)$.

II.6 Emergence of the root system of a semisimple Lie algebra.

Assume $\Bbbk$ is algebraically closed of characteristic 0.

At this point, it is time to introduce the root system underlying any semisimple Lie algebra. It is a combinatorial tool whose importance is central. It has two applications. First, it will allow us to classify finite dimensional simple Lie algebras (and, as a consequence, finite dimensional semisimple Lie algebras). It will also allow to go deeper into the structure of the Cartan-Chevalley decomposition by exhibiting a *positive* and a *negative* part which will turn out very important latter on.

Fix a finite dimensional semisimple Lie algebra $\mathfrak{g}$ and a maximal toral subalgebra $\mathfrak{h}$.

Recall the Cartan-Chevalley decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right\}$ where, for $\alpha \in \mathfrak{h}^*$, we put $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, \forall h \in \mathfrak{h},\ [h, x] = \alpha(h)x\}$ and $\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \,|\, \mathfrak{g}_\alpha \neq (0)\}$. Recall also that $\mathfrak{g}_0 = \mathfrak{h}$.

It is useful to tranfer to $\mathfrak{h}^*$ the restriction to $\mathfrak{h}$ of the Killing form on $\mathfrak{g}$. Recall that the restriction to $\mathfrak{h}$ of $\kappa_{\mathfrak{g}}$ being nondegenerate, we have an isomorphisme of vector spaces

$$\begin{array}{rccc} \iota & : & \mathfrak{h} & \longrightarrow & \mathfrak{h}^* \\ & & h & \mapsto & \kappa_{\mathfrak{g}}(h, -) \end{array}$$

We may then transfer $(\kappa_{\mathfrak{g}})_{|\mathfrak{h}\times\mathfrak{h}}$ to $\mathfrak{h}^*$ as follows

$$\begin{array}{rccc} (-,-) & : & \mathfrak{h}^*\times\mathfrak{h}^* & \longrightarrow & \Bbbk \\ & & (\alpha,\beta) & \mapsto & \kappa_{\mathfrak{g}}(\iota^{-1}(\alpha), \iota^{-1}(\beta)) \end{array}.$$

Recall that, for $\alpha \in \mathfrak{h}^*$, we put: $t_\alpha = \iota^{-1}(\alpha)$. Since, by Proposition II.5.12, Φ generates $\mathfrak{h}^*$, the form $(-,-)$ is then completely described by:

$$\forall\,\alpha,\beta \in \Phi, \quad (\alpha,\beta) = \kappa_{\mathfrak{g}}(t_\alpha, t_\beta).$$

At this stage, notice that Point 1 of Proposition II.5.16 shows that,

$$\forall\,\alpha,\beta \in \Phi, \quad \beta(h_\alpha) = \beta\left(\frac{2}{\kappa_{\mathfrak{g}}(t_\alpha,t_\alpha)}t_\alpha\right) = 2\frac{\kappa_{\mathfrak{g}}(t_\beta,t_\alpha)}{\kappa_{\mathfrak{g}}(t_\alpha,t_\alpha)} = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}. \tag{II.6.11}$$

To go further, let us give a more detailled account on the way Φ spans $\mathfrak{h}^*$.

Let $\{\alpha_1,\dots,\alpha_\ell\} \subseteq \Phi$ be a basis of $\mathfrak{h}^*$. If α is an element of Φ, there exist $c_i \in \Bbbk$, $1 \le i \le \ell$, such that $\alpha = \sum_{1\le i\le\ell} c_i\alpha_i$. For $1 \le j \le \ell$, we thus have the following equation:

$$\frac{2(\alpha,\alpha_j)}{(\alpha_j,\alpha_j)} = \sum_{1\le i\le\ell} c_i\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}.$$

The above ℓ equations show that $(c_1,\dots,c_\ell)$ is a solution to a $\ell\times\ell$ linear system of equations whose coefficients are in $\mathbb{Z}$, by (II.6.11). But, the matrix of this system is, up to left multiplication by an obviously invertible matrix, the matrix of the nondegenerate form $(-,-)$ in the basis $\{\alpha_1,\dots,\alpha_\ell\}$. So, it has to be invertible. From this, using Kramer's formula, we get that $c_i \in \mathbb{Q}$, for all $1 \leq i \leq \ell$. We have shown that any element of Φ is a linear combination with rational coefficients of the elements of $\{\alpha_1, \dots, \alpha_\ell\}$. Now, put

$$\mathsf{E}_{\mathbb{Q}} = \mathrm{Span}_{\mathbb{Q}}(\Phi) \subseteq \mathfrak{h}^*.$$

The above discusion shows that $\mathsf{E}_{\mathbb{Q}} = \mathrm{Span}_{\mathbb{Q}}\{\alpha_1, \dots, \alpha_\ell\}$, so that:

$$\dim_{\mathbb{Q}}(\mathsf{E}_{\mathbb{Q}}) = \dim_{\Bbbk}(\mathfrak{h}^*). \tag{II.6.12}$$

On the other hand, recall that, for all for $h, k \in \mathfrak{h}$, $\kappa_{\mathfrak{g}}(h,k) = \sum_{\gamma \in \Phi} \gamma(h)\gamma(k)$, by Exercise II.5.18. Translated into $\mathfrak{h}^*$, this gives that,

$$\forall\, \alpha, \beta \in \mathfrak{h}^*, \quad (\alpha, \beta) = \sum_{\gamma \in \Phi} (\gamma, \alpha)(\gamma, \beta). \tag{II.6.13}$$

Applying (II.6.13) with $\beta = \alpha$, and taking (II.6.11) into account, it follows that

$$\forall\, \beta \in \Phi, \quad \frac{1}{(\beta, \beta)} = \sum_{\gamma \in \Phi} \frac{(\gamma, \beta)^2}{(\beta, \beta)^2} \in \mathbb{Q}. \tag{II.6.14}$$

Using (II.6.11) again, we end up with

$$\forall\, \alpha, \beta \in \Phi, \quad (\alpha, \beta) \in \mathbb{Q}. \tag{II.6.15}$$

This shows that the restriction of $(-,-)$ to $\mathsf{E}_{\mathbb{Q}} \times \mathsf{E}_{\mathbb{Q}}$ takes its values in $\mathbb{Q}$. Hence, we have endowed $\mathsf{E}_{\mathbb{Q}}$ with a symmetric bilinear form

$$(-,-)_{\mathbb{Q}} \quad : \quad \mathsf{E}_{\mathbb{Q}} \times \mathsf{E}_{\mathbb{Q}} \quad \longrightarrow \quad \mathbb{Q}$$

which is nondegenerate since $\mathsf{E}_{\mathbb{Q}}$ contains a basis of the $\Bbbk$-vector space $\mathfrak{h}^*$ and the bilinear form $(-,-)$ on $\mathfrak{h}^*$ is. Actually, (II.6.13) gives

$$\forall\, \alpha, \beta \in \mathsf{E}_{\mathbb{Q}}, \quad (\alpha, \beta)_{\mathbb{Q}} = \sum_{\gamma \in \Phi} (\gamma, \alpha)_{\mathbb{Q}}(\gamma, \beta)_{\mathbb{Q}}, \tag{II.6.16}$$

from which it follows easily that $(-,-)_{\mathbb{Q}}$ is positive definite.

The picture will be perfect with a last small effort. Consider the $\mathbb{R}$-vector space

$$\mathsf{E}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} \mathsf{E}_{\mathbb{Q}},$$

the canonical, $\mathbb{Q}$-linear, injective map $\mathsf{E}_{\mathbb{Q}} \longrightarrow \mathsf{E}_{\mathbb{R}}$, $\phi \mapsto 1 \otimes \phi$, and identify $\mathsf{E}_{\mathbb{Q}}$ to a $\mathbb{Q}$-subspace of $\mathsf{E}_{\mathbb{R}}$ by means of this map. It is not difficult to show that there exists a symmetric $\mathbb{R}$-bilinear map

$$(-,-)_{\mathbb{R}} \quad : \quad \mathsf{E}_{\mathbb{R}} \times \mathsf{E}_{\mathbb{R}} \quad \longrightarrow \quad \mathbb{R}\ ,$$

such that, under the above identification:

$$\forall\, \alpha, \beta \in \mathsf{E}_{\mathbb{Q}}, \quad (\alpha, \beta)_{\mathbb{R}} = (\alpha, \beta)_{\mathbb{Q}}.$$

Now, by construction, Φ (identified with a subset of $\mathsf{E}_{\mathbb{R}}$) generates the $\mathbb{R}$-vector space $\mathsf{E}_{\mathbb{R}}$ and (II.6.16) gives:

$$\forall\, \alpha, \beta \in \Phi, \quad (\alpha, \beta)_{\mathbb{R}} = \sum_{\gamma \in \Phi} (\gamma, \alpha)_{\mathbb{R}}(\gamma, \beta)_{\mathbb{R}}, \tag{II.6.17}$$

which entails (by bilinearity)

$$\forall\, \alpha, \beta \in \mathsf{E}_{\mathbb{R}}, \quad (\alpha, \beta)_{\mathbb{R}} = \sum_{\gamma \in \Phi} (\gamma, \alpha)_{\mathbb{R}} (\gamma, \beta)_{\mathbb{R}}, \tag{II.6.18}$$

so that $(-,-)_{\mathbb{R}}$ is positive definite. In other words, the pair $(\mathsf{E}_{\mathbb{R}}, (-,-)_{\mathbb{R}})$ is a euclidean space.

Summing up, we have the following statement.

Theorem II.6.1 – *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{h}$, Φ be the associated set of roots and $(\mathsf{E}_{\mathbb{R}}, (-,-)_{\mathbb{R}})$ be the euclidean space attached to these data. The following holds.*
1. The $\mathbb{R}$-vector space $\mathsf{E}_{\mathbb{R}}$ is generated by Φ and $0 \notin \Phi$.
2. Let $\alpha \in \Phi$. The only scalar multiple of α in Φ (seen as a subset of $\mathsf{E}_{\mathbb{R}}$) are $\pm\alpha$.
3. For $\alpha, \beta \in \Phi$, $2\dfrac{(\alpha,\beta)_{\mathbb{R}}}{(\alpha,\alpha)_{\mathbb{R}}} \in \mathbb{Z}$.
4. For $\alpha, \beta \in \Phi$, $\beta - 2\dfrac{(\alpha,\beta)_{\mathbb{R}}}{(\alpha,\alpha)_{\mathbb{R}}}\alpha \in \Phi$.

Proof. Point 1 is clear, by construction of $\mathsf{E}_{\mathbb{R}}$ and definition of Φ. Points 3 and 4 are contained in Proposition II.5.16, as we already noticed.

Now, Point 2 claims that, for all $\alpha \in \Phi$, $\{\pm\alpha\} = \mathbb{R}\alpha \cap \Phi \subseteq \mathsf{E}_{\mathbb{R}}$.

Let $\alpha \in \Phi$ and $\lambda \in \mathbb{R}$ such that $\lambda\alpha \in \Phi$. Choose a basis of $\mathsf{E}_{\mathbb{Q}}$ which contains α. As $\lambda\alpha \in \Phi \subseteq \mathsf{E}_{\mathbb{Q}}$, it may be written in a unique way as a linear combination with coefficients in $\mathbb{Q}$ of the elements of this basis. This clearly implies that $\lambda \in \mathbb{Q}$. So, $\lambda\alpha \in \Phi \cap \mathbb{Q}\alpha \subseteq \Phi \cap \Bbbk\alpha \subseteq \mathfrak{h}^*$. Therefore, Proposition II.5.15 leads to $\lambda = \pm 1$. ∎

In the vocabulary of Part III, Theorem II.6.1 states that Φ, seen as a subset of the euclidean space $\mathsf{E}_{\mathbb{R}}$, is a root system (see Definition III.2.1).

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. By Theorem I.7.19, $\mathfrak{g}$ enjoys a decomposition as a direct sum of its simple ideals. By Theorem II.6.1, $\mathfrak{g}$ enjoys a Cartan-Chevalley decomposition (associated to the choice of a maximal toral subalgebra). The aim of the following Remark is to investigate the compatibility between these two decompositions.

Remark II.6.2 – Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Let $t \in \mathbb{N}^*$ and, for $1 \leq i \leq t$, let $\mathfrak{g}_i$ be semisimple Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \oplus_{1 \leq i \leq t} \mathfrak{g}_i$ and, for all $1 \leq i \neq j \leq t$, $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$ and let κ be the Killing form of $\mathfrak{g}$.
1. Recall from Exercise II.5.3 that, if we put $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$, $1 \leq i \leq t$, then we have

$$\mathfrak{h} = \oplus_{1 \leq i \leq t} \mathfrak{h}_i$$

and $\mathfrak{h}_i$ is a maximal toral subalgebra of $\mathfrak{g}_i$, $1 \leq i \leq t$.
2. Let $\Phi \subseteq \mathfrak{h}^*$ be the set of roots associated to the pair $(\mathfrak{g}, \mathfrak{h})$. Let $\alpha \in \Phi$. Since $\alpha \neq 0$, we cannot have $\alpha(\mathfrak{h}_i) = 0$ for all $1 \leq i \leq t$. Let $1 \leq i \leq t$ be such that $\alpha(\mathfrak{h}_i) \neq 0$. Then, consider $x = \sum_{1 \leq i \leq t} x_i \in \mathfrak{g}_\alpha$, $x_i \in \mathfrak{g}_i$, $1 \leq i \leq t$. Consider $h \in \mathfrak{h}_i$ such that $\alpha(h) \neq 0$, We have that

$$\mathfrak{g}_i \ni [h, x_i] = [h, x] = \alpha(h)x$$

and hence, $x \in \mathfrak{g}_i$. We have shown that $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$. Since in addition, the sum of the $\mathfrak{g}_i$, $1 \leq i \leq t$, is direct, such an i must be unique.

We have shown that, for all $\alpha \in \Phi$, there exists a unique $1 \leq i \leq t$, such that $\alpha(\mathfrak{g}_i) \neq 0$ and that $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$.
3. In the light of Point 2, for all $1 \leq i \leq t$, we let Φ_i be the subset of Φ of those elements $\alpha \in \Phi$ such that $\alpha(\mathfrak{h}_i) \neq 0$ and $\alpha(\mathfrak{h}_j) = 0$ whenever $j \neq i$. By Point 2, we have that

$$\Phi = \bigsqcup_{1 \leq i \leq t} \Phi_i. \tag{II.6.19}$$

In addition, we have

$$\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right) = \bigoplus_{1 \leq i \leq t} \left(\mathfrak{h}_i \bigoplus \left(\bigoplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha \right) \right).$$

But, for $1 \leq i \leq t$, we have $\mathfrak{h}_i \oplus (\oplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha) \subseteq \mathfrak{g}_i$. Hence, since $\mathfrak{g} = \oplus_{1 \leq i \leq t} \mathfrak{g}_i$, we end up with

$$\forall\, 1 \leq i \leq t, \quad \mathfrak{g}_i = \mathfrak{h}_i \bigoplus \left(\bigoplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha \right). \tag{II.6.20}$$

Actually, the above equality gives the Cartan-Chevalley decomposition of the pair $(\mathfrak{g}_i, \mathfrak{h}_i)$ as we now proceed to show. Let $1 \leq i \leq t$, $\alpha \in \Phi_i$, then

$$\begin{array}{rcl}
\mathfrak{g}_\alpha & = & \{x \in \mathfrak{g} \,|\, \forall\, h \in \mathfrak{h},\ [h,x] = \alpha(h)x\} \\
& = & \{x \in \mathfrak{g}_i \,|\, \forall\, h \in \mathfrak{h},\ [h,x] = \alpha(h)x\} \\
& = & \{x \in \mathfrak{g}_i \,|\, \forall\, h \in \mathfrak{h}_i,\ [h,x] = \alpha(h)x\} \\
& = & \{x \in \mathfrak{g}_i \,|\, \forall\, h \in \mathfrak{h}_i,\ [h,x] = \alpha_{|\mathfrak{h}_i}(h)x\} \\
& = & (\mathfrak{g}_i)_{\alpha_{|\mathfrak{h}_i}}.
\end{array}$$

Indeed, the second equality follows from $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_i$. As to the third, the inclusion $\subseteq$ is trivial, and the inclusion $\supseteq$ follows from the fact that $[\mathfrak{h}_j, \mathfrak{g}_i] = 0$ and $\alpha(\mathfrak{h}_j) = 0$ whenever $j \neq i$. The other equalities are clear.

Hence, (II.6.20) is in fact the Cartan-Chevalley decomposition of the pair $(\mathfrak{g}_i, \mathfrak{h}_i)$, and the set $\{\alpha_{|\mathfrak{h}_i},\ \alpha \in \Phi_i\}$ is the correponding set of roots.
4. Let $1 \leq i \neq j \leq t$. It is clear that $\kappa(\mathfrak{h}_i, \mathfrak{h}_j) = 0$, hence, $\oplus_{i \neq j} \mathfrak{h}_j \subseteq (\mathfrak{h}_i)^\perp$. Let now h be an element of $(\mathfrak{h}_i)^\perp$ and write $h = \sum_{1 \leq j \leq t} h_j$, $h_j \in \mathfrak{h}_j$, $1 \leq j \leq t$. Then, clearly, h_i is also in $(\mathfrak{h}_i)^\perp$. It follows that h_i is in $\mathfrak{h}^\perp$. Hence, κ being nondegenerate, $h_i = 0$. We have proved that $\oplus_{i \neq j} \mathfrak{h}_j = (\mathfrak{h}_i)^\perp$ or, equivalently κ being nondegenerate:

$$\left(\bigoplus_{i \neq j} \mathfrak{h}_j \right)^\perp = \mathfrak{h}_i. \tag{II.6.21}$$

Now, recall that, κ being nondegenerate on $\mathfrak{h}$, it induces a nondegenrate symmetric bilinear form $(-,-)$ on $\mathfrak{h}^*$ under the canonical identification $\mathfrak{h} \longrightarrow \mathfrak{h}^*$, $x \mapsto \kappa(x,-)$. Recall also that we denote t_α the element of $\mathfrak{h}$ corresponding to $\alpha \in \Phi$ under this identification, so that $\alpha = \kappa(t_\alpha, -)$. It follows that, if $\alpha \in \Phi_i$, $1 \leq i \leq t$, then for $j \neq i$, we must have $t_\alpha \in (\mathfrak{g}_j)^\perp$, as α vanishes on $\mathfrak{h}_j$. So, by (II.6.21), $t_\alpha \in \mathfrak{h}_i$. This shows that

$$\forall\, 1 \leq i \neq j \leq t, \quad (\Phi_i, \Phi_j) = 0.$$

So, the partition (II.6.19) is a decomposition of Φ into pairwise orthogonal subsets of $\mathfrak{h}^*$. As a consequence, we have the following direct sum decomposition of $\mathfrak{h}^*$ into orthogonal subspaces:

$$\mathfrak{h}^* = \sum_{1\leq i\leq t} \mathrm{Span}_{\Bbbk}(\Phi_i) = \bigoplus_{1\leq i\leq t} \mathrm{Span}_{\Bbbk}(\Phi_i), \tag{II.6.22}$$

indeed, the first equality follows from the fact that Φ generates $\mathfrak{h}^*$ and the second from the fact that $(-,-)$ is nondegenerate and the summands pairwise orthogonal.

5. We now complete the picture by expliciting the link between the euclidean spaces associated to $(\mathfrak{g},\mathfrak{h})$, on the one hand, and the t euclidean spaces associated to the pairs $(\mathfrak{g}_i,\mathfrak{h}_i)$, $1\leq i\leq t$, on the other hand.

Beware: we will have to consider orthogonal in the sense of duality. If V is a vector space and W a subspace of V, its orthogonal in the sense of duality will be denoted $W^{*\perp}$, to avoid ambiguity.

We will have to make use of the following restriction maps, $1\leq i\leq t$:

$$\begin{array}{cccc} \mathrm{res}_i & : & \mathfrak{h}^* & \longrightarrow & \mathfrak{h}_i^* \\ & & \lambda & \mapsto & \lambda_{|\mathfrak{h}_i}. \end{array}.$$

Clearly, by restriction, it gives rise to an isomorphism of $\Bbbk$-vector spaces: $(\oplus_{j\neq i}\mathfrak{h}_j)^{*\perp} \xrightarrow{\mathrm{res}_i} \mathfrak{h}_i^*$.

5.1. To start with, we have the following diagram

$$\begin{array}{ccccccc} & & \Phi & \subseteq & \mathsf{E}_{\mathbb{Q}} = \mathrm{Span}_{\mathbb{Q}}(\Phi) & \subseteq & \mathfrak{h}^* \\ & & \cup\!| & & \cup\!| & & \cup\!| \\ \alpha & \in & \Phi_i & \subseteq & \mathsf{E}_{i,\mathbb{Q}} = \mathrm{Span}_{\mathbb{Q}}(\Phi_i) & \subseteq & (\oplus_{j\neq i}\mathfrak{h}_j)^{*\perp} \\ \downarrow & & \downarrow & & \cong\downarrow & & \cong\downarrow \mathrm{res}_i \\ \alpha_{|\mathfrak{h}_i} & \in & \Psi_i & \subseteq & \mathsf{F}_{i,\mathbb{Q}} = \mathrm{Span}_{\mathbb{Q}}(\Psi_i) & \subseteq & \mathfrak{h}_i^* \end{array}$$

Further, by Point 3 above, $\Psi_i = \mathrm{res}_i(\Phi_i)$ is the set of roots of the Cartan-Chevalley decomposition of the pair $(\mathfrak{g}_i,\mathfrak{h}_i)$.

5.2. For $1\leq i\leq t$, put $\mathsf{F}_{i,\mathbb{R}} = \mathbb{R}\otimes_{\mathbb{Q}} \mathsf{F}_{i,\mathbb{Q}}$. We now describe a natural isomorphism between $\mathsf{E}_{\mathbb{R}}$ and the direct sum of the $\mathsf{F}_{i,\mathbb{R}}$, $1\leq i\leq t$. Notice first that, by (II.6.22):

$$\mathsf{E}_{\mathbb{Q}} = \bigoplus_{1\leq i\leq t} \mathsf{E}_{i,\mathbb{Q}} \subseteq \mathfrak{h}^*.$$

This allows to construct an $\mathbb{R}$-linear isomorphism δ as follows

$$\begin{array}{rcl} \mathsf{E}_{\mathbb{R}} & = & \mathbb{R}\otimes_{\mathbb{Q}} \mathsf{E}_{\mathbb{Q}} \\ & = & \mathbb{R}\otimes_{\mathbb{Q}} \left(\bigoplus_{1\leq i\leq t} \mathsf{E}_{i,\mathbb{Q}}\right) \\ & \xrightarrow{\cong} & \left(\bigoplus_{1\leq i\leq t} \mathbb{R}\otimes_{\mathbb{Q}} \mathsf{E}_{i,\mathbb{Q}}\right) \\ & \xrightarrow{\cong} & \left(\bigoplus_{1\leq i\leq t} \mathbb{R}\otimes_{\mathbb{Q}} \mathsf{F}_{i,\mathbb{Q}}\right) \\ & = & \bigoplus_{1\leq i\leq t} \mathsf{F}_{i,\mathbb{R}} \end{array}$$

where, in the above display, the first isomorphism is the canonical isomorphism (distributivity of the tensor product on the direct sum) and the second is the direct sum of the maps $\mathrm{id}_{\mathbb{R}}\otimes_{\mathbb{Q}}\mathrm{res}_i$. Hence

$$\delta \ : \ \mathsf{E}_{\mathbb{R}} \longrightarrow \bigoplus_{1\leq i\leq t} \mathsf{F}_{i,\mathbb{R}}$$

sends any $1 \otimes \alpha$, $\alpha \in \Phi_i$ to $1 \otimes \alpha_{|\mathfrak{h}_i} \in 1 \otimes \Psi_i$.

So, identifying Φ with its image in $\mathsf{E}_{\mathbb{R}}$ and, for $1 \leq i \leq t$, Ψ_i with its image in $\mathsf{F}_{i,\mathbb{R}}$, we have that

$$\forall\, 1 \leq i \leq t, \quad \delta(\Phi_i) = \Psi_i.$$

5.3. In order to complete the picture, it remains to link the euclidean structures on the spaces $\mathsf{E}_{\mathbb{R}}$ and $\mathsf{F}_{i,\mathbb{R}}$. This is straightforward. We actually do it at the level of the duals of the maximal toral subalgebras involved. Recall the natural isomorphism

$$\begin{array}{ccc} \mathfrak{h}^* & \longrightarrow & \bigoplus_{1 \leq i \leq t} \mathfrak{h}_i^* \\ \lambda & \mapsto & (\lambda_{|\mathfrak{h}_i})_{1 \leq i \leq t} \end{array}. \tag{II.6.23}$$

We have the nondegenerate bilinear form $(-,-)$ on $\mathfrak{h}^*$, dual to the Killing form on $\mathfrak{h}$, and, for $1 \leq i \leq t$, the nondegenerate bilinear form, denoted $(-,-)_i$, dual to the Killing form on $\mathfrak{h}_i$. The latter give rise to a nondegenerate bilinear form, that we denote $\oplus_{1 \leq i \leq t}(-,-)_i$ on $\bigoplus_{1 \leq i \leq t} \mathfrak{h}_i^*$ relative to which the components of different indices are pairwise orthogonal. It is not difficult to check that the isomorphism (II.6.23) commutes with the bilinear forms $(-,-)$ and $\oplus_{1 \leq i \leq t}(-,-)_i$.

As an immediate consequence, we get that δ is an isometry between the euclidean spaces $\mathsf{E}_{\mathbb{R}}$ and $\bigoplus_{1 \leq i \leq t} \mathsf{F}_{i,\mathbb{R}}$, where $\bigoplus_{1 \leq i \leq t} \mathsf{F}_{i,\mathbb{R}}$ is endowed with the euclidean structure which extends the natural euclidean structure of each summand in such a way that distinct summands be orthogonal.

Part III

Root systems.

III.1 Reflexions in euclidean spaces.

For all this section, E stands for a euclidean vector space which scalar product we denote $(-,-) : \mathsf{E} \times \mathsf{E} \longrightarrow \mathbb{R}$. We let $O(\mathsf{E})$ be the group of orthogonal linear automorphisms of $(\mathsf{E}, (-,-))$.

Notation III.1.1 – The following notation will be useful. For $\alpha, \beta \in \mathsf{E}$, $\beta \neq 0$, put

$$\langle \alpha, \beta \rangle = 2\frac{(\alpha,\beta)}{(\beta,\beta)}.$$

To any $\alpha \in \mathsf{E} \setminus \{0\}$, we associate the map

$$\begin{array}{cccc} \sigma_\alpha & : & \mathsf{E} & \longrightarrow & \mathsf{E} \\ & & x & \mapsto & x - \langle x, \alpha \rangle \alpha \end{array}.$$

As is well known, σ_α is a reflection with reflecting hyperplane $(\mathbb{R}\alpha)^\perp$.

We start with an elementary Lemma.

Lemma III.1.2 – *Let V be an $\mathbb{R}$-vector space of finite dimension n, Φ a finite subset which spans V and $\alpha \in V$ a nonzero element. There exists at most one endomorphism f of V such that $f(\alpha) = -\alpha$, $\dim_{\mathbb{R}}(\ker(f - \mathrm{id}_V)) \geq n-1$ and $f(\Phi) \subseteq \Phi$.*

Proof. Let $f, g : V \longrightarrow V$ be such endomorphisms. First, observe that $V = \ker(f - \mathrm{id}_V) \oplus \mathbb{R}\alpha = \ker(g - \mathrm{id}_V) \bigoplus \mathbb{R}\alpha$, that f and g are automorphisms of ordre 2 and that they both induce the identity on $V/\mathbb{R}\alpha$.

Put $h = f \circ g$. Since $h(\alpha) = \alpha$ and h induces the identity on $V/\mathbb{R}\alpha$, its characteristic polynomial must be $(X-1)^n$, as me see by computing its matrix in a basis of V starting by α. Its minimal polynomial thus divides $(X-1)^n$. On the other hand, as h stabilises the finite set Φ, for all $x \in \Phi$, the set $\{h^k(x),\, k \in \mathbb{N}\}$ must be finite. So, h being invertible, there exists an integer $k \in \mathbb{N}$ such that $h^k(x) = x$. It follows, Φ being a finite generating set of V, that there exists $m \in \mathbb{N}$ such that $h^m = \mathrm{id}$. The minimal polynomial of h thus divides $X^m - 1$. As a consequence of the above, the minimal polynomial of h is $X - 1$; that is: $h = 1$. But $f^2 = \mathrm{id}_V$, thus $f = g$. ∎

Lemma III.1.3 – *Let $\alpha \in \mathsf{E} \setminus \{0\}$ and consider a subspace F of E. If $\sigma_\alpha(F) \subseteq F$, then, either $\alpha \in F$, or $F \subseteq (\mathbb{R}\alpha)^\perp$.*

Proof. Suppose $\sigma_\alpha(F) \subseteq F$ and $\alpha \notin F$. For all $x \in F$, $\sigma_\alpha(x) = x - \langle x, \alpha \rangle \alpha \in F$, so that $(x,\alpha)\alpha \in F$. But $\alpha \notin F$, so $(x,\alpha) = 0$. ∎

Lemma III.1.4 – *Let Φ be a finite generating set of E such that, for all $\alpha \in \Phi \setminus \{0\}$, $\sigma_\alpha(\Phi) \subseteq \Phi$. Let $\sigma \in \mathrm{GL}(\mathsf{E})$ such that $\sigma(\Phi) \subseteq \Phi$ and $\beta \in \Phi \setminus \{0\}$. Then, $\sigma = \sigma_\beta$ if and only if there exists a hyperplane H of E pointwise fixed by σ and $\sigma(\beta) = -\beta$.*

Proof. This follows immediately from Lemma III.1.2. ∎

Corollary III.1.5 – *Let Φ be a finite generating set of E such that, for all $\alpha \in \Phi \setminus \{0\}$, $\sigma_\alpha(\Phi) \subseteq \Phi$. Let $\sigma \in \mathrm{GL}(E)$ such that $\sigma(\Phi) \subseteq \Phi$. The following holds:*
1. for all $\alpha \in \Phi \setminus \{0\}$, $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$;
2. for all $\alpha \in \Phi \setminus \{0\}$ and for all $\beta \in \Phi$, $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$.

Proof. Notice first that, for $\alpha \in \Phi \setminus \{0\}$, since Φ is finite and σ_α is an automorphism, the hypotheses actually give $\sigma_\alpha(\Phi) = \Phi$ and thus $\sigma_\alpha^{-1}(\Phi) = \Phi$.
1. Let $\alpha \in \Phi \setminus \{0\}$. The automorphism $\sigma\sigma_\alpha\sigma^{-1}$ fixes pointwise the hyperplane $\sigma((\mathbb{R}\alpha)^\perp)$, sends Φ into itself and maps $\sigma(\alpha)$ to $-\sigma(\alpha)$. Thus, by Lemma III.1.4, $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$.
2. Applying the above identity to $\sigma(\beta)$, $\beta \in \Phi$, gives the result. ∎

III.2 Definition of Root Systems.

In this section, $(\mathsf{E}, (-,-))$ is a euclidean space. We start with the definition of *root system* in the euclidean space E.

Definition III.2.1 – *Let Φ be a subset of E. Then Φ is called a* root system *of E if it satisfies the following conditions.*
1. The set Φ is finite, generates E as an $\mathbb{R}$-vector space and does not contain 0.
2. For all $\alpha \in \Phi$, the scalar multiples of α in Φ are $\pm\alpha$.
3. For all $\alpha \in \Phi$, $\sigma_\alpha(\Phi) \subseteq \Phi$.
4. For all $\alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle \in \mathbb{Z}$.
The rank *of Φ is defined to be the dimension of E.*

Example III.2.2 – Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and $\mathfrak{h}$ a maximal toral subalgebra. Theorem II.6.1 exactly says that Φ, seen as a subset of $\mathsf{E}_\mathbb{R}$, is a root system of $\mathsf{E}_\mathbb{R}$.

Remark III.2.3 – Let Φ be a root system of $(\mathsf{E}, (-,-))$. If we consider any $\lambda \in \mathbb{R}_{>0}$, $\lambda(-,-)$ is a scalar product on the vector space E and Φ is still a root system for $(\mathsf{E}, \lambda(-,-))$.

Definition III.2.4 – *Let Φ be a root system of E. The* Weyl group *associated to Φ is the subgroup of $O(\mathsf{E})$ generated by the reflections σ_α, $\alpha \in \Phi$. We denote it by W_Φ.*

Remark III.2.5 – Let Φ be a root system of E.
1. By the third condition of the definition of root system, we have a natural morphism of groups

$$W_\Phi \longrightarrow \mathfrak{S}(\Phi),$$

where $\mathfrak{S}(\Phi)$ stands for the symmetric group of Φ. In addition, since Φ generates E, this morphism must be injective. Since, on the other hand, Φ is finite, then W_Φ must also be finite.
2. By Corollary III.1.5, the action of $\mathrm{GL}(\mathsf{E})$ on itself by conjugation restricts to an action of its subgroup $\{\sigma \in \mathrm{GL}(\mathsf{E}) \,|\sigma(\Phi) \subseteq \Phi\}$ on W_Φ.

Definition III.2.6 – **Isomorphism of root systems** – *Let Φ be a root system of E. Let E' be a euclidean space and Φ' be a root system of E'. The pairs (E, Φ) and (E', Φ') are said to be isomorphic if there exists an isomorphism $\varphi : \mathsf{E} \longrightarrow \mathsf{E}'$ of vector spaces such that $\varphi(\Phi) = \Phi'$ and, for all $\alpha, \beta \in \Phi$,*

$$\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle.$$

An automorphism of (E, Φ) is an isomorphism between (E, Φ) and itself.

Exercise III.2.7 – Let Φ be a root system of E. Let E' be a euclidean space and Φ' be a root system of E' and let $\varphi : \mathsf{E} \longrightarrow \mathsf{E}'$ be an isomorphism of root systems between (E, Φ) and (E', Φ').
1. For all $\alpha, \beta \in \Phi$, $\sigma_{\varphi(\alpha)}(\varphi(\beta)) = \varphi(\sigma_\alpha(\beta))$.
2. The map $W_\Phi \longrightarrow W_{\Phi'}$, $\varphi \circ \sigma \circ \varphi^{-1}$ is an isomorphism of groups.

Exercise III.2.8 – Let Φ be a root system of E. Any automorphism of the vector space E that leaves Φ invariant is an automorphism of (E, Φ). (See Corollary III.1.5.)

Exercise III.2.9 – **Dual (or inverse) of a root system** –
Let Φ be a root system. To $\alpha \in \Phi$, associate

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

1. For all $\alpha, \beta \in \Phi$,
1.1. $\alpha^\vee = \beta^\vee$ if and only if $\alpha = \beta$,
1.2. $\sigma_{\alpha^\vee}(\beta^\vee) = (\sigma_\alpha(\beta))^\vee$,
1.3. $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$.
2. The set $\Phi^\vee = \{\alpha^\vee,\ \alpha \in \Phi\} \subseteq \mathsf{E}$ is a root system of E, called the dual (or inverse) of Φ.

Example III.2.10 – **Root systems of rank one** – Up to isomorphism, there is a unique root system of rank one. It is the root system of $\mathfrak{sl}_2(\mathbb{k})$.

Example III.2.11 – **Examples of root systems of rank two** – LAURENT. A compléter en suivant [Humphreys ; p.44].

As will be seen latter, root systems can be classified. This is due to the very strong constraint put on them by the fourth condition in the definition. We now examine this condition.

Remark III.2.12 – Let $\alpha, \beta \in \mathsf{E} \setminus \{0\}$. The Cauchy-Schwartz inequality states that

$$-1 \leq \frac{(\alpha, \beta)}{||\alpha||\,||\beta||} \leq 1.$$

In addition,

$$\frac{(\alpha, \beta)}{||\alpha||\,||\beta||} = \pm 1 \qquad \text{iff} \qquad \exists \lambda \in \mathbb{R}^* \,|\, \beta = \lambda\alpha.$$

(Here, $||-||$ stands for the euclidean norm associated to $(-,-)$.) Therefore, there exists a unique real number θ in the interval $[0, \pi]$ such that $(\alpha, \beta) = \cos(\theta)||\alpha||\,||\beta||$. We call this real number the *angle* between α and β.

Remark III.2.13 – Let Φ be a root system of E. Let $\alpha, \beta \in \Phi$.
1. The integers $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ have the same sign since:

$$\langle \alpha, \beta \rangle ||\beta||^2 = 2(\alpha, \beta) = 2(\beta, \alpha) = \langle \beta, \alpha \rangle ||\alpha||^2.$$

2. The Cauchy-Schwartz inequality then gives

$$0 \leq \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \leq 4 \qquad \text{and} \qquad \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \Longleftrightarrow \exists \lambda \in \mathbb{R}^* \,|\, \beta = \lambda\alpha.$$

But, α and β being roots, $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ must be in $\mathbb{Z}$ and, in addition, they are proportional if and only if $\alpha = \pm\beta$. It follows that, if we assume $||\alpha|| \leq ||\beta||$, the possible values for these integers are as follows:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	angle	$\lVert\beta\rVert^2/\lVert\alpha\rVert^2$	
0	0	$\pi/2$	*undetermined*	case where α and β are orthogonal
1	1	$\pi/3$	1	
-1	-1	$2\pi/3$	1	
1	2	$\pi/4$	2	
-1	-2	$3\pi/4$	2	
1	3	$\pi/6$	3	
-1	-3	$5\pi/6$	3	
2	2	0	1	case where $\alpha = \beta$
-2	-2	π	1	case where $\alpha = -\beta$

As a consequence, we get the following Lemma.

Lemma III.2.14 – *Let Φ be a root system of E. Let $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$.*
1. If $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$.
2. If $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$.

Proof. Suppose that $(\alpha, \beta) > 0$, by Remark III.2.13, either $\langle \alpha, \beta \rangle = 1$ or $\langle \beta, \alpha \rangle = 1$. In the first case, $\alpha - \beta = \sigma_\beta(\alpha) \in \Phi$, in the second case, $\beta - \alpha = \sigma_\alpha(\beta) \in \Phi$ (by the third condition on root systems). By the second condition, we get point 1. Point 2 follows from point 1. ∎

We now introduce the notion of *string* of roots.

Proposition III.2.15 – *Let $\alpha, \beta \in \Phi$ be nonproportional roots.*
Put $I = \{i \in \mathbb{Z} \,|\, \beta + i\alpha \in \Phi\}$ and $S = \{\beta + i\alpha,\, i \in I\}$. Then:
1. I is an bounded interval of $\mathbb{Z}$ containing 0,
2. if $r, q \in \mathbb{N}$ are the integers such that $I = [-r, q]$, then $r - q = \langle \beta, \alpha \rangle$.

Proof. Consider the map

$$\begin{array}{cccc} \rho & : & \mathbb{Z} & \longrightarrow & \mathsf{E} \\ & & i & \mapsto & \beta + i\alpha \end{array}.$$

As $\alpha \neq 0$, ρ is injective. Since Φ is finite, I is a finite subset of $\mathbb{Z}$ which clearly contains 0. We may thus consider $r, q \in \mathbb{N}$ such that $q = \max(I)$ et $-r = \min(I)$ and we have $I \subseteq [-r, q]$. Suppose I is not an interval of $\mathbb{Z}$. Then, there exists integers p, s avec $p < s$ such that $p, s \in I$, $p + 1, s - 1 \notin I$. Lemma III.2.14 then implies that $(\beta + p\alpha, \alpha) \geq 0$ and $(\beta + s\alpha, \alpha) \leq 0$. From which it follows that $-p(\alpha, \alpha) \leq (\beta, \alpha) \leq -s(\alpha, \alpha)$, contradicting $p < s$. This proves Point 1.

It is clear that $\sigma_\alpha(S) \subseteq S$. Moreover, σ_α is injective and S finite. So, $\sigma_\alpha(S) = S$. More precisely, for $i \in \mathbb{Z}$,

$$\sigma_\alpha(\beta + i\alpha) = \beta - \langle \beta, \alpha \rangle \alpha - i\alpha = \beta + (-\langle \beta, \alpha \rangle - i)\alpha.$$

It follows that the map $\mathbb{Z} \longrightarrow \mathbb{Z}$, $i \mapsto -i - \langle \beta, \alpha \rangle$ induces a map

$$\begin{array}{ccc} I & \longrightarrow & I \\ i & \mapsto & -i - \langle \beta, \alpha \rangle \end{array}$$

obviously bijective and decreasing. It thus maps $-r$ to q. Hence, $r - \langle \beta, \alpha \rangle = q$. ∎

Definition III.2.16 – **Strings of roots** – *Let $\alpha, \beta \in \Phi$ be nonproportional roots. In the notation of Proposition III.2.15, S is called the α-string through β, $\beta - r\alpha$ its origine, $\beta + q\alpha$ its extremity and $q + r$ its length.*

Proposition III.2.17 – **Length of strings of roots** – *Let $\alpha, \beta \in \Phi$ be nonproportional roots. The length of the α-string through β is bounded by 3.*

Proof. Let γ be the origine of the α-string through β. It is clear that the α-string through β and the α-string through γ coincide. But, by Point 2 of Proposition III.2.15, the length of the latter is $-\langle \gamma, \alpha \rangle$. Point 2 in Remark III.2.13 then gives the result. ∎

III.3 Bases of root systems.

In this section, $(\mathsf{E}, (-,-))$ is a euclidean space.

Definition III.3.1 – *Let Φ be a root system of E. A subset Δ of Φ is called a base of Φ if:*
(i) Δ is a basis of E;
(ii) for all $\alpha \in \Phi$, the coefficients of α as a linear combination of elements of Δ are integers which are all in $\mathbb{N}$ or all in $-\mathbb{N}$.

Remark III.3.2 – Let Φ be a root system of E, let E' be a euclidean space and Φ' a root system of E' and let $\varphi : \mathsf{E} \longrightarrow \mathsf{E}'$ be an isomorphism between (E, Φ) and (E', Φ'). Then if Δ is a base of Φ, $\varphi(\Delta)$ is a base of Φ'.

Remark III.3.3 – Let Φ be a root system of E. A base Δ of Φ gives rise to a partition of Φ. Indeed, put:

$$\Phi^+ = \left\{ \sum_{\alpha \in \Delta} n_\alpha \alpha,\ n_\alpha \in \mathbb{N},\ \forall \alpha \in \Delta \right\} \cap \Phi \quad \text{and} \quad \Phi^- = \left\{ \sum_{\alpha \in \Delta} n_\alpha \alpha,\ n_\alpha \in (-\mathbb{N}),\ \forall \alpha \in \Delta \right\} \cap \Phi$$

Then $\Phi = \Phi^+ \sqcup \Phi^-$ (disjointe union). In addition, $\Phi^- = -\Phi^+$.

Definition III.3.4 – *Let Φ be a root system of E and Δ a base of Φ. The elements of Δ are called simple roots; the elements of Φ^+ (resp. Φ^-) are called positive (resp. negative) roots. Further, the height of $\alpha \in \Phi$, denoted $\mathrm{ht}(\alpha)$, is the sum of its coefficients in its expression as a linear combination of elements of Δ.*

Definition III.3.5 – *Let Φ be a root system of E and Δ a base of Φ. We define a binary relation on E, denoted $\preceq$, as follows:*

$$\forall x, y \in \mathsf{E}, \quad x \preceq y \quad \textit{if} \quad y - x \in \mathrm{Span}_{\mathbb{N}}(\Phi^+) = \mathrm{Span}_{\mathbb{N}}(\Delta).$$

Proposition III.3.6 – *Let Φ be a root system of E and Δ a base of Φ. The binary relation $\preceq$ of Definition III.3.5 is an ordre on E.*

Proof. This is clear. ∎

Lemma III.3.7 – *Let Φ be a root system of E and Δ a base of Φ. If $\alpha, \beta \in \Delta$ are distinct, then $(\alpha, \beta) \leq 0$.*

Proof. Suppose, on the contrary, that $(\alpha, \beta) > 0$. Then, Lemma III.2.14 shows that $\alpha - \beta$ is a root, which contradicts the second condition of the definition of base. ∎

At this stage, however, it is not even clear that a root system does admit a base. We now proceed to show that, indeed, it does.

Let Φ be a root system. To any $\alpha \in \Phi$, associate the hyperplane $P_\alpha = (\mathbb{R}\alpha)^\perp$. This hyperplane determines two half-spaces of E:

$$P_\alpha^+ = \{x \in \mathsf{E} \,|\, (\alpha, x) > 0\} \quad \text{and} \quad P_\alpha^- = \{x \in \mathsf{E} \,|\, (\alpha, x) < 0\}.$$

Exercise III.3.8 – Let E be a finite dimensional nonzero vector space over an infinite field. Let $r \in \mathbb{N}^*$. For $1 \leq i \leq r$, let P_i be an hyperplane of E. Then we have the strict inclusion: $\bigcup_{1 \leq i \leq r} P_i \subset E$.

Definition III.3.9 – *Let Φ be a root system of E. An element of E is called regular if it belongs to $\mathsf{E} \setminus \bigcup_{\alpha \in \Phi} P_\alpha$.*

Remark III.3.10 – In the above notation, by Exercise III.3.8, we have the strict inclusion

$$\bigcup_{\alpha \in \Phi} P_\alpha \subset \mathsf{E}.$$

Hence, there exist regular elements.

Definition III.3.11 – *Let Φ be a root system of E and x a regular element of E. Put*

$$\Phi^+(x) = \{\alpha \in \Phi \,|\, (x, \alpha) > 0\} \quad \textit{and} \quad \Phi^-(x) = \{\alpha \in \Phi \,|\, (x, \alpha) < 0\}.$$

An element of $\Phi^+(x)$ is indecomposable if it cannot be written as the sum of two elements of $\Phi^+(x)$. We denote by $\Delta(x)$ the subset of $\Phi^+(x)$ of indecomposable elements.

Remark III.3.12 – Let Φ be a root system of E and x a regular element of E.
1. It is clear that:
1.1. $\Phi = \Phi^+(x) \sqcup \Phi^-(x)$;
1.2. $\Phi^-(x) = -\Phi^+(x)$.
2. Suppose $\Delta(x)$ is a base of Φ, and let Φ^+ and Φ^- be the set of positive and negative roots relative to the choice of $\Delta(x)$ as a basis of Φ, as defined in Remark III.3.3. Then,

$$\Phi^+ = \Phi^+(x) \qquad \text{and} \qquad \Phi^- = \Phi^-(x).$$

Exercise III.3.13 – Let E be a euclidean space and let $\mathcal{B}$ be a basis of E.
For all $b \in \mathcal{B}$, let p_b be the orthogonal projection of b on the line $(\mathrm{Span}_{\mathbb{R}}(\mathcal{B} \setminus \{b\}))^\perp$. Put

$$\delta = \sum_{b \in \mathcal{B}} p_b.$$

Then, for all $b \in \mathcal{B}$, $(b, \delta) > 0$.

In particular, there exists an element $x \in E$ such that $(b, x) > 0$, for all $b \in \mathcal{B}$.

Lemma III.3.14 – *Let E be a euclidean space, $v \in E \setminus \{0\}$, K a nonempty set and $\mathcal{X} = \{x_k,\, k \in K\}$ a familly of elements of E. If, for all $k \in K$, $(x_k, v) > 0$, and, for all $i, j \in K$, $i \neq j$, $(x_i, x_j) \leq 0$, then $\mathcal{X}$ is linearly independant.*

Proof. Suppose $\mathcal{X}$ is linearly dependant. From the existence of a nontrivial equation of linear dependence between elements of $\mathcal{X}$, we deduce the existence of an equality $\sum_{i \in I} r_i x_i = \sum_{i \in J} r_i x_i$, where I, J are disjoint subsets of K and $r_i \in \mathbb{R}_{>0}$, for all $i \in I \sqcup J$. Notice that I or J may be empty (in which case the corresponding sum is understood to be zero) but that one of the two at least is not. Now, let $\varepsilon = \sum_{i \in I} r_i x_i$. Then,

$$(\varepsilon, \varepsilon) = \left(\sum_{i \in I} r_i x_i, \sum_{j \in J} r_j x_j \right) = \sum_{(i,j) \in I \times J} r_i r_j \,(x_i, x_j) \leq 0.$$

So $\varepsilon = 0$. But then, $0 = (\varepsilon, v) = \sum_{i \in I} r_i (x_i, v)$, with $r_i > 0$ and $(x_i, v) > 0$. This entails $I = \emptyset$. In the same manner, we get $J = \emptyset$. A contradiction. ∎

Theorem III.3.15 – **Existence of bases** – *Let Φ be a root system of E.*
1. If x is a regular element of E, then $\Delta(x)$ is a base of Φ.
2. If Δ is a base of Φ, then there exists a regular element x of E such that $\Delta = \Delta(x)$.

Proof. 1. We proceed in four steps.
(1) *We have $\Phi^+(x) \subseteq \mathrm{Span}_{\mathbb{N}}(\Delta(x))$.*
Suppose the contrary and choose $\alpha \in \Phi^+(x) \setminus \mathrm{Span}_{\mathbb{N}}(\Delta(x))$ with (x, α) minimal. In particular, $\alpha \notin \Delta(x)$, so that there exist $\alpha_1, \alpha_2 \in \Phi^+(x)$ satisfying $\alpha = \alpha_1 + \alpha_2$. Thus, we have $(\alpha, x) = (\alpha_1, x) + (\alpha_2, x)$ with $(\alpha_1, x), (\alpha_2, x) > 0$. Hence, by minimality of (α, x), $\alpha_1, \alpha_2 \in \mathrm{Span}_{\mathbb{N}}(\Delta(x))$, which entails $\alpha \in \mathrm{Span}_{\mathbb{N}}(\Delta(x))$. A contradiction.
(2) *If $\alpha, \beta \in \Delta(x)$, then either $(\alpha, \beta) \leq 0$, or $\alpha = \beta$.*
Suppose $(\alpha, \beta) > 0$ and $\alpha \neq \beta$. Since we cannot have $\alpha = -\beta$ as $\alpha, \beta \in \Phi^+(x)$, Lemma III.2.14 applies and shows that $\alpha - \beta \in \Phi$. If $\alpha - \beta \in \Phi^+(x)$, $\alpha = (\alpha - \beta) + \beta$ and α is decomposable; otherwise, $\beta - \alpha \in \Phi^+(x)$, $\beta = (\beta - \alpha) + \alpha$ and β is decomposable. A contradiction.
(3) *The set $\Delta(x)$ is linearly independant.*
By (2), we are in position to apply Lemma III.3.14, which gives the result.
(4) *The set $\Delta(x)$ is a base of Φ.*
By (1), any element of $\Phi^+(x)$ is a linear combination with coefficients in $\mathbb{N}$ of elements of $\Delta(x)$. It follows that any element of $\Phi^-(x)$ is a linear combination with coefficients in $(-\mathbb{N})$ of elements of $\Delta(x)$, since $\Phi^-(x) = -\Phi^+(x)$. Since $\Phi = \Phi^+(x) \sqcup \Phi^-(x)$ the second condition in the definition of a base is fulfilled. In particular, any element of Φ is in $\mathrm{Span}_{\mathbb{R}}(\Delta(x))$. And, since Φ generates E as a vector space, so does $\Delta(x)$. So, by (3), $\Delta(x)$ is a basis of the vector space E.
2. Let Δ be a base of Φ. Let Φ^+ and Φ^- be the sets of positive and negative roots with respect to Δ. By Exercise III.3.13, there exists an element $x \in \mathsf{E}$ such that $(x, \alpha) > 0$, for all $\alpha \in \Delta$. By the second condition of the definition of base, such an x must be regular. More precisely:

$$\forall\, \alpha \in \Phi^+,\ (x, \alpha) > 0 \qquad \text{and} \qquad \forall\, \alpha \in \Phi^-,\ (x, \alpha) < 0,$$

so that $\Phi^+ \subseteq \Phi^+(x)$ and $\Phi^- \subseteq \Phi^-(x)$. But, since $\Phi^+(x) \sqcup \Phi^-(x) = \Phi = \Phi^+ \sqcup \Phi^+$, we actually have $\Phi^+ = \Phi^+(x)$ and $\Phi^- = \Phi^-(x)$. In particular, $\Delta \subseteq \Phi^+(x)$. More is true: for all $\alpha \in \Delta$, α must be indecomposable (as an element of $\Phi^+(x)$). Indeed, otherwise α could be written as the sum of two elements in $\Phi^+(x) = \Phi^+$, each of which, in turn, is a linear combination with coefficient in $\mathbb{N}^*$ of elements of Δ. This would contradict the linear independance of Δ. So $\Delta \subseteq \Delta(x)$. But, as Point 1 shows, $\Delta(x)$ is a base of Φ. As Δ is also a base of Φ, they both are bases of the vector space E and hence have the same cardinality. So $\Delta(x) = \Delta$. ∎

Remark III.3.16 – Let Φ be a root system of E and Δ a base of Φ. Let Φ^+ and Φ^- be the sets of positive and negative roots with respect to Δ.
1. By Exercise III.3.13, there exists an element x of E such that $(x, \alpha) > 0$, for all $\alpha \in \Delta$.
2. By the proof of Theorem III.3.15, an element x as in Point 1 must be regular.
3. By the proof of Theorem III.3.15, an element x as in Point 1 satisfies $\Delta = \Delta(x)$, $\Phi^+ = \Phi^+(x)$ and $\Phi^- = \Phi^-(x)$. In particular

$$\Phi^+ \subseteq \{y \in \mathsf{E} \,|\, (y, x) > 0\}.$$

4. If y is a regular element of E (with respect to Δ) and $\Delta(y) = \Delta$, then $\Delta \subseteq \Phi^+(y)$ and, thus: $(y, \alpha) > 0$, for all $\alpha \in \Delta$.
5. We have shown that the regular elements x such that $\Delta = \Delta(x)$ are those satisfying $(x, \alpha) > 0$, for all $\alpha \in \Delta$.

At this stage, we are in position to discuss bases for dual root systems. Recall Exercise III.2.9 for the definition of the dual root system $\Phi^\vee$ of Φ.

Proposition III.3.17 – *Let Φ be a root system and Δ a base of Φ. Put*

$$\Delta^\vee = \{\alpha^\vee,\, \alpha \in \Delta\} \subseteq \mathsf{E}.$$

Then, $\Delta^\vee$ is a base of the root system $\Phi^\vee$ of E.

Proof. Notice first that the set of regular elements relative to Φ and $\Phi^\vee$ is the same.

By Remark III.3.16, there exists an element x in E such that $(x,\alpha) > 0$, for all $\alpha \in \Delta$, such an element is regular (with respect to Φ) and we have $\Delta = \Delta(x)$ and $\Phi^+ = \Phi^+(x)$. Now, as pointed above, x is regular with respect to $\Phi^\vee$. Observe in addition that

$$\Delta^\vee \subseteq (\Phi^\vee)^+(x) = \{\alpha^\vee,\, \alpha \in \Phi \,|\, (\alpha, x) > 0\} = \{\alpha^\vee,\, \alpha \in \Phi^+\}.$$

Consider now $\alpha \in \Delta$ and suppose that $\alpha^\vee$ is decomposable as an element of $(\Phi^\vee)^+(x)$. Then, by definition, there exists $\beta, \gamma \in \Phi^+$ such that $\alpha^\vee = \beta^\vee + \gamma^\vee$. Put $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, $\alpha_1 = \alpha$. Since $\beta, \gamma \in \Phi^+$, there exists $n_i, m_i \in \mathbb{N}$, $1 \le i \le \ell$, such that

$$\beta = \sum_{1\le i\le \ell} n_i \alpha_i \quad \text{and} \quad \gamma = \sum_{1\le i\le \ell} m_i \alpha_i.$$

The equality $\alpha^\vee = \beta^\vee + \gamma^\vee$ then gives

$$\frac{\alpha}{(\alpha,\alpha)} = \sum_{1\le i\le \ell} \left(\frac{n_i}{(\beta,\beta)} + \frac{m_i}{(\gamma,\gamma)} \right) \alpha_i.$$

From which it follows, Δ being a basis of E, that $n_i = m_i = 0$ whenever $i \neq 1$. Therefore, β and γ are positive roots, proportional to α; that is, $\beta = \gamma = \alpha$. This leads to $\alpha = 2\alpha$, a contradiction.

At this stage, we have proved that $\Delta^\vee$ is included in the set of indecomposable elements of $(\Phi^\vee)^+(x)$. But, by Theorem III.3.15, the latter set is a base of the root system $\Phi^\vee$ and thus, in particular, a basis of E. As $\Delta^\vee$ is also a basis of E, the previous inclusion must be an equality, which proves that $\Delta^\vee$ is a base of $\Phi^\vee$. ∎

The notion of base of a root system allows to refine the Cartan-Chevalley decomposition as we now show.

Example III.3.18 – **Application to the Cartan-Chevalley decomposition** – Recall the setup of Section II.6. The field $\Bbbk$ is assumed to be algebraically closed of characteristic 0. We consider a pair $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra and $\mathfrak{h}$ a maximal toral subalgebra of $\mathfrak{g}$.

We then have the Cartan-Chevalley decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha\right\}$ where, for $\alpha \in \mathfrak{h}^*$, we put $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, \forall h \in \mathfrak{h},\ [h,x] = \alpha(h)x\}$ and $\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \,|\, \mathfrak{g}_\alpha \neq (0)\}$. Recall also that $\mathfrak{g}_0 = \mathfrak{h}$.

The Killing form on $\mathfrak{h}$ gives rise to a nondegenerate form on $\mathfrak{h}^*$: $(-,-) : \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \Bbbk$, via the identification $\iota : \mathfrak{h} \longrightarrow \mathfrak{h}^*$. Then, putting $\mathsf{E}_\mathbb{Q} = \mathrm{Span}_\mathbb{Q}(\Phi) \subseteq \mathfrak{h}^*$, we get a $\mathbb{Q}$-subspace of dimension $\dim_\Bbbk(\mathfrak{h}^*)$ and on which $(-,-)$ induces a positive, definite, symmetric bilinear form $(-,-)_\mathbb{Q} : \mathsf{E}_\mathbb{Q} \times \mathsf{E}_\mathbb{Q} \longrightarrow \mathbb{Q}$ wich, in turn, defines a positive, definite, symmetric bilinear form $(-,-)_\mathbb{R} : \mathsf{E}_\mathbb{R} \times \mathsf{E}_\mathbb{R} \longrightarrow \mathbb{R}$ on the $\mathbb{R}$-vector space $\mathsf{E}_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Q} \mathsf{E}_\mathbb{Q}$, turning it into a euclidean space.

Then, Theorem II.6.1 shows that, seen as a subspace of $\mathsf{E}_{\mathbb{R}}$, Φ is a root system of $\mathsf{E}_{\mathbb{R}}$.

At this stage, the results of the present section allow us to refine the Cartan-Chevalley decomposition as follows. Choose a basis Δ of Φ, and write $\Phi = \Phi^+ \sqcup \Phi^-$. Then, we can put:

$$\mathfrak{n}^- = \oplus_{\alpha\in\Phi^-}\mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n} = \mathfrak{n}^+ = \oplus_{\alpha\in\Phi^+}\mathfrak{g}_\alpha\,, \quad \text{so that} \quad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

The first point of Lemma II.5.6 shows that $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are Lie subalgebras of $\mathfrak{g}$. The second point of the same lemma, together with Engel's Theorem, then shows that $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent Lie algebras. Put now

$$\mathfrak{b} = \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+.$$

The same argument as above shows that $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{g}$ and that $[\mathfrak{b},\mathfrak{b}] \subseteq \mathfrak{n}^+$. It follows that the Lie subalgebra $[\mathfrak{b},\mathfrak{b}]$ of $\mathfrak{b}$ is nilpotente (hence solvable), which entails that $\mathfrak{b}$ is solvable.

Actually, more is true, we have: $[\mathfrak{b},\mathfrak{b}] = \mathfrak{n}$. Indeed, let $\alpha \in \Phi^+$. There exists $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$. Then, if $x \in \mathfrak{g}_\alpha$, the identity $[h,x] = \phi(h)x$ shows that $x \in [\mathfrak{h},\mathfrak{n}]$. The inclusion $\mathfrak{n} \subseteq [\mathfrak{b},\mathfrak{b}]$ follows.

III.4 Properties of root systems.

In this section, $(\mathsf{E},(-,-))$ is a euclidean space.

Proposition III.4.1 – *Let Φ be a root system of E and Δ be a base of Φ.*
1. If α is a positive but not simple root, there exists a simple root β such that $\alpha-\beta$ is a positive root.
2. If α is a positive root, there exists $t \in \mathbb{N}^*$ and a finite sequence $(\alpha_i)_{1\le i\le t}$ of simple roots such that:
(i) $\alpha = \sum_{1\le i\le t}\alpha_i$ and,
(ii) for all $1 \le s \le t$, $\sum_{1\le i\le s}\alpha_i \in \Phi^+$.

Proof. Remark III.3.16 shows there exists $x \in \mathsf{E}$, regular, such that $\Phi^+ \subseteq \{y \in \mathsf{E} \,|\, (y,x) > 0\}$.

Let α be a nonsimple, positive root. Suppose that $(\alpha,\beta) \le 0$ for all simple root β, by Lemme III.3.7, we are in position to apply Lemma III.3.14 which implies that $\Delta \cup \{\alpha\}$ is linearly independant. This is a contradiction. Hence, there exists a simple root β such that $(\alpha,\beta) > 0$. But, α and β are not proportional since α is positive but not simple. So, Lemma III.2.14 applies and shows that $\alpha-\beta$ is a root.

Now, α being positive, for all $\gamma \in \Delta$, there exists $n_\gamma \in \mathbb{N}$ such that $\alpha = \sum_{\gamma\in\Delta} n_\gamma\gamma$. As α and β are not proportional, there exists $\gamma \in \Delta\setminus\{\beta\}$ such that $n_\gamma > 0$. But then, since $\alpha-\beta$ is a root, the definition of base implies that $n_\beta \ge 1$. Point 1 follows.

Point 2 is an immediate consequence of Point 1. ■

We are now in position to give a better set of generators for a semisimple Lie algebra than the one given in Proposition II.5.17. (See Example III.3.18 for comments on the context.)

Proposition III.4.2 – *Suppose $\Bbbk$ is algebraically closed of characteristic 0. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, let $\mathfrak{h}$ be a maximal toral subalgebra, let Φ be the set of roots for the pair $(\mathfrak{g},\mathfrak{h})$ and let Δ be a base of the root system $(\mathsf{E}_{\mathbb{R}},\Phi)$. Then, the set $\sum_{\alpha\in\Delta}(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$ generates $\mathfrak{g}$ as a Lie algebra.*

Proof. Let $\mathfrak{l}$ be the Lie subalgebra of $\mathfrak{g}$ generated by the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, $\alpha \in \Delta$. By Proposition II.5.17, it is enough to show that any root space $\mathfrak{g}_\beta$, $\beta \in \Phi$, is in $\mathfrak{l}$.

Suppose first that β is a positive root. We proceed by induction on the height of β. The result is trivial if the height of β is 1 since then $\beta \in \Delta$. Consider now any β with height at least equal to 2. By Proposition III.4.1, we know that there is a simple root α and a positive root γ with $\mathrm{ht}(\gamma) = \mathrm{ht}(\beta) - 1$ such that $\beta = \alpha + \gamma$. By the induction hypothesis, both $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\gamma$ are in $\mathfrak{l}$. On the other hand, by Proposition II.5.16, we have that $\mathfrak{g}_\beta = [\mathfrak{g}_\alpha, \mathfrak{g}_\gamma]$. So, $\mathfrak{g}_\beta \in \mathfrak{l}$.

Clearly, a similar argument works for negative roots β, using induction on $-\mathrm{ht}(\beta)$. ∎

Notation III.4.3 – Let Φ be a root system of E and Δ a base of Φ. We put

$$\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \mathsf{E}.$$

Proposition III.4.4 – *Let Φ be a root system of E and Δ a base of Φ. Let $\alpha \in \Delta$.*
1. If $\beta \in \Phi^+ \setminus \{\alpha\}$, then $\sigma_\alpha(\beta) \in \Phi^+ \setminus \{\alpha\}$.
2. The restriction of σ_α to $\Phi^+ \setminus \{\alpha\}$ induces a bijection of $\Phi^+ \setminus \{\alpha\}$ into itself.
3. We have $\sigma_\alpha(\delta) = \delta - \alpha$.

Proof. Denote $\alpha_1, \ldots, \alpha_r$ the simple roots, with $\alpha_1 = \alpha$. There exist $n, n_2, \ldots, n_r \in \mathbb{N}$ such that $\beta = n\alpha + \sum_{2 \le i \le r} n_i \alpha_i$. Moreover, since $\alpha \neq \beta$, there exists $2 \le i \le r$ such that $n_i > 0$. Then,

$$\sigma_\alpha(\beta) = \left(-n - \sum_{2 \le i \le r} n_i \langle \alpha_i, \alpha \rangle \right) \alpha + \sum_{2 \le i \le r} n_i \alpha_i.$$

Now, $\sigma_\alpha(\beta) \in \Phi$ by definition of a root system, and one of its coefficients n_i, $2 \le i \le r$, in its decomposition over Δ is in $\mathbb{N}^*$ by the above observation. Hence, $\sigma_\alpha(\beta) \in \Phi^+$ and it is different from α. This proves the first point. The two others follow immediately. ∎

Proposition III.4.5 – *Let Φ be a root system of E and Δ a base of Φ. Let $t \in \mathbb{N}$, $t \ge 2$ and, for $1 \le i \le t$, a simple root α_i and the reflection $\sigma_i = \sigma_{\alpha_i}$ attached to it.*

If $\sigma_1 \ldots \sigma_{t-1}(\alpha_t) \in \Phi^-$, then there exists an integer s, $1 \le s < t$, such that

$$\sigma_1 \ldots \sigma_t = \prod_{1 \le i \le t,\, i \neq s,\, i \neq t} \sigma_i.$$

(With the convention that the above product is the identity if $t = 2$.)

Proof. Let $\beta_0, \ldots, \beta_{t-1}$ be the roots defined by $\beta_i = \sigma_{i+1} \ldots \sigma_{t-1}(\alpha_t)$, $0 \le i \le t-2$, and $\beta_{t-1} = \alpha_t$.

By hypothesis, $\beta_0 \in \Phi^-$ et $\beta_{t-1} \in \Phi^+$. Hence, there exists a least integer s such that $1 \le s \le t-1$ and $\beta_s \in \Phi^+$. We have $\beta_{s-1} = \sigma_s(\beta_s) \in \Phi^-$. Proposition III.4.4 thus implies that $\beta_s = \alpha_s$.

If $s = t-1$, we thus have $\alpha_t = \beta_{t-1} = \alpha_{t-1}$ and the result is clear.

Otherwise, we have $\sigma_{s+1} \ldots \sigma_{t-1}(\alpha_t) = \alpha_s$. Thus, by Corollary III.1.5, we have

$$(\sigma_{s+1} \ldots \sigma_{t-1}) \sigma_t (\sigma_{s+1} \ldots \sigma_{t-1})^{-1} = \sigma_s,$$

that is $\sigma_1 \ldots \sigma_t = \sigma_1 \ldots \sigma_{s-1} \sigma_{s+1} \ldots \sigma_{t-1}$. ∎

Corollary III.4.6 – *Let Φ be a root system of E, W_Φ its Weyl group and Δ a base of Φ. Let $\sigma \in W_\Phi$, $\sigma \neq \mathrm{id}$. If t is the least element of $\mathbb{N}^*$ such that σ may be written as product of t reflections σ_α, $\alpha \in \Delta$ and if $\alpha_1, \dots, \alpha_t$ are elements of Δ such that $\sigma = \sigma_{\alpha_1} \dots \sigma_{\alpha_t}$, then $\sigma(\alpha_t) \in \Phi^-$.*

Proof. If $t = 1$, the result is clear.

Suppose $t = 2$. Then $\sigma = \sigma_{\alpha_1}\sigma_{\alpha_2}$, $\alpha_1, \alpha_2 \in \Delta$. If we suppose that $\sigma(\alpha_2) \in \Phi^+$, then $\sigma_{\alpha_1}(\alpha_2) \in \Phi^-$ which implies, by Proposition III.4.4, that $\alpha_1 = \alpha_2$. But $\sigma \neq \mathrm{id}$, hence a contradiction.

Suppose now that $t \geq 3$. Then $\sigma = \sigma_{\alpha_1} \dots \sigma_{\alpha_t}$, $\alpha_1, \dots, \alpha_t \in \Delta$. If we suppose that $\sigma(\alpha_t) \in \Phi^+$, then $\sigma_{\alpha_1} \dots \sigma_{\alpha_{t-1}}(\alpha_t) \in \Phi^-$. We are then in position to apply Proposition III.4.5 which shows that σ may be written as a product of $t-2$ reflections associated to simple roots, which contradicts the minimality of t. Hence, $\sigma(\alpha_t) \in \Phi^-$. ∎

III.5 Weyl chambers.

In this section, $(\mathsf{E}, (-,-))$ is a euclidean space.

Let Φ be a root system of E. Recall the set

$$\mathcal{T} = \mathsf{E} \setminus \bigcup_{\alpha \in \Phi} P_\alpha$$

of regular elements of E (cf. Definition III.3.9) relative to Φ, which we know is not empty (cf. Remark III.3.10).

Remark III.5.1 –
1. Clearly, $\mathcal{T} = \{x \in \mathsf{E} \,|\, (x, \alpha) \neq 0, \, \forall \alpha \in \Phi\}$.
2. As the Weyl group relative to Φ stabilises Φ, it also stabilises $\mathcal{T}$.

Lemma III.5.2 – *Let $x, y \in \mathcal{T}$. The following statements are equivalent:*
(i) for all $\alpha \in \Phi$, $(x, \alpha)(y, \alpha) > 0$;
(ii) $\Phi^+(x) = \Phi^+(y)$;
(iii) $\Delta(x) = \Delta(y)$.

Proof. Statement (i) means, for all $\alpha \in \Phi$, the sign of the nonzero real numbers (x, α) and (y, α) is the same. The equivalence between (i) and (ii) is therefore immediate. It is clear also that (ii) implies (iii), by definition of $\Delta(x)$ and $\Delta(y)$.

Let us now suppose that (iii) holds. By Theorem III.3.15, $\Delta(x)$ and $\Delta(y)$ are bases of Φ, so we are in position to apply Point 2 of Remark III.3.12, which gives $\Phi^+(x) = \Phi^+(y)$. ∎

Lemma III.5.2 suggests an equivalence relation on $\mathcal{T}$, denoted $\sim$, defined as follows. Let $x, y \in \mathcal{T}$, put

$$x \sim y \quad \text{if} \quad \forall \alpha \in \Phi, \; (x, \alpha)(y, \alpha) > 0.$$

In other terms, two elements of $\mathcal{T}$ are in relation if, for every root α, they are in the same half-space relative to P_α.

Lemma III.5.3 –
1. Relation $\sim$ is an equivalence relation.
2. This equivalence relation is compatible with the action of W_Φ on $\mathcal{T}$ (that is, $\forall\, x, y \in \mathcal{T}$ and $w \in W_\Phi$, if $x \sim y$, then $w(x) \sim w(y)$).

Proof. The first statement is clear. The second follows easily from the fact that W_Φ stabilise Φ. ∎

Definition III.5.4 – *Let Φ be a root system of E.*
1. Equivalence classes for the equivalence relation $\sim$ are called Weyl chambres.
2. If $x \in \mathcal{T}$, the Weyl chambre to which x belongs will be denoted $\mathrm{Ch}(x)$.

Remark III.5.5 – It can be shown that Weyl chambres are the connected components of the topological space $\mathcal{T}$ (equiped with the topology induced from that of the euclidean space E).

Remark III.5.6 – Let $\mathcal{P}(\Phi)$ stand for the set of subsets of Φ. Then, we have a map $\mathcal{T} \longrightarrow \mathcal{P}(\Phi)$, $x \mapsto \Delta(x)$ whose image is, by Theorem III.3.15, the set of all the bases of Φ. Lemma III.5.2 then shows that it induces an injection

$$\begin{array}{ccc} \mathcal{T}/\sim & \longrightarrow & \mathcal{P}(\Phi) \\ \mathrm{Ch}(x) & \mapsto & \Delta(x) \end{array}. \tag{III.5.24}$$

Hence, the set of Weyl chambres is in one-to-one correspondance with the set of bases of Φ.

Definition III.5.7 – *Let Δ be a base of Φ. The inverse image of Δ by the injective map (III.5.24) is called the* fundamental chambre *relative to* (Φ, Δ). *It will be denoted* $\mathrm{Ch}(\Delta)$.

Lemma III.5.8 – *Let Δ be a base of the root system Φ. Then* $\mathrm{Ch}(\Delta) = \{y \in \mathcal{T} \,|\, \forall \alpha \in \Delta,\, (y, \alpha) > 0\}$.

Proof. By definition, $\mathrm{Ch}(\Delta) = \{x \in \mathcal{T} \,|\, \Delta(x) = \Delta\}$. Hence, the result is just Point 5 in Remark III.3.16. ∎

Remark III.5.9 –
1. The Weyl group stabilises Φ, hence acts on Φ. It follows that it also acts on $\mathcal{P}(\Phi)$. Actually, it is easy to check that:

$$\forall\, x \in \mathcal{T} \quad \text{and} \quad \forall\, w \in W_\Phi, \qquad w(\Phi^+(x)) = \Phi^+(w(x)) \quad \text{and} \quad w(\Delta(x)) = \Delta(w(x)).$$

Hence, the map

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{P}(\Phi) \\ x & \mapsto & \Delta(x) \end{array}$$

is W_Φ-equivariant.
2. Recall from Lemma III.5.3 that the action of W_Φ on $\mathcal{T}$ is compatible with $\sim$. Hence, the action of W_Φ on $\mathcal{T}$ induces an action of W_Φ on $\mathcal{T}/\sim$. It follows from the above that:

$$\begin{array}{ccc} \mathcal{T}/\sim & \longrightarrow & \mathcal{P}(\Phi) \\ \mathrm{Ch}(x) & \mapsto & \Delta(x) \end{array}$$

is W_Φ equivariant. That is: the set of Weyl chambers is in bijection with the set of bases of Φ and this bijection commutes with the action of W_Φ.

III.6 Weyl group, generators and action.

In this section, $(\mathsf{E}, (-,-))$ is a euclidean space.

Let Φ be a root system of E and Δ a base of Φ. Lemma III.5.8 gives the first equality below and the second is clear by the definition of $\mathcal{T}$:

$$\mathrm{Ch}(\Delta) = \{y \in \mathcal{T} \,|\, \forall \alpha \in \Delta,\, (y, \alpha) > 0\} = \{y \in \mathsf{E} \,|\, \forall \alpha \in \Delta,\, (y, \alpha) > 0\}.$$

We now introduce the following notation.

Notation III.6.1 – Let Φ be a root system of E and Δ a base of Φ. Put:

$$\overline{\mathrm{Ch}(\Delta)} = \{y \in \mathsf{E} \,|\, \forall \alpha \in \Delta,\, (y, \alpha) \geq 0\}.$$

The following theorem collects fundamental results about the Weyl group. We prepare its proof with two exercises.

Exercise III.6.2 – Let E be a finite dimensional nonzero vector space over an infinite field. Let $r \in \mathbb{N}$, $r \geq 2$. For $1 \leq i \leq r$, let P_i be an hyperplane of E. If $P_1, \dots, P_r$ are pairwise distinct, then there exists an element of P_1 which is not in $\bigcup_{2 \leq i \leq r} P_i$. (This is a consequence of the result in Exercise III.3.8.)

Exercise III.6.3 – 1. Let T be a topological space, $r \in \mathbb{N}^*$, and $f : T \longrightarrow \mathbb{R}$, $f_i : T \longrightarrow \mathbb{R}$, $1 \leq i \leq r$ be continuous maps. Suppose $y \in T$ satisfies $f(y) = 0$ and, $f_i(y) \neq 0$, for $1 \leq i \leq r$. Then there exists an open subset U of T containing y such that, for all $x \in U$, and for all $1 \leq i \leq r$, $f(x) < |f_i(x)|$.
2. Let E be a euclidean vector space, $r \in \mathbb{N}^*$, and v, v_i be nonzero elements of E, $1 \leq i \leq r$. Suppose there exists $y \in E$ such that $(v, y) = 0$ and $(v_i, y) \neq 0$, for all $1 \leq i \leq r$. Then, there exists $x \in E$ such that $0 < (x, v) < |(x, v_i)|$ for all $1 \leq i \leq r$.

Theorem III.6.4 – *Let Φ be a root system of E and W_Φ its Weyl group. Let Δ be a base of Φ and W' the subgroup of W_Φ generated by the reflexions σ_α, $\alpha \in \Delta$.*
1. Let x be a regular element of E. There exists $w \in W'$ such that $w(x) \in \mathrm{Ch}(\Delta)$. In particular, the Weyl group acts transitively on the set $\mathcal{T}/\sim$ of Weyl Chambers.
2. If Δ' is a base of Φ, there exists $w \in W'$ such that $w(\Delta') = \Delta$. In particular, the Weyl group acts transitively on the set of bases of Φ.
3. Let $\alpha \in \Phi$. There exists $w \in W'$ such that $w(\alpha) \in \Delta$.
4. The Weyl group is generated by the reflections σ_α, $\alpha \in \Delta$; that is, $W_\Phi = W'$.
5. If w is an element of the Weyl group such that $w(\Delta) = \Delta$, then $w = \mathrm{id}$. In particular, the Weyl group acts simply transitively on the set of bases of Φ.

Proof. 1. Recall the element $\delta = \dfrac{1}{2} \displaystyle\sum_{\beta \in \Phi^+} \beta$ (see Notation III.4.3). As the set $\{(w(x), \delta),\, w \in W'\} \subseteq \mathbb{R}$ is finite, we may chose $w \in W'$ such that $(w(x), \delta)$ is its maximum. Now, let $\alpha \in \Delta$. Then $\sigma_\alpha w \in W'$ and thus

$$(w(x), \delta) \geq (\sigma_\alpha w(x), \delta) = (w(x), \sigma_\alpha(\delta)) = (w(x), \delta - \alpha)$$

(see Proposition III.4.4), which entails $(w(x), \alpha) \geq 0$. The second point of Remark III.5.1 then shows that $(w(x), \alpha) > 0$. Therefore, by Lemma III.5.8, $w(x) \in \mathrm{Ch}(\Delta)$.
2. This follows immediately from Point 1 by Remark III.5.9.
3. By Point 2, it suffices to show that α belongs to a base. By Exercise III.6.2, there exists $x \in P_\alpha$ such that, for all $\beta \in \Phi \setminus \{\pm\alpha\}$, $x \notin P_\beta$. Now, by Exercise III.6.3, it follows that there exists $y \in \mathsf{E}$ such that $0 < (y, \alpha) < |(y, \beta)|$, for all $\beta \in \Phi \setminus \{\pm\alpha\}$. Clearly, y must be regular and, in addition, we have $\alpha \in \Phi^+(y)$. It is easy to check that, actually, α is an indecomposable element of $\Phi^+(y)$, so that $\alpha \in \Delta(y)$.
4. Let $\alpha \in \Phi$. By Point 3, there exists $w \in W'$ such $w(\alpha) \in \Delta$. Corollary III.1.5 then shows that $w\sigma_\alpha w^{-1} = \sigma_{w(\alpha)} \in W'$. It follows that $\sigma_\alpha \in W'$.
5. Let w be an element of the Weyl group such that $w(\Delta) = \Delta$. By Point 4, w may be written as a product of reflections σ_α, $\alpha \in \Delta$. Suppose $w \neq \mathrm{id}$, Corollary III.4.6 shows that there exists a simple root sent by w to a negative root, which contradicts the hypothesis on w. Therefore, $w = \mathrm{id}$. The rest is clear since the action of W_Φ on the set of bases of Φ is transitive. ■

Definition III.6.5 – **Simple reflections** –
Let Φ be a root system of E and Δ a base of Φ. A simple reflection is a reflection σ_α with $\alpha \in \Delta$.

Definition III.6.6 – **Length of a Weyl group element** –
Let Φ be a root system of E, W_Φ its Weyl group and Δ a base of Φ.
1. Let $w \in W_\Phi$, $w \neq \mathrm{id}$. The least integer $t \in \mathbb{N}^*$ such that w may be written as the product of t simple reflections is denoted $\ell(w)$ and called the length *of w relative to Δ.*
2. In addition, we put $\ell(\mathrm{id}) = 0$.

Definition III.6.7 – **Reduced expression of a Weyl group element** –
Let Φ be a root system of E, W_Φ its Weyl group and Δ a base of Φ. Let $w \in W_\Phi$, $w \neq \mathrm{id}$. A reduced expression of w is a decomposition of w as a product of $\ell(w)$ simple reflections.

Notation III.6.8 – *Let Φ be a root system of E, W_Φ the Weyl group of Φ and Δ a base of Φ. If $w \in W_\Phi$, we denote $n(w)$ the cardinality of the set $\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$.*

Proposition III.6.9 – *Let Φ be a root system of E, W_Φ its Weyl group and Δ a base of Φ.*
1. If w is a Weyl group element of nonzero length, then there exists $w' \in W$ and $\alpha \in \Delta$ such that:
(i) $w = w'\sigma_\alpha$;
(ii) $\ell(w) = \ell(w') + 1$;
(iii) $w(\alpha) \in \Phi^-$.
2. For all $w \in W_\Phi$, $\ell(w) = n(w)$.

Proof. 1. Let $w = \sigma_{\alpha_1} \dots \sigma_{\alpha_\ell}$ be a reduced expression of w; hence $\alpha_1, \dots, \alpha_\ell \in \Delta$ and $\ell = \ell(\sigma)$. Put $\alpha = \alpha_\ell$ and $w' = w\sigma_\alpha$. Then, clearly, $w = w'\sigma_\alpha$ and $\ell(w') = \ell(w) - 1$. On the other hand, by Corollary III.4.6, $w(\alpha) \in \Phi^-$.
2. We proceed by induction on the length. The result is obvious for elements of length 0. (It is also true for elements of length 1 by Proposition III.4.4, Point 2.) Suppose now that $w \in W_\Phi$, $\ell(w) \geq 1$. Decompose w as Point 1 allows to. We have $w(\alpha) \in \Phi^-$. But then, Proposition III.4.4, Point 2, implies that $n(w\sigma_\alpha) = n(w) - 1$. But, $\ell(w\sigma_\alpha) = \ell(w) - 1$, so that the induction hypothesis, yields $n(w\sigma_\alpha) = \ell(w\sigma_\alpha)$. So, $n(w) = \ell(w)$. ■

Proposition III.6.10 – *Let Φ be a root system of E, W_Φ its Weyl group and Δ a base of Φ. The set $\overline{\mathrm{Ch}(\Delta)}$ is a fundamental domain for the action of W_Φ on E. That is, each W_Φ-orbit for this action intersect $\overline{\mathrm{Ch}(\Delta)}$ in exactly one point.*

Proof. Define a binary relation on E, denoted $\trianglerighteq$, by: for all $x, y \in \mathsf{E}$, $y \trianglerighteq x$ if $y-x \in \operatorname{Span}_{\mathbb{R}_{\geq 0}}(\Delta) = \operatorname{Span}_{\mathbb{R}_{\geq 0}}(\Phi^+)$. It is clear that this binary relation is an order on E.

Let $x \in \mathsf{E}$. As W_Φ is finite, so is the orbit $W_\Phi.x$ of x. It follows that $W_\Phi.x$ has a maximal element; let y be such an element. Let $\alpha \in \Delta$. Then, $\sigma_\alpha(y) = y - \langle y, \alpha \rangle \alpha$. Now suppose $(y, \alpha) \leq 0$, then $\sigma_\alpha(y) - y = -\langle y, \alpha \rangle \alpha \in \mathbb{R}^+ \Delta$. The maximality of y with respect to the above order entails $(y, \alpha) = 0$. This shows that $y \in \overline{\operatorname{Ch}(\Delta)}$. We have shown that any W_Φ-orbit intersect $\overline{\operatorname{Ch}(\Delta)}$.

Suppose now that $x, y \in \mathsf{E}$ are elements of $\overline{\operatorname{Ch}(\Delta)}$ such that there exists $w \in W_\Phi$ with $y = w(x)$. We wich to show that $x = y$. For this, we proceed by induction on the length of w. The result is trivial if $\ell(w) = 0$. Suppose now that w is an element of W_Φ such that $\ell(w) > 0$. By Proposition III.6.9, there exists $w' \in W_\Phi$ and $\alpha \in \Delta$ such that $w = w'\sigma_\alpha$, $\ell(w) = \ell(w') + 1$ and $w(\alpha) \in \Phi^-$. But, as $x, y \in \overline{\operatorname{Ch}(\Delta)}$, $0 \leq (x, \alpha) = (w^{-1}(y), \alpha) = (y, w(\alpha)) \leq 0$. Hence, $(x, \alpha) = 0$ and thus $y = w\sigma_\alpha(x)$. But, $\ell(w\sigma_\alpha) = \ell(w) - 1$. Thus, the induction hypothesis gives $x = y$. This terminates the proof. ∎

Exercise III.6.11 – Let Φ be a root system of E and Δ be a base of Φ. Let in addition $\lambda = \sum_{\alpha \in \Delta} k_\alpha \alpha$, with $k_\alpha \in \mathbb{Z}$, for all $\alpha \in \Delta$ and suppose that either $k_\alpha \geq 0$ for all $\alpha \in \Delta$ or $k_\alpha \leq 0$ for all $\alpha \in \Delta$. Then, either $\lambda \in \mathbb{R}\beta$, for some $\beta \in \Phi$, or there exists $w \in W_\Phi$ such that if $w(\lambda) = \sum_{\alpha \in \Delta} k'_\alpha \alpha$, with $\mathbb{k}'_\alpha \in \mathbb{Z}$, then there exists $\alpha, \beta \in \Delta$ such that $k'_\alpha > 0$ and $\mathbb{k}'_\beta < 0$.

III.7 Irreducible root systems.

In this section, $(\mathsf{E}, (-,-))$ is a euclidean space.

Lemma III.7.1 – *Let Φ be a root system of E. Suppose there exists a partition $\Phi = \Phi_1 \sqcup \Phi_2$ of Φ such that $\Phi_1 \perp \Phi_2$ and put $\mathsf{E}_i = \operatorname{Span}(\Phi_i)$, $i = 1, 2$. Then $\mathsf{E} = \mathsf{E}_1 \oplus \mathsf{E}_2$, $\mathsf{E}_1 \perp \mathsf{E}_2$ and $\Phi \subseteq \mathsf{E}_1 \cup \mathsf{E}_2$.*

Proof. This is clear. ∎

Lemma III.7.2 – *Let Φ be a root system of E. Suppose there exists a decomposition $\mathsf{E} = \mathsf{E}_1 \oplus \mathsf{E}_2$ of E into subspaces such that $\Phi \subseteq \mathsf{E}_1 \cup \mathsf{E}_2$ and put $\Phi_i = \Phi \cap \mathsf{E}_i$, $i = 1, 2$. Then, $\mathsf{E}_1 \perp \mathsf{E}_2$, $\Phi = \Phi_1 \sqcup \Phi_2$, and, for $i = 1, 2$, Φ_i is a root system of E_i.*

Proof. The hypotheses imply that $\Phi_1 \cap \Phi_2 = \emptyset$ and $\Phi = \Phi_1 \sqcup \Phi_2$.

Let x be an element of E. As Φ spans E, there exists $x_i \in \operatorname{Span}(\Phi_i) \subseteq \mathsf{E}_i$, $i = 1, 2$, such that $x = x_1 + x_2$. As $\mathsf{E} = \mathsf{E}_1 \oplus \mathsf{E}_2$, if $x \in E_1$, then $x = x_1$, which proves that $\operatorname{Span}(\Phi_1) = \mathsf{E}_1$. Similarly, $\operatorname{Span}(\Phi_2) = \mathsf{E}_2$.

Let $\alpha_i \in \Phi_i$, $i = 1, 2$. We have that $\sigma_{\alpha_1}(\alpha_2) = \alpha_2 - \langle \alpha_2, \alpha_1 \rangle \alpha_1 \in \Phi = \Phi_1 \sqcup \Phi_2$. As the sum of E_1 and E_2 is direct, the only possibility is $\sigma_{\alpha_1}(\alpha_2) = \alpha_2 \in \Phi_2$; that is $(\alpha_1, \alpha_2) = 0$. Whence, $\mathsf{E}_1 \perp \mathsf{E}_2$.

Let $\alpha \in \mathsf{E}_1$. The above shows that σ_α leaves Φ_2 (pointwise) fixed. However, as σ_α stabilises Φ, it must stabilise Φ_1. Similarly, if $\alpha \in \mathsf{E}_2$, σ_α must stabilise Φ_2.

The proof is complete. ∎

Definition III.7.3 – *Let Φ be a root system of E. We say that Φ is irreducible if $\mathsf{E} \neq (0)$ (equivalently $\Phi \neq \emptyset$) and there exist no partition $\Phi = \Phi_1 \sqcup \Phi_2$ of Φ with Φ_1 and Φ_2 nonempty and orthogonal.*

Remark III.7.4 – Let Φ be a root system of E, let E' be a euclidean space and Φ' a root system of E' and let $\varphi : \mathsf{E} \longrightarrow \mathsf{E}'$ be an isomorphism between (E, Φ) and (E', Φ'). Then Φ is irreducible if and only if Φ' is.

Proposition III.7.5 – **Reducibility of root systems** – *Suppose* $\mathsf{E} \neq (0)$*. Let* Φ *be a root system of* E*. There exists* $k \in \mathbb{N}^*$ *and subspaces* E_i *of* E*,* $1 \leq i \leq k$*, such that, if we put* $\Phi_i = \Phi \cap \mathsf{E}_i$*, then:*
1. $\Phi = \sqcup_{1 \leq i \leq k} \Phi_i$ *;*
2. Φ_i *is an irreducible root system of* E_i*, for* $1 \leq i \leq k$*;*
3. E *is the orthogonal direct sum of the subspaces* E_i*,* $1 \leq i \leq k$*.*
Further, such a decomosition of (E, Φ) *is unique (up to the permutation of indices).*

Proof. To prove the existence, we proceed by induction on the dimension of E.

The result is clear if $\dim_{\mathbb{R}}(\mathsf{E}) = 1$ since then any root system is irreducible.

Suppose now that E has dimension $d \in \mathbb{N}$ with $d \geq 2$. If Φ is irreducible, then there is nothing to do. Otherwise, there exists a partition $\Phi = \Phi_1 \sqcup \Phi_2$ of Φ into nonempty subsets such that $\Phi_1 \perp \Phi_2$. Put $\mathsf{E}_i = \mathrm{Span}(\Phi_i)$, $i = 1, 2$. By Lemma III.7.1, $\mathsf{E} = \mathsf{E}_1 \oplus \mathsf{E}_2$, $\mathsf{E}_1 \perp \mathsf{E}_2$ and $\Phi \subseteq \mathsf{E}_1 \cup \mathsf{E}_2$. Clearly, $\Phi_i = \Phi \cap \mathsf{E}_i$, $i = 1, 2$. By Lemma III.7.2, for $i = 1, 2$, Φ_i is a root system of E_i. Applying the induction hypothesis to (E_1, Φ_1) and (E_2, Φ_2) shows that (E, Φ) enjoys a decompositon as required.

Let us now prove the unicity. We begin with an observation. Suppose $\Phi = \Phi' \sqcup \Phi''$ is a partition of Φ with $\Phi' \perp \Phi''$. Then, for all $1 \leq i \leq k$, $\Phi_i = (\Phi' \cap \Phi_i) \sqcup (\Phi'' \cap \Phi_i)$, and $(\Phi' \cap \Phi_i) \perp (\Phi'' \cap \Phi_i)$. Now, Φ_i being an irreducible root system of E_i, this forces $\Phi' \cap \Phi_i = \emptyset$ or $\Phi'' \cap \Phi_i = \emptyset$.

Suppose now that we are given $l \in \mathbb{N}^*$ and subspaces E'_i of E, $1 \leq i \leq l$, such that, if we put $\Phi'_i = \Phi \cap \mathsf{E}'_i$ then $\Phi = \sqcup_{1 \leq i \leq l} \Phi'_i$, Φ'_i is an irreducible root system of E'_i, for $1 \leq i \leq l$ and E is the orthogonal direct sum of the subspaces E'_i, $1 \leq i \leq l$. Given $1 \leq i \leq k$, it is clear that there must exist $1 \leq j \leq l$ such that $\Phi_i \cap \Phi'_j \neq \emptyset$. Now, the above observation applied with $\Phi' = \Phi'_j$ and $\Phi'' = \sqcup_{p \neq j} \Phi'_p$ shows that $\Phi_i \subseteq \Phi'_j$. Further, a similar argument gives $\Phi_i \supseteq \Phi'_j$. In particular, such a j must be unique. We have therefore defined a map $\{1, \ldots, k\} \longrightarrow \{1, \ldots, l\}$ that associates to i the unique j such that $\Phi_i = \Phi'_j$. The injectivity of this map is obvious, its surjectivity follows from the fact that, for $1 \leq i \leq l$, (E'_i, Φ'_i) is irreducible and hence Φ'_i not empty.

It follows that $k = l$ and that there exists a permutation $\sigma \in \mathfrak{S}_k$ such that $\Phi'_i = \Phi_{\sigma(i)}$, $1 \leq i \leq k$. In addition, for all $1 \leq i \leq k$, $\mathsf{E}_{\sigma(i)} = \mathrm{Span}_{\mathbb{R}}(\Phi_{\sigma(i)}) = \mathrm{Span}_{\mathbb{R}}(\Phi'_i) = \mathsf{E}'_i$. The result is proved. ∎

Definition III.7.6 – *Suppose* $\mathsf{E} \neq (0)$ *and let* Φ *be a root system of* E*. The decomposition of* (E, Φ) *given by Proposition III.7.5 is called the decomposition of* (E, Φ) *into irreducible components.*

Exercise III.7.7 – Suppose $\mathsf{E} \neq (0)$ and let Φ be a root system of E. Let E' be a euclidean space and Φ' a root system of E'. If φ is an isomorphism from (E, Φ) to (E', Φ'), then φ sends the decomposition of (E, Φ) into irreducible components onto the decomposition of (E', Φ') into irreducible components.

Proposition III.7.8 – *Let* Φ *be a root system of* E *and* Δ *a base of* E*. The root system* Φ *is irreductible if and only if* Δ *cannot be partitionned into two nonempty orthogonal subsets.*

Proof. Suppose Φ is reductible. By definition, there exists a partition $\Phi = \Phi_1 \sqcup \Phi_2$ of Φ into non empty and orthogonal subsets. Suppose $\Delta \subseteq \Phi_1$. Then, Δ being a basis of E, an element of Φ_2 must be orthogonal to itself, hence zero; a contradiction. So, $\Delta \not\subseteq \Phi_1$. Similarly, $\Delta \not\subseteq \Phi_2$. Whence, Δ is the disjoint union of the nonempty subsets $\Delta \cap \Phi_1$ and $\Delta \cap \Phi_2$.

Conversely, suppose Δ is the disjoint union of two nonempty subsets Δ_1 and Δ_2. Put $\Phi_i = W_\Phi.\Delta_i$, $i = 1, 2$.

Point 3 of Theorem III.6.4 together with the stability of Φ under W_Φ imply that $\Phi = \Phi_1 \cup \Phi_2$. Further, for $i = 1, 2$,

$$\Phi_i \subseteq \mathrm{Span}(\Delta_i).$$

Indeed, let $\alpha \in \Delta_1$ and $w \in W_\Phi$. By Theorem III.6.4, w is the product of reflexions associated to simple roots. But, reflections associated to orthogonal vectors commute and α is invariant under any reflexion associated to an element of Δ_2. Hence, $w(\alpha)$ is the image of α under a product of reflections associated to elements of Δ_1. It follows that $\alpha \in \mathrm{Span}(\Delta_1)$. Hence, $\Phi_1 \subseteq \mathrm{Span}(\Delta_1)$. Similarly, $\Phi_2 \subseteq \mathrm{Span}(\Delta_2)$. As a consequence $\Phi_1 \perp \Phi_2$. It follows that Φ is reducible. ■

Lemma III.7.9 – *Let Φ be an irreductible root system of E and Δ a base of E.*
1. The ordered set $(\Phi, \preceq)$ has a maximum element (see Definition III.3.5).
2. Let μ be the maximum element of $(\Phi, \preceq)$, then:
2.1. $\forall \alpha \in \Phi$, $\alpha \neq \mu$, $\mathrm{ht}(\alpha) < \mathrm{ht}(\mu)$ (see Definition III.3.4);
2.2. $\forall \alpha \in \Delta$, $(\mu, \alpha) \geq 0$;
2.3. if $\mu = \sum_{\alpha \in \Delta} k_\alpha \alpha$, $k_\alpha \in \mathbb{N}$, $\forall \alpha \in \Delta$, then $k_\alpha \in \mathbb{N}^*$.

Proof. Observe that the result is easy if $\dim_{\mathbb{R}}(\mathsf{E}) = 1$. We thus suppose now that $\dim_{\mathbb{R}}(\mathsf{E}) \geq 2$.

The ordered set $(\Phi, \preceq)$ must have a maximal element since Φ is finite. Let μ be such an element. Observe that $\mu \preceq 0$ would then force $\mu \prec \alpha$ for any simple root α, which contradicts the maximality of μ. So that $0 \preceq \mu$ and thus $\mu = \sum_{\alpha \in \Delta} k_\alpha \alpha$, $k_\alpha \in \mathbb{N}$, $\forall \alpha \in \Delta$. Let

$$\Delta_1 = \{\alpha \in \Delta \,|\, k_\alpha > 0\} \quad \text{and} \quad \Delta_2 = \{\alpha \in \Delta \,|\, k_\alpha = 0\},$$

so that $\Delta = \Delta_1 \sqcup \Delta_2$.

Suppose that $\Delta_2 \neq \emptyset$. By Lemme III.3.7, for all $\alpha \in \Delta_2$, $(\alpha, \mu) \leq 0$. But, as Φ is irreductible, Proposition III.7.8 implies that there exists an element of Δ_2 which is not orthogonal to all the elements of Δ_1. Hence, there exists $\alpha \in \Delta_2$ such that $(\alpha, \mu) < 0$. But then, Lemma III.2.14 implies that $\mu + \alpha \in \Phi$, which contradicts the maximality of μ. Hence, we must have $\Delta_2 = \emptyset$.

Now, Lemma III.2.14, together with the maximality of μ implies that, for all $\alpha \in \Delta$, $(\mu, \alpha) \geq 0$. Further, as Δ spans E, there exists $\alpha \in \Delta$ such that $(\mu, \alpha) > 0$.

Let μ' be any maximal element of $(\Phi, \preceq)$. The above applies to it: there exits $k'_\alpha \in \mathbb{N}^*$, $\alpha \in \Delta$, such that $\mu' = \sum_{\alpha \in \Delta} k'_\alpha \alpha$. And, since there exists $\alpha \in \Delta$ such that $(\mu, \alpha) > 0$, we have $(\mu, \mu') > 0$. If we suppose $\mu \neq \mu'$, then μ and μ' are not proportianal since they are both positive roots (see above). Thus Lemma III.2.14 applies and shows that $\mu - \mu'$ is a root, with implies that $\mu \preceq \mu'$ or $\mu' \preceq \mu$ and hence $\mu = \mu'$; a contradiction. therefore $\mu = \mu'$.

At this stage, we have shown Points 1, 2.2 and 2.3. In addition, Point 2.1 is clear. ■

Lemma III.7.10 – *Let Φ be an irreducible root system of E.*
1. The natural action of the Weyl group on E is irreductible.
2. For all $\alpha \in \Phi$, $W_\Phi.\alpha$ spans E.

Proof. Let F be a subspace of E stable under the action of W_Φ.

Let $\alpha \in \Phi$. Suppose $\alpha \notin F$. As $\sigma_\alpha(F) \subseteq F$, Lemma III.1.3 implies that $F \subseteq (\mathbb{R}\alpha)^\perp$. As a consequence, $\alpha \in F^\perp$. This shows that $\Phi \subseteq F \cup F^\perp$. So $\Phi = (\Phi \cap F) \sqcup (\Phi \cap F^\perp)$. As Φ is irreductible, we must have $\Phi = \Phi \cap F$ or $\Phi = \Phi \cap F^\perp$. But Φ spans E, so $F = \mathsf{E}$ or $F = (0)$. Point 1 is proved.

Let $\alpha \in \Phi$. It is clear that $\mathrm{Span}(W_\Phi.\alpha)$ is a nonzero subspace of E stable under the action of W_Φ. Point 1 then gives Point 2. ■

Remark III.7.11 – Let Φ be an irreducible root system of E. Let $\alpha, \beta \in \Phi$. By Lemma III.7.10, $W_\Phi.\alpha$ spans E, so there must exist $w \in W_\Phi$ such that $w(\alpha) \not\perp \beta$.

Exercise III.7.12 – **Isomorphisms of irreducible root systems** – Let (E, Φ) and (E', Φ') be root systems and let φ be an isomorphism between them.
1. For all $\alpha, \beta \in \Phi$, with $\alpha \not\perp \beta$,

$$\frac{(\varphi(\alpha), \varphi(\alpha))}{(\alpha, \alpha)} = \frac{(\varphi(\beta), \varphi(\beta))}{(\beta, \beta)}.$$

2. Suppose Φ is irreducible.
2.1. The equality of question 1 holds for all $\alpha, \beta \in \Phi$.
2.2. The isomorphism φ is an isometry, up to multiplication by an element of $\mathbb{R}_{>0}$.

Lemma III.7.13 – *Let Φ be an irreductible root system of E.*
1. The set of lengths of elements of Φ is of cardinality at most 2.
2. If $\alpha, \beta \in \Phi$ have the same length, there exists $w \in W_\Phi$ such that $\beta = w(\alpha)$.

Proof. Let $\alpha, \beta \in \Phi$. According to Remark III.7.11 there exists $w \in W_\Phi$ such that $w(\alpha) \not\perp \beta$. As the lenght of α and $w(\alpha)$ are the same, it follows from Remark III.2.13 that $||\alpha||^2/||\beta||^2 \in \{1/3, 1/2, 1, 2, 3\}$. Suppose that there exists three roots α, β, γ with pairwise distinct lengths. We can order them so that $||\alpha||^2 < ||\beta||^2 < ||\gamma||^2$. This implies that $||\beta||^2/||\alpha||^2 = 2$ and $||\gamma||^2/||\alpha||^2 = 3$. Which entails that $||\gamma||^2/||\beta||^2 = 3/2$; a contradiction. Point 1 is proved.

Let us now prove Point 2. If $\alpha, \beta \in \Phi$ have the same length, by Remark III.7.11, there exists $w \in W_\Phi$ such that $w(\alpha)$ and β be nonorthogonal roots with the same length. Hence, to prove Point 2, we may assume that α and β are not orthogonal. The case where $\alpha = \beta$ is trivial. The case where $\alpha = -\beta$ is easy since then $\sigma_\beta(\beta) = -\beta$. Suppose now that α and β are nonorthogonal and nonproportionnal. By Remark III.2.13, $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \pm 1$. Changing β in $-\beta$ if necessary (which we can do without loss of generality since opposit roots are in the same W_Φ-orbit), we may suppose that $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = 1$. Then $\sigma_\alpha \sigma_\beta \sigma_\alpha(\beta) = \alpha$. Point 2 is proved. ■

Definition III.7.14 – *Let Φ be an irreductible root system of E. (See Lemma III.7.13.)*
1. If the set of lengths of elements of Φ has cardinality 2, the roots with the greatest length are called long roots*, the others are called* short roots*.*
2. If the set of lengths of elements of Φ has cardinality 1, all the roots are called long roots*.*

Lemma III.7.15 – *Let Φ be an irreductible root system of E with two root lengths. The maximum root of Φ (see Lemma III.7.9) is long.*

Proof. Fix a base Δ of Φ. Let μ be the maximum root of Φ. We must show that $(\mu, \mu) \geq (\alpha, \alpha)$, for all $\alpha \in \Phi$.

Let $\alpha \in \Phi$. Then, by Proposition III.6.10, we know that the W_Φ-orbit of α contains an element of the set $\overline{\mathrm{Ch}(\Delta)}$, which clearly is a root. For this reason, we can assume without loss of generality that $\alpha \in \overline{\mathrm{Ch}(\Delta)}$. By Lemma III.7.9 and our assumption on α, $\alpha, \mu \in \overline{\mathrm{Ch}(\Delta)}$. Since $0 \preceq \mu - \alpha$, it follows that $(\mu, \mu) - (\alpha, \alpha) = (\mu - \alpha, \mu + \alpha) = (\mu - \alpha, \mu) + (\mu - \alpha, \alpha) \geq 0$.

The result is proved. ■

III.8 Examples.

Type A_ℓ***,*** $\ell \in \mathbb{N}^*$***.*** Consider the euclidean space $\mathbb{R}^{\ell+1}$ equipped with the standard scalar product. Let $(\varepsilon_1, \ldots, \varepsilon_{\ell+1})$ be the canonical basis of $\mathbb{R}^{\ell+1}$. Put

$$I = \bigoplus_{1 \leq i \leq \ell+1} \mathbb{Z}\varepsilon_i.$$

Let E be the hyperplane of $\mathbb{R}^{\ell+1}$ defined by:

$$\mathsf{E} = (\mathbb{R}(\varepsilon_1 + \ldots + \varepsilon_{\ell+1}))^{\perp};$$

hence $(\mathsf{E}, (-,-))$ is a euclidean space of dimension ℓ. Consider the set Φ of elements of E belonging to I and whose norm is $\sqrt{2}$:

$$\Phi = \{x \in \mathsf{E} \cap I \,|\, (x,x) = 2\}.$$

It is emediate that

$$\Phi = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i \neq j \leq \ell+1\}.$$

Put now

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1},\, 1 \leq i \leq \ell\}.$$

It is clear that Δ is a linearly independant family of $\mathbb{R}^{\ell+1}$ and a basis of the $\mathbb{R}$-vector space E.

Let $1 \leq i \leq \ell$. Put $\sigma_i = \sigma_{\varepsilon_i - \varepsilon_{i+1}} \in \mathcal{O}(\mathsf{E})$. Clearly, σ_i is the restriction to E of the reflection τ_i of $\mathbb{R}^{\ell+1}$ associated to $\varepsilon_i - \varepsilon_{i+1}$.

An easy computation shows that, for $1 \leq i \leq \ell$,

$$\tau_i(\varepsilon_i) = \varepsilon_{i+1}, \quad \tau_i(\varepsilon_{i+1}) = \varepsilon_i, \quad \text{and} \quad \tau_i(\varepsilon_k) = \varepsilon_k,\, 1 \leq k \leq \ell+1,\, k \neq i,\, k \neq i+1.$$

Lemma III.8.1 – *Keep the above notation. Then,*
1. Φ *is an irreducible root system and* Δ *a base of* Φ*;*
2. W_Φ *is isomorphic to* $\mathfrak{S}_{\ell+1}$*.*

Proof. The fact that Φ is a root system and Δ a base of Φ is a straightforward verification. The irreducibility of Δ is easy to prove using Proposition III.7.8. (Suppose that we are given a partition $\Delta = \Delta_1 \sqcup \Delta_2$ with $\Delta_1 \perp \Delta_2$ and $\varepsilon_1 - \varepsilon_2 \in \Delta_1$. Since $\varepsilon_1 - \varepsilon_2$ and $\varepsilon_2 - \varepsilon_3$ are not orthogonal, we must have $\varepsilon_2 - \varepsilon_3 \in \Delta_1$, etc; so that $\Delta_2 = \emptyset$.)

By Theorem III.6.4, W_Φ is generated by the simple reflections $\sigma_1, \ldots, \sigma_\ell$. On the other hand, there exists a morphism of groups

$$\mathcal{O}(\mathsf{E}) \longrightarrow \mathcal{O}(\mathbb{R}^{\ell+1})$$

mapping an orthogonal automorphism of E onto its extension as an orthogonal automorphism of $\mathbb{R}^{\ell+1}$ acting by the identity on $\mathsf{E}^\perp$. This morphism is clearly injective. Hence, it identifies W_Φ with the subgroup of $\mathcal{O}(\mathbb{R}^{\ell+1})$ generated by the reflections τ_i, $1 \leq i \leq \ell$. But, the above shows that, for $1 \leq i \leq \ell$, τ_i permutes ε_i and ε_{i+1} leaving invariant any other vector of the canonical basis of $\mathbb{R}^{\ell+1}$. Thus, the image of W_Φ is the subgroup of $\mathcal{O}(\mathbb{R}^{\ell+1})$ of those automorphisms that permute the canonical basis of $\mathbb{R}^{\ell+1}$. Therefore, W_Φ identifies with the symmetric group $\mathfrak{S}_{\ell+1}$, as requierred. ■

Definition III.8.2 – *The root system of Lemma III.8.1 is called the root system of type* A_ℓ*,* $\ell \in \mathbb{N}^*$*.*

Type B_ℓ***,*** $\ell \in \mathbb{N}$***,*** $\ell \geq 2$***.** We put $\mathsf{E} = \mathbb{R}^\ell$ and endow it with its standard euclidean structure. We let $\varepsilon_1, \ldots, \varepsilon_\ell$ be the canonical basis of $\mathbb{R}^\ell$ and put

$$I = \bigoplus_{1 \leq i \leq \ell} \mathbb{Z}\varepsilon_i.$$

We then denote Φ the set of those elements in I whose norm is 1 or $\sqrt{2}$. Clearly,

$$\Phi = \{\pm\varepsilon_i,\, 1 \leq i \leq \ell\} \sqcup \{\pm\varepsilon_i \pm \varepsilon_j \,|\, 1 \leq i < j \leq \ell\}.$$

Put now

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1},\, 1 \leq i \leq \ell - 1\} \sqcup \{\varepsilon_\ell\}.$$

It is clear that Δ is a basis of the $\mathbb{R}$-vector space $\mathbb{R}^\ell$.

For $1 \leq i \leq \ell$, put $\tau_i = \sigma_{\varepsilon_i}$. For $1 \leq i \leq \ell - 1$, put $\sigma_i = \sigma_{\varepsilon_i - \varepsilon_{i+1}}$. The action of the above reflections on the canonical basis is follows.

$$\text{For } 1 \leq i \leq \ell - 1, \quad \sigma_i(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if} \quad k \neq i, i+1 \\ \varepsilon_{i+1} & \text{if} \quad k = i \\ \varepsilon_i & \text{if} \quad k = i+1 \end{cases}. \tag{III.8.25}$$

$$\text{For } 1 \leq i \leq \ell, \quad \tau_i(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if} \quad k \neq i \\ -\varepsilon_i & \text{if} \quad k = i \end{cases}. \tag{III.8.26}$$

There is an injective morphism of groups $\mathfrak{S}_\ell \longrightarrow \mathcal{O}(\mathbb{R}^\ell)$, which sends a permutation p to the linear automorphism, denoted f_p, that sends ε_k to $\varepsilon_{p(k)}$. Let S be its image.

There is an injective morphism of groups $(\mathbb{Z}/2\mathbb{Z})^\ell \longrightarrow \mathcal{O}(\mathbb{R}^\ell)$, which, for $(z_1, \ldots, z_\ell) \in \mathbb{Z}^\ell$, sends $\chi = (z_1 + 2\mathbb{Z}, \ldots, z_\ell + 2\mathbb{Z}) \in \mathbb{Z}/2\mathbb{Z}^\ell$ to the linear automorphism, denoted f_χ, that sends ε_k to $(-1)^{z_k}\varepsilon_k$. Let Z be its image.

Denote by $\mathrm{Aut}_{\mathrm{group}}\left((\mathbb{Z}/2\mathbb{Z})^\ell\right)$ the group of automorphisms of group of $(\mathbb{Z}/2\mathbb{Z})^\ell$. There is a morphism of groups as follows:

$$\varphi \ : \ \mathfrak{S}_\ell \ \longrightarrow \ \mathrm{Aut}_{\mathrm{group}}\left((\mathbb{Z}/2\mathbb{Z})^\ell\right)$$

where, for $p \in \mathfrak{S}_\ell$ and $(\chi_1, \ldots, \chi_\ell) \in (\mathbb{Z}/2\mathbb{Z})^\ell$, $\varphi(p)((\chi_1, \ldots, \chi_\ell)) = (\chi_{p^{-1}(1)}, \ldots, \chi_{p^{-1}(\ell)})$. It allows to form the semidirect product $(\mathbb{Z}/2\mathbb{Z})^\ell \rtimes_\varphi \mathfrak{S}_\ell$. Combining the two maps of groups above, we can define a map

$$\begin{array}{ccc} (\mathbb{Z}/2\mathbb{Z})^\ell \rtimes_\varphi \mathfrak{S}_\ell & \longrightarrow & \mathcal{O}(\mathbb{R}^\ell) \\ (\chi, p) & \mapsto & f_\chi f_p \end{array} \tag{III.8.27}$$

which, as one easily verifies, is an injective morphism of groups.

Lemma III.8.3 – *Keep the above notation. Then,*
1. Φ *is an irreducible root system and* Δ *a base of* Φ*;*
2. W_Φ *is isomorphic to the semi-direct product* $(\mathbb{Z}/2\mathbb{Z})^\ell \rtimes_\varphi \mathfrak{S}_\ell$*, where* $\mathfrak{S}_\ell$ *acts on* $(\mathbb{Z}/2\mathbb{Z})^\ell$ *via the map* φ *above (that is, by permutation of factors).*

Proof. 1. It is not difficult to show that Φ is a root system with base Δ. The argument used in type A works again to show that Φ is irreducible.
2. The group morphism (III.8.27) sends the canonical generators of the group $(\mathbb{Z}/2\mathbb{Z})^\ell$ to the orthogonal automorphisms τ_i, $1 \leq i \leq \ell$, and the elementary transposition $(i, i+1)$ to the orthogonal automorphism σ_i, $1 \leq i \leq \ell - 1$. Hence, its image is included in W_Φ, since all these automorphisms belong to W_Φ. On the other hand, W_Φ is generated by the simple reflections (cf. Theorem III.6.4) which are $\tau_\ell, \sigma_1, \ldots, \sigma_{\ell-1}$ and all belong to the image of the group morphism (III.8.27). Therefore, the image of this injective group morphism is W_Φ. This proves Point 2. ∎

Definition III.8.4 – *The root system of Lemma III.8.3 is called the root system of type B_ℓ, $\ell \in \mathbb{N}$, $\ell \geq 2$.*

Type C_ℓ, $\ell \in \mathbb{N}$, $\ell \geq 2$. We put $\mathsf{E} = \mathbb{R}^\ell$ and endow it with its standard euclidean structure. We retain the notation used for the description of the root system of type B_ℓ, $\ell \in \mathbb{N}$, $\ell \geq 2$.

Definition III.8.5 – *Let $\ell \in \mathbb{N}^*$. The dual root system (see Exercise III.2.9) of the root system of type B_ℓ is called the root system of type C_ℓ.*

Remark III.8.6 – Let $\ell \in \mathbb{N}$, $\ell \geq 2$. It is easy to describe the root system of type C_ℓ. Denote it by Φ, then, the following holds.
1. We have
$$\Phi = \{\pm 2\varepsilon_i,\, 1 \leq i \leq \ell\} \sqcup \{\pm\varepsilon_i \pm \varepsilon_j \,|\, 1 \leq i < j \leq \ell\}.$$
2. The subset
$$\Delta = \{\varepsilon_i - \varepsilon_{i+1},\, 1 \leq i \leq \ell - 1\} \sqcup \{2\varepsilon_\ell\}$$
is a base of Φ, by Proposition III.3.17.
3. The root system Φ is irreducible (since the root system of type B_ℓ is).
4. The Weyl group of Φ is isomorphic to the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^\ell \rtimes \mathfrak{S}_\ell$, where $\mathfrak{S}_\ell$ acts on $(\mathbb{Z}/2\mathbb{Z})^\ell$ by permutation of factors (because it is the same as the Weyl group of the root system of type B_ℓ).

Type D_ℓ, $\ell \in \mathbb{N}$, $\ell \geq 4$. We put $\mathsf{E} = \mathbb{R}^\ell$ and endow it with its standard euclidean structure. We let $\varepsilon_1, \ldots, \varepsilon_\ell$ be the canonical basis of $\mathbb{R}^\ell$ and put $I = \bigoplus_{1 \leq i \leq \ell} \mathbb{Z}\varepsilon_i$. We then let:
$$\Phi = \{x \in I \,|\, (x, x) = 2\} = \{\pm\varepsilon_i \pm \varepsilon_j \,|\, 1 \leq i < j \leq \ell\},$$
$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \leq i \leq \ell - 1$, $\alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$,
$$\Delta = \{\alpha_1, \ldots, \alpha_\ell\},$$
and $\sigma_i = \sigma_{\alpha_i}$, for $1 \leq i \leq \ell$. It is easy to see that Δ is a basis of the $\mathbb{R}$-vector space $\mathbb{R}^\ell$.

Using direct calculations, the action of the reflections associated to elements of Φ on the canonical basis are as follows.
$$\text{For } 1 \leq i < j \leq \ell, \quad \sigma_{\varepsilon_i - \varepsilon_j} : \begin{cases} \varepsilon_k & \mapsto & \varepsilon_k & \text{if} \quad k \neq i, j \\ \varepsilon_i & \mapsto & \varepsilon_j & \\ \varepsilon_j & \mapsto & \varepsilon_i & \end{cases}. \tag{III.8.28}$$
$$\text{For } 1 \leq i < j \leq \ell, \quad \sigma_{\varepsilon_i + \varepsilon_j} : \begin{cases} \varepsilon_k & \mapsto & \varepsilon_k & \text{if} \quad k \neq i, j \\ \varepsilon_i & \mapsto & -\varepsilon_j & \\ \varepsilon_j & \mapsto & -\varepsilon_i & \end{cases}. \tag{III.8.29}$$
Consider the two maps
$$\begin{array}{ccc} (\mathbb{Z}/2\mathbb{Z})^\ell & \xrightarrow{i} & \mathcal{O}(\mathsf{E}) \\ (\overline{z_1}, \ldots, \overline{z_\ell}) & \mapsto & [\varepsilon_i \mapsto (-1)^{z_i}\varepsilon_i,\, 1 \leq i \leq \ell] \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathfrak{S}_\ell & \xrightarrow{j} & \mathcal{O}(\mathsf{E}) \\ p & \mapsto & \left[\varepsilon_i \mapsto \varepsilon_{p(i)},\, 1 \leq i \leq \ell\right] \end{array}.$$
These are injective morphisms of groups. We have the following identity:
$$\forall\, (\overline{z_1}, \ldots, \overline{z_\ell}) \in (\mathbb{Z}/2\mathbb{Z})^\ell,\ \forall p \in \mathfrak{S}_\ell,\ j(p)i((\overline{z_1}, \ldots, \overline{z_\ell}))j(p)^{-1} = i(p.(\overline{z_1}, \ldots, \overline{z_\ell})) \tag{III.8.30}$$

where the dot in the rightmost term is the natural action of $\mathfrak{S}_\ell$ on $(\mathbb{Z}/2\mathbb{Z})^\ell$ by permutation of factors:

$$\begin{array}{ccc} \mathfrak{S}_\ell \times (\mathbb{Z}/2\mathbb{Z})^\ell & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^\ell \\ (p, (\overline{z_1}, \ldots, \overline{z_\ell}) & \mapsto & (\overline{z_{p^{-1}(1)}}, \ldots, \overline{z_{p^{-1}(\ell)}}) \end{array} ; \tag{III.8.31}$$

notice that this is an action by automorphisms of groups.

Let P denote the subgroup of $(\mathbb{Z}/2\mathbb{Z})^\ell$ of those elements $(\overline{z_1}, \ldots, \overline{z_\ell})$ such that $(-1)^{z_1+\cdots+z_\ell} = 1$ (that is, among the coordinates, $\overline{1}$ appears an even number of times). The action of $\mathfrak{S}_\ell$ clearly stabilises P. So, we may form the semi-direct product relative to this action:

$$P \rtimes \mathfrak{S}_\ell.$$

In addition, relations (III.8.30) show that the subgroup of $\mathcal{O}(\mathsf{E})$ generated by $i(P)$ and $j(\mathfrak{S}_\ell)$ is just $i(P)j(\mathfrak{S}_\ell)$ and that $i(P)$ is a normal subgroup of $i(P)j(\mathfrak{S}_\ell)$. Since, in addition, the intersection of $i(P)$ and $j(\mathfrak{S}_\ell)$ is clearly reduced to the identity, we get that the map

$$\begin{array}{ccc} P \rtimes \mathfrak{S}_\ell & \longrightarrow & \mathcal{O}(\mathsf{E}) \\ ((\overline{z_1}, \ldots, \overline{z_\ell}), p) & \mapsto & i((\overline{z_1}, \ldots, \overline{z_\ell}))j(p) \end{array} \tag{III.8.32}$$

is an injective group morphism with image $i(P)j(\mathfrak{S}_\ell)$.

Lemma III.8.7 – *Keep the above notation. Then,*
1. Φ is an irreducible root system and Δ a base of Φ;
2. W_Φ is isomorphic to the semi-direct product $P \rtimes \mathfrak{S}_\ell$, where $\mathfrak{S}_\ell$ acts on P by permutation of factors.

Proof. 1. It is easy to verify that Φ is a root system and Δ a base of Φ. If we suppose that Δ is the disjoint union of two orthogonal subsets : $\Delta = \Delta_1 \sqcup \Delta_2$ with $\alpha_1 \in \Delta_1$, the orthogonality condition forces $\alpha_2, \ldots, \alpha_{\ell-2}$ to be in Δ_1 and then $\alpha_{\ell-1}$ and α_ℓ as well. Hence, we must have $\Delta_2 = \emptyset$ from which the irreducibility of Φ follows, by Proposition III.7.8.
2. Recall the injective group morphism of (III.8.32). It follows easily from (III.8.28) and (III.8.29) that the image of this morphism is just W_Φ. Hence the result. ■

Definition III.8.8 – *The root system of Lemma III.8.7 is called the root system of type D_ℓ, $\ell \in \mathbb{N}$, $\ell \geq 4$.*

III.9 Weights associated to a root system.

In this section, $(\mathsf{E}, (-,-))$ is a euclidean space, Φ a root system and W_Φ the Weyl group of Φ.

Recall the notation:

$$\langle x, y \rangle = 2\frac{(x,y)}{(y,y)}, \quad \forall (x,y) \in \mathsf{E}^2, \quad y \neq 0.$$

Definition III.9.1 – *A weight is an element $\lambda \in \mathsf{E}$ satisfying the following property:*

$$\forall\, \alpha \in \Phi, \quad \langle \lambda, \alpha \rangle \in \mathbb{Z}.$$

The set of weights of (E, Φ) will be denoted Λ_Φ.

Remark III.9.2 – Clearly, Λ_Φ is a subgroup of E and $\Phi \subseteq \Lambda_\Phi \subseteq \mathsf{E}$.

Lemma III.9.3 – *Let Δ be a base of Φ. The following description of Λ_Φ holds: $\Lambda_\Phi = \{\lambda \in \mathsf{E} \,|\, \langle \lambda, \alpha \rangle \in \mathbb{Z},\ \forall \alpha \in \Delta\}$.*

Proof. We may reformulate the definition of Λ_Φ via the dual root system of Φ (see Exercise III.2.9). We have to show that

$$\{\lambda \in \mathsf{E} \,|\, (\lambda, \alpha^\vee) \in \mathbb{Z},\ \forall \alpha \in \Phi\} = \{\lambda \in \mathsf{E} \,|\, (\lambda, \alpha^\vee) \in \mathbb{Z},\ \forall \alpha \in \Delta\}.$$

The inclusion $\subseteq$ is trivial. The converse inclusion follows immediately from the fact that $\Delta^\vee$ is a base of $\Phi^\vee$, as Proposition III.3.17 establishes. ∎

Remark III.9.4 – It follows immediately from the existence of a base of Φ that $\mathrm{Span}_\mathbb{Z}(\Phi)$ is a free $\mathbb{Z}$-module. More precisely, any base of Φ is a $\mathbb{Z}$-basis of $\mathrm{Span}_\mathbb{Z}(\Phi)$.

Definition III.9.5 – The set $\mathrm{Span}_\mathbb{Z}(\Phi)$ is called the root lattice of Φ (see Remarque III.9.4); it is denoted by $\Lambda_{\Phi,r}$.

Definition III.9.6 – *Let Δ be a base of Φ.*
1. A dominant weight *(relative to Δ) is a weight $\lambda \in \Lambda_\Phi$ such that $\langle \lambda, \alpha \rangle \geq 0$, $\forall \alpha \in \Delta$. We denote by Λ_Φ^+ the set of dominant weights.*
2. A strongly *dominant weight (relative to Δ) is a weight $\lambda \in \Lambda_\Phi$ such that $\langle \lambda, \alpha \rangle > 0$, $\forall \alpha \in \Delta$.*

Remark III.9.7 – Let Δ be a base of Φ.
1. The set of dominant weights is $\Lambda_\Phi \cap \overline{\mathrm{Ch}(\Delta)}$. That is, $\Lambda_\Phi^+ = \Lambda_\Phi \cap \overline{\mathrm{Ch}(\Delta)}$. Further, Λ_Φ^+ is a submonoide of Λ_Φ.
2. The set of strongly dominant weights is $\Lambda_\Phi \cap \mathrm{Ch}(\Delta)$.
(See Lemma III.5.8 and Notation III.6.1).

Remark III.9.8 – Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a base of Φ.
1. The set $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ is a basis of E. We denote $\{\varpi_1, \ldots, \varpi_n\}$ the dual basis of $\Delta^\vee$ with respect to the euclidean structure of E; in other words, for $1 \leq j \leq n$, we define ϖ_j as the unique element of E such that

$$\forall\, i \in \{1, \ldots, n\}, \quad \langle \varpi_j, \alpha_i \rangle = (\varpi_j, \alpha_i^\vee) = \delta_{i,j}.$$

2. By Lemma III.9.3, $\forall\, j \in \{1, \ldots, n\}$, $\varpi_j \in \Lambda_\Phi^+$.
3. For $1 \leq i \leq n$, put $\sigma_i = \sigma_{\alpha_i}$. Then

$$\forall \quad 1 \leq i, j \leq n, \quad \sigma_i(\varpi_j) = \varpi_j - \delta_{i,j}\alpha_i.$$

Definition III.9.9 – *Let Δ be a base of Φ. The weights ϖ_j, $1 \leq j \leq n$, are called the* fundamental *dominant weights relative to Δ. (See Remark III.9.8.)*

Lemma III.9.10 – *Let Δ be a base of Φ.*
1. The set $\{\varpi_1, \ldots, \varpi_n\}$ is a basis of the $\mathbb{Z}$-module Λ_Φ.
2. For all $\lambda \in \Lambda_\Phi$:

$$\lambda = \sum_{1 \leq i \leq n} \langle \lambda, \alpha_i \rangle \varpi_i.$$

3. We have:

$$\Lambda_\Phi^+ = \bigoplus_{1 \leq i \leq n} \mathbb{N}\varpi_i.$$

Proof. By definition, the set $\{\varpi_1, \dots, \varpi_n\}$ is a basis of the $\mathbb{R}$-vector space E and, by Remark III.9.8, its elements all are in Λ_Φ^+. Further, for all $\lambda \in \Lambda_\Phi$,

$$\lambda = \sum_{1 \le i \le n} \langle \lambda, \alpha_i \rangle \varpi_i \in \bigoplus_{1 \le i \le n} \mathbb{Z}\varpi_i.$$

The result follows. ∎

Definition III.9.11 – *Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a base of Φ. The Cartan matrix of Φ relative to Δ is the $n \times n$ matrix with coefficients in $\mathbb{Z}$ whose coefficient in row i and column j is $\langle \alpha_i, \alpha_j \rangle$, $1 \le i, j \le n$.*

Remark III.9.12 – Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a base of Φ. The above shows that we have an inclusion of free $\mathbb{Z}$-modules of rank n as follows:

$$(0) \subseteq \Lambda_{\Phi,r} \subseteq \Lambda_\Phi.$$

1. By the structure Theorem of finitely generated abelian groups, the quotient group $\Lambda_\Phi/\Lambda_{\Phi,r}$ is finite.
2. By Lemma III.9.10:

$$\alpha_j = \sum_{1 \le i \le n} \langle \alpha_j, \alpha_i \rangle \varpi_i, \quad \forall j \in \{1, \dots, n\}.$$

Hence, the j-th column of the transpose of the Cartan matrix relative to Δ gives the coordinates of α_j in the basis $\{\varpi_1, \dots, \varpi_n\}$.
3. It follows from Point 2 that the absolute value of the determinant of the Cartan matrix bounds the order of any element of $\Lambda_\Phi/\Lambda_{\Phi,r}$.

Lemma III.9.13 – *The Weyl group stabilises the set of weights.*

Proof. Let $\lambda \in \Lambda_\Phi$ and $w \in W_\Phi$. For all $\alpha \in \Phi$, $\langle w(\lambda), \alpha \rangle = \langle \lambda, w^{-1}(\alpha) \rangle$. The result follows, since the Weyl group stabilises Φ. ∎

Let Δ be a base of Φ. Recall the order on E defined by:

$$\forall\, x, y \in E,\ x \preceq y \quad \text{si} \quad y - x \in \mathbb{N}\Delta = \mathbb{N}\Phi^+.$$

(see Definition III.3.5 and Proposition III.3.6).

Proposition III.9.14 – *Let Δ be a base of Φ.*
1. Let $\lambda \in \Lambda_\Phi$. The orbit of λ under the action of W_Φ contains exactly one dominant weight.
2. If λ is a dominant weight, for all $w \in W_\Phi$, $w(\lambda) \preceq \lambda$.
3. If λ is a strongly dominant weight, for $w \in W_\Phi$, $w(\lambda) = \lambda$ implies $w = \mathrm{id}$.

Proof. 1. By Lemma III.9.13 and Remark III.9.7, it is an immediate consequence of Proposition III.6.10.
2. We proceed by induction on the length of w. If $\ell(w) = 0$, the result is trivial. Suppose $\ell(w) \ge 1$. By Proposition III.6.9, there exists $w' \in W_\Phi$ and $\alpha \in \Delta$ such that $w = w'\sigma_\alpha$, $\ell(w) = \ell(w') + 1$ and $w(\alpha) \in \Phi^-$. We then have:

$$\lambda - w(\lambda) = \lambda - w'(\lambda) + w'(\lambda) - w(\lambda) = \lambda - w'(\lambda) + w(\sigma_\alpha(\lambda) - \lambda) = \lambda - w'(\lambda) - \langle \lambda, \alpha \rangle w(\alpha).$$

The induction hypothesis gives $0 \preceq \lambda - w'(\lambda)$ and, since $w(\alpha) \in \Phi^-$, $0 \prec -\langle \lambda, \alpha \rangle w(\alpha)$. So, $0 \preceq \lambda - w(\lambda)$. This terminates the proof of Point 2.
3. Suppose $\ell(w) \ge 1$. Proceeding as in Point 2 and with the same notation, we have that $0 < (\lambda, \alpha) = (w^{-1}(\lambda), \alpha) = (\lambda, w(\alpha)) < 0$, which is absurd. Hence, $\ell(w) = 0$, that is $w = \mathrm{id}$. ∎

Exercise III.9.15 – Let E be an $\mathbb{R}$-vector space of finite dimension $n \in \mathbb{N}^*$, equipped with a norm N. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis of E. For all $r \in \mathbb{R}_{\geq 0}$, the set

$$\left(\bigoplus_{1 \leq i \leq n} \mathbb{N} b_i \right) \cap \{x \in \mathsf{E} \,|\, N(x) \leq r\}$$

is finite.

Lemma III.9.16 – *Let $\lambda \in \Lambda_\Phi^+$. The set of dominant weights μ such that $\mu \preceq \lambda$ is finite.*

Proof. Let μ be a dominant weight such that $\mu \preceq \lambda$. Then, $\lambda + \mu$ is a dominant weight and $\lambda - \mu$ is a sum of simple roots. It follows that $0 \leq (\lambda + \mu, \lambda - \mu) = ||\lambda||^2 - ||\mu||^2$. Hence μ belongs to the set $\Lambda_\Phi^+ \cap \{x \in \mathsf{E} \,|\, ||x|| \leq ||\lambda||\}$. But, by Lemma III.9.10 and Exercise III.9.15 the latter set is finite. ■

Recall from Notation III.4.3 the element

$$\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta \in \mathsf{E}.$$

Lemma III.9.17 – **δ is a dominant weight** – *Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of Φ. Then, δ is a strongly dominant weight and*

$$\delta = \sum_{1 \leq i \leq n} \varpi_i.$$

Proof. Let $1 \leq i \leq n$. By Proposition III.4.4, we have that $\sigma_{\alpha_i}(\delta) = \delta - \alpha_i$ or, equivalently, $\langle \delta, \alpha_i \rangle = 1$. Hence, δ is a strongly dominant weight. In addition, by Lemma III.9.10, $\delta = \sum_{1 \leq i \leq n} \langle \delta, \alpha_i \rangle \varpi_i = \sum_{1 \leq i \leq n} \varpi_i$. ■

We now come to the notion of *saturated set of weights* which will turn out to be very useful in the representation theory of semisimple Lie algebras.

Definition III.9.18 – **Saturated set of weights** – *A subset Π of Λ_Φ is called* saturated *if, for all $\lambda \in \Pi$ and all $\alpha \in \Phi$, we have $\lambda - i\alpha \in \Pi$ for all integers i between 0 and $\langle \lambda, \alpha \rangle$.*

Remark III.9.19 – It is clear from the definition that any saturated set of weights is stable under W_Φ. Indeed, if Π is such a set, for all $\lambda \in \Pi$ and all $\alpha \in \Phi$, $\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha \in \Pi$.

Definition III.9.20 – **Highest weight of a saturated set of weights** – *Let Δ be a base of Φ. Let Π be a saturated set of weights and $\lambda \in \Lambda_\Phi^+$. We say that Π has highest weight λ if λ is a maximum element of Π with respect to $\preceq$, that is, $\lambda \in \Pi$ and for all $\mu \in \Pi$, $\mu \preceq \lambda$.*

Example III.9.21 – Let Δ be a base of Φ.
1. It is clear that the set $\{0\}$ is a saturated set of weights, with highest weight 0.
2. It is easy to deduce from Proposition III.2.15 that $\Phi \cup \{0\}$ is a saturated set of weights. Suppose in addition that Φ is irreducible. By Lemma III.7.9, the ordered set $(\Phi, \preceq)$ has a maximum element; denote it by μ. Clearly, $0 \prec \mu$. We have that $\mu \in \Lambda_\Phi^+$, by Lemma III.2.14 and the maximality of μ. So, Φ is a saturated set of weights with highest weight μ.

Lemma III.9.22 – *Let Δ be a base of Φ. A saturated set of weights with highest weight must be finite.*

Proof. Let Π be a saturated set of weights with highest weight $\lambda \in \Lambda_\Phi^+$.

Being stable under W_Φ (see Remark III.9.19), Π is a union of W_Φ-orbits, each of which contains exactly one element in Λ_Φ^+, by Proposition III.9.14. Hence, there exists a (non empty) set I and dominant weights λ_i, $i \in I$ such that

$$\Pi = \bigsqcup_{i\in I} W_\Phi(\lambda_i).$$

But, λ being an highest weight for Π, we have $\lambda_i \preceq \lambda$ for all $i \in I$. So, by Lemma III.9.16, I must be finite. Since W_Φ-orbits are finite, the statement is proved. ∎

Lemma III.9.23 – *Let Δ be a base of Φ. Let Π be a saturated set of weights with highest weight $\lambda \in \Lambda_\Phi^+$. Then, any dominant weight μ such that $\mu \preceq \lambda$ belongs to Π.*

Proof. The proof relies on the study of the set

$$(\mu + \mathrm{Span}_{\mathbb{N}}(\Delta)) \cap \Pi \subseteq \Lambda_\Phi,$$

to which λ belongs, since $\mu \preceq \lambda$.

Suppose we are given an element $\mu' = \mu + \sum_{\alpha\in\Delta} n_\alpha \alpha$ in $\Pi \cap (\mu + \mathrm{Span}_{\mathbb{N}}(\Delta))$ ($n_\alpha \in \mathbb{N}$, for all $\alpha \in \Delta$). Suppose in addition that $\mu' \neq \mu$. Then $\sum_{\alpha\in\Delta} n_\alpha\alpha$ is a nonzero vector, so that $(\sum_{\alpha\in\Delta} n_\alpha\alpha, \sum_{\alpha\in\Delta} n_\alpha\alpha) > 0$, which entails that there exists $\beta \in \Delta$ such that $(\sum_{\alpha\in\Delta} n_\alpha\alpha, \beta) > 0$ and $n_\beta > 0$. Therefore, since μ is dominant,

$$\langle \mu', \beta\rangle = \langle \mu, \beta\rangle + \left\langle \sum_{\alpha\in\Delta} n_\alpha\alpha, \beta \right\rangle > 0.$$

Now, by definition of a saturated set of weights, and since $\langle \mu', \beta\rangle > 0$, we must have that, for all $0 \leq i \leq \langle \mu', \beta\rangle$, $\mu' - i\beta \in \Pi$. We may apply this with $i = 1$ and thus get that

$$\mu' - \beta = \mu + \sum_{\alpha\in\Delta, \alpha\neq\beta} n_\alpha\alpha + (n_\beta - 1)\beta \in \Pi \cap (\mu + \mathrm{Span}_{\mathbb{N}}(\Delta)).$$

By an easy induction, we deduce that $\mu \in \Pi$, since $\lambda \in \Pi \cap (\mu + \mathrm{Span}_{\mathbb{N}}(\Delta))$. ∎

Exercise III.9.24 – **Saturated sets of weights with prescribed highest weight** – Let $\lambda \in \Lambda_\Phi^+$. Put

$$\Pi = \bigsqcup_{\mu\in\Lambda_\Phi^+, \mu\preceq\lambda} W_\Phi.\mu.$$

1. The set Π is stable under W_Φ and $\Pi = \{\nu \in \Lambda_\Phi \,|\, w(\nu) \preceq \lambda, \forall\, w \in W_\Phi\}$.
2. Let $\mu \in \Pi$, $\alpha \in \Phi$ such that $\langle \mu, \alpha\rangle \geq 0$. Let $C = \{\mu - i\alpha, 0 \leq i \leq \langle \mu, \alpha\rangle\}$.
2.1. Let $w \in W_\Phi$. All the elements of $w(C)$ are bounded above by λ with respect to $\preceq$.
2.2. We have $C \subseteq \Pi$.
3. Let $\mu \in \Pi$, $\alpha \in \Phi$ such that $\langle \mu, \alpha\rangle \leq 0$. Let $C = \{\mu - i\alpha, \langle \mu, \alpha\rangle \leq i \leq 0\}$. Then $C \subseteq \Pi$.
4. The set Π is saturated.

Remark III.9.25 – **Structure of saturated sets of weights with highest weight** – Let Δ be a base of Φ.
1. Suppose Π is a saturated set of weights with highest weight $\lambda \in \Lambda_\Phi^+$. By Remark III.9.19, Π is a union of W_Φ-orbits. If $\mathcal{O}$ is such an orbit, it meets Λ_Φ^+ exactly once (Proposition III.9.14),

say in $\mu \in \Pi$ and λ being a highest weight for Π, $\mu \preceq \lambda$. But, conversaly, any dominant weight ν such that $\nu \preceq \lambda$ must be in Π, by Lemma III.9.23. All in all, we get that

$$\Pi = \bigsqcup_{\mu \in \Lambda_\Phi^+, \mu \preceq \lambda} W_\Phi.\mu.$$

2. Conversaly, let $\lambda \in \Lambda_\Phi^+$. Put

$$\Pi = \bigsqcup_{\mu \in \Lambda_\Phi^+, \mu \preceq \lambda} W_\Phi.\mu.$$

By Exercise III.9.24, Π is a saturated set of weights and, by Point 2 of Proposition III.9.14, it has highest weight λ.
3. The two points above show that there is a one-to-one correspondance between Λ_Φ^+ and saturated sets of weights with highest weight, given by $\lambda \mapsto \bigsqcup_{\mu \in \Lambda_\Phi^+, \mu \preceq \lambda} W_\Phi.\mu$.

Proposition III.9.26 – *Let Δ be a base of Φ. Let Π be a saturated set of weights with highest weight λ. Then, for all $\mu \in \Pi$:*
1. $(\mu + \delta, \mu + \delta) \leq (\lambda + \delta, \lambda + \delta)$;
2. $(\mu + \delta, \mu + \delta) = (\lambda + \delta, \lambda + \delta)$ implies $\mu = \lambda$.

Proof. [1]

III.10 Classification of root systems.

Our aim in this section is to classify the root systems (up to isomorphism). By Proposition III.7.5, any root system can be partitioned into irreducible root systems (see that proposition for a precise statement). Hence, the study of root systems reduces to that of irreducible ones.

It turns out that the key ingredient to this aim is a data consisting of so-called *Cartan integers*. More precisely, the knowledge of the Cartan integers for simple roots is enough to recover the root system.

Definition III.10.1 – *Let $(\mathsf{E}, (-,-))$ be a euclidean space and Φ a root system of E. The Cartan integers of the root system Φ are the integers $\langle \alpha, \beta \rangle$, $\alpha, \beta \in \Phi$.*

Remark III.10.2 – Let Φ be a root system of E and Δ a base of Φ. Let $\alpha, \beta \in \Delta$, $\alpha \neq \beta$.
1. By Lemma III.3.7, we have that $(\alpha, \beta) \leq 0$.
2. Suppose now that $||\alpha|| \leq ||\beta||$, by Point 1 above, the possible values for the Cartan integers involving α and β are as follows:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	angle	$\|\|\beta\|\|^2/\|\|\alpha\|\|^2$	
0	0	$\pi/2$	*undetermined*	case where α and β are orthogonal
-1	-1	$2\pi/3$	1	
-1	-2	$3\pi/4$	2	
-1	-3	$5\pi/6$	3	

[1] LAURENT. A écrire en suivant [Humphreys ; Lemma 13.3.B et Lemma 13.4.C, pp. 70-71]. Il n'y a pas de difficulté. D'après les commentaires de Humphreys, cette Proposition est utile pour la formule de multiplicités de Freudenthal.

3. It is clear from the above array that, if we know that $||\alpha|| \leq ||\beta||$, then we can recover each of the Cartan integers $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ from their product $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$. This easy observation will be at the origine of the definition of the Coxeter graph of the pair (Φ, Δ).

Definition III.10.3 – *Let $(\mathsf{E}, (-,-))$ be a euclidean space of dimension $\ell \in \mathbb{N}^*$, Φ a root system of E and Δ a base of Φ. Given an ordering $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ of Δ, the Cartan matrix of the pair (Φ, Δ) (with respect to this ordering) is defined to be the $\ell \times \ell$ matrix with coefficients in $\mathbb{Z}$: $(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq \ell}$.*

Remark III.10.4 – **Independence of the Cartan matrix with respect to the base** – Keep notation as in Definition III.10.3.
1. Clearly, the Cartan matrix associated to (Φ, Δ) depends on the ordering of the elements of Δ.
2. Suppose Δ' is a base of Φ. By Theorem III.6.4, there exists $w \in W_\Phi$ such that $\Delta' = w(\Delta)$. It follows that the Cartan matrix of the pair (Φ, Δ') is the same (up to the orderings of Δ and Δ'). Hence, *the Cartan matrix only depends on* Φ. For this reason, from now on, we will speak of *the Cartan matrix of* Φ.
3. We have

$$(\langle \alpha_i, \alpha_j \rangle)_{1 \leq i,j \leq \ell} = ((\alpha_i, \alpha_j))_{1 \leq i,j \leq \ell} \operatorname{diag}\left(\frac{2}{(\alpha_1, \alpha_1)}, \dots, \frac{2}{(\alpha_\ell, \alpha_\ell)} \right).$$

Hence, the Cartan matrix of Φ is invertible.

The following statement shows that the Cartan matrix determines the root system up to isomorphism.

Proposition III.10.5 – *Let $(\mathsf{E}, (-,-))$ and $(\mathsf{E}', (-,-))$ be euclidean spaces of dimension $\ell \in \mathbb{N}^*$. Let Φ and Φ' be root systems of E and E', respectively. Let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ and $\Delta' = \{\alpha'_1, \dots, \alpha'_\ell\}$ be ordered bases of Φ and Φ', respectively.*

If, for all $1 \leq i, j \leq \ell$, $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$, then the isomorphism $\varphi : \mathsf{E} \longrightarrow \mathsf{E}'$, $\alpha_i \mapsto \alpha'_i$ is an isomorphism of the pairs (E, Φ) and (E', Φ').

Proof. Observe that, since Δ and Δ' are bases of the vector spaces E and E', respectively, φ is well-defined and an isomorphism of vector spaces. By the hypotheses, we have that, for all $\alpha \in \Delta$, the following diagram is commutative:

$$\begin{tikzcd} \mathsf{E} \arrow[r, "\varphi"] \arrow[d, "\sigma_\alpha"] & \mathsf{E}' \arrow[d, "\sigma_{\varphi(\alpha)}"] \\ \mathsf{E} \arrow[r, "\varphi"] & \mathsf{E}' \end{tikzcd}$$

Since W_Φ and $W_{\Phi'}$ are generated by the simple reflections (cf. Theorem III.6.4), it follows that,

$$\begin{array}{ccc} W_\Phi & \longrightarrow & W_{\Phi'} \\ w & \mapsto & \varphi \circ w \circ \varphi^{-1} \end{array}$$

is an isomorphism of groups (which sends σ_α to $\sigma_{\varphi(\alpha)}$, for all $\alpha \in \Delta$).

Now, let $\beta \in \Phi$. By Theorem III.6.4, there exists $\alpha \in \Delta$ and $w \in W_\Phi$ such $\beta = w(\alpha)$ and we have:

$$\varphi(\beta) = \varphi(w(\alpha)) = \varphi \circ w \circ \varphi^{-1}(\varphi(\alpha)),$$

which shows that $\varphi(\beta)$ is the image under $\varphi \circ \sigma \circ \varphi^{-1} \in W_{\Phi'}$ of $\varphi(\alpha) \in \Delta'$. Hence, $\varphi(\beta) \in \Phi'$. We have shown that $\varphi(\Phi) \subseteq \Phi'$. Exchanging the role of (E, Φ) and (E', Φ'), we get the reverse inclusion. So $\varphi(\Phi) = \Phi'$.

It remains to show that, for all $\alpha, \beta \in \Phi$, $\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle$. This is true by hypothesis whenever $\alpha, \beta \in \Delta$. Observe then that the result follows when $\alpha \in \Phi$ and $\beta \in \Delta$, by the linearity of $\langle -, - \rangle$ with respect to its first entry and the linearity of φ. Let us now consider any element $\beta \in \Phi$. By Theorem III.6.4, there exists $w \in W_\Phi$ such that $w(\beta) \in \Delta$. We then have,

$$\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \varphi \circ w(\alpha), \varphi \circ w(\beta) \rangle = \langle \varphi(w(\alpha)), \varphi(w(\beta)) \rangle = \langle w(\alpha), w(\beta) \rangle = \langle \alpha, \beta \rangle.$$

Indeed, the first (resp. fourth) equality holds since $\varphi \circ w \circ \varphi^{-1} \in W_{\Phi'}$ (resp. $w \in W_\Phi$) and the third because of the above observation, since $w(\beta) \in \Delta$. This completes the proof. ∎

We now introduce the Coxeter graph of a root system. It is a first step in encoding the Cartan matrix of a root system into a diagramatic form. The Dynkin diagram, to be introduced a little latter, will complete it.

We adopt a somewhat intuitive definition, as a formal one would requierre to specify what we mean by a graph. However, this approach is quite common and do not create serious problems: a formal definition adapted to our context is quite easy to concoct.

Definition III.10.6 – *Let $(\mathsf{E}, (-,-))$ be a euclidean space of dimension $\ell \in \mathbb{N}^*$, Φ a root system of E and Δ a base of Φ. The Coxeter graph associated to (Φ, Δ) is the graph with vertex set Δ and, for all $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges joining α and β.*

Remark III.10.7 – Keep the notation of Definition III.10.6. Making use of Remark III.10.2, we see that, for distinct simple roots α, β, the Coxeter graph has 0 edges linking α and β if and only if they are orthogonal. Further, if they are not orthogonal, it has 1 edge linking them when they have the same length, 2 edges when one has length equal to $\sqrt{2}$ times the length of the other, and 3 edges when one has length equal to $\sqrt{3}$ times the length of the other. However, in the case of simple roots linked by 2 or 3 edges, the Coxeter graph does not allow to decide which is the longest, which the shortest. (This lack will be repared in the Dynkin diagram.)

Remark III.10.8 – Let $(\mathsf{E}, (-,-))$ be a euclidean space, Φ a root system of E and Δ, Δ' bases of Φ. It is easy to see that the Coxeter graphs (Φ, Δ) and (Φ, Δ') are the same (or rather, are isomorphic, in a sense that would need to be made precise). Indeed, by Theorem III.6.4, there is an element w of the Weyl group of Φ such that $\Delta' = w(\Delta)$. The rest follows from the obvious equality $\langle w(\alpha), w(\beta) \rangle = \langle \alpha, \beta \rangle$. Hence, in the sequel, we will often speak of the Coxeter graph of Φ (when dealing with properties of graphs which are invariant under graph isomorphisms).

Remark III.10.9 – **Dynkin diagram of a root system** –
1. Let E be a euclidean space and Φ a root system of E. Choose an arbitrary base Δ of Φ. Then, we have the Coxeter graph of $(\mathsf{E}, \Phi, \Delta)$. On this graph, for each pair of dots linked by at least 2 edges, add between the dots an inequality sign $<$ pointing to the dot coming from the short root. This new diagram will be called the Dynkin diagram of Φ. (Indeed, the diagram obtained that way do not depend on the choice of Δ).
2. In the context of Point 1 above, it is clear, using Remark III.10.2 that we can recover the Cartan integers associated to simple roots of the pair (Φ, Δ). Hence, the Dynkin diagram contains enough information to reconstruct the Cartan matrix of Φ.
3. Let E and E' be euclidean spaces, Φ a root system of E and Φ' a root system of E'. The

Dynkin diagrams obtained from (E, Φ) and (E', Φ') are the same if and only if these root systems are isomorphic.

In one direction, we want to prove that, if (E, Φ) and (E', Φ') are isomorphic, then their Dynkin diagram is the same. Now, using Exercise III.7.7, the problem reduces to the case where these root systems are irreducible. Hence, suppose (E, Φ) and (E', Φ') are isomorphic and irreducible. We already saw that their associated Coxeter graphs are the same. But, on the other hand, by Exercise III.7.12, the isomorphism between them must be an isometry, up to multiplication by a positive real number, so that their associated Dynkin diagrams must coincide.

Conversally, suppose the two Dynkin diagrams are the same. Then, consider arbitrary bases Δ and Δ' of Φ and Φ', respectively. By hypothesis, applying the process described in Point 1 to (Φ, Δ) and (Φ', Δ') leeds to the same Dynkin diagram. This means that there exists an ordering of Δ and Δ' leading to the same Cartan matrices. Hence, Proposition III.10.5 proves that (E, Φ) and (E', Φ') are isomorphic.

The irreducibility of a root system can be read out of its Coxeter graph. Intuitively, we may define a path between two vertices of a Coxeter graph as a finite sequence of vertices of this graph with the property that any two consecutive vertices in the sequence are linked by at least one edge (that is, any two consecutive vertices in the sequence are not orthogonal). Then, we may define a equivalence relation in the set of vertices: two vertices being equivalent if there exists a path starting with one of the vertices and ending with the other. This gives rise to equivalence classes, which we call *connected components* of the graph. Then, the graph is called *connected* if it has a unique connected component.

Lemma III.10.10 – *Let $(\mathsf{E}, (-,-))$ be a euclidean space and Φ be a root system of E. The root system Φ is irreducible if and only if its Coxeter graph is connected.*

Proof. Suppose the Coxeter graph is not connected. Consider a connected component, which we denote Δ_1 and put $\Delta_2 = \Delta \setminus \Delta_1$. Consider $\alpha \in \Delta_1$ and $\beta \in \Delta_2$. By hypothesis, they belong to distinct connected components of the graph and hence are not linked by a path. In particular, there must be no edge of the graph linking these roots. This means that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0$, that is, $(\alpha, \beta) = 0$. This shows that $\Delta_1 \perp \Delta_2$. By Proposition III.7.8, we have that Φ is reducible.

Suppose now that Φ is reducible. By Proposition III.7.8, there exists a partition $\Delta = \Delta_1 \sqcup \Delta_2$ of Δ into nonempty subsets Δ_1 and Δ_2 such that $\Delta_1 \perp \Delta_2$. Let $\alpha \in \Delta_1$ and $\beta \in \Delta_2$. Suppose there exists a path between α and β. This means that there is a finite sequence $\alpha_1, \ldots, \alpha_{n+1}$, $n \in \mathbb{N}^*$, such that $\alpha_1 = \alpha$, $\alpha_{n+1} = \beta$ and, for all $1 \leq i \leq n$, $\alpha_i \not\perp \alpha_{i+1}$. An obvious induction shows that $\alpha_i \in \Delta_1$ for all $1 \leq i \leq n+1$. In particuler, $\beta \in \Delta_1$ which is absurd. Thus, α and β are not in the same connected component and the Coxeter graph of Φ must be disconnected. ∎

As mentionned before, the study of root systems reduces to that of irreducible ones. For this reason, by Lemma III.10.10, we will be primarily interested in connected Coxeter graphs.

Our objective now is the classification of Coxeter graphs associated to irreducible root systems. To reach this aim, we first introduce the convenient notion of *admissible set* of a euclidean space.

Definition III.10.11 – *Let $(\mathsf{E}, (-,-))$ be a euclidean space.*
1. An admissible set of E is a subset, $\mathfrak{A}$, of linearly independant unit vectors such that, for all $\epsilon, \epsilon' \in \mathfrak{A}$, $\epsilon \neq \epsilon'$, $(\epsilon, \epsilon') \leq 0$ and $4(\epsilon, \epsilon')^2 \in \{0, 1, 2, 3\}$.
2. The graph associated to an admissible set $\mathfrak{A}$ is the graph with vertex set $\mathfrak{A}$ and, for all distinct $\epsilon, \epsilon' \in \mathfrak{A}$, $4(\epsilon, \epsilon')^2 \in \{0, 1, 2, 3\}$ edges linking ϵ and ϵ'.

Remark III.10.12 – Let $(\mathsf{E}, (-,-))$ be a euclidean space, Φ a root system of E and Δ a base of Φ.
1. Denote by $\mathfrak{A}$ the subset of E whose elements are $\alpha/||\alpha||$, $\alpha \in \Delta$. Let ϵ and ϵ' be distinct elements of $\mathfrak{A}$. By Lemma III.3.7, we have that $(\epsilon, \epsilon') \leq 0$. In addition, if α, α' are elements of Δ such that $\epsilon = \alpha/||\alpha||$ and $\epsilon' = \alpha'/||\alpha'||$, then

$$4(\epsilon, \epsilon')^2 = 2\frac{(\alpha', \alpha)}{(\alpha, \alpha)} 2\frac{(\alpha, \alpha')}{(\alpha', \alpha')} = \langle \alpha', \alpha \rangle \langle \alpha, \alpha' \rangle \in \{0, 1, 2, 3\}.$$

Since, in addition, Δ is linearly independent, $\mathfrak{A}$ is an admissible set.
2. Obviously, the Coxeter graph of (Φ, Δ) is isomorphic to the graph of $\mathfrak{A}$.

Theorem III.10.13 – *Let* $(\mathsf{E}, (-,-))$ *be a euclidean space,* Φ *an irreducible root system of* E *and* Δ *a base of* Φ*. The Dynkin diagram of* (Φ, Δ) *is one, and only one, of the following list.*
1. Type A_ℓ*,* $\ell \geq 1$*:*
2. Type B_ℓ*,* $\ell \geq 2$*:*
3. Type C_ℓ*,* $\ell \geq 3$*:*
4. Type D_ℓ*,* $\ell \geq 4$*:*
5. Type E_6*:*
6. Type E_7*:*
7. Type E_8*:*
8. Type F_4*:*
9. Type G_2*:*

Proof. See [Humphreys; section 11.4]. ∎

Part IV

Classification of semi-simple Lie algebras.

IV.1 From semisimple Lie algebras to root systems.

Assume $\Bbbk$ is algebraically closed of characteristic 0.

We come back to the context of Section II.6, taking into account Example III.3.18.

Hence, we consider a semisimple Lie algebra $\mathfrak{g}$, a maximal toral subalgebra (equivalently a Cartan subalgebra) $\mathfrak{h}$ of $\mathfrak{g}$, the correponding set $\Phi \subseteq \mathfrak{h}^*$ of non zero linear forms α on $\mathfrak{h}$ such that $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, \forall h \in \mathfrak{h},\ [h,x] = \alpha(h)x\} \neq (0)$. Recall that $\mathfrak{g}_0 = \{x \in \mathfrak{g} \,|\, \forall h \in \mathfrak{h},\ [h,x] = 0\} = \mathfrak{h}$.

The Killing form on $\mathfrak{h}$ gives rise to a nondegenerate form on $\mathfrak{h}^*$: $(-,-) : \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \Bbbk$, via the identification $\iota : \mathfrak{h} \longrightarrow \mathfrak{h}^*$. Then, putting $\mathsf{E}_\mathbb{Q} = \mathrm{Span}_\mathbb{Q}(\Phi) \subseteq \mathfrak{h}^*$, we get a $\mathbb{Q}$-subspace of dimension $\dim_\Bbbk(\mathfrak{h}^*)$ on which $(-,-)$ induces a positive, definite, symmetric bilinear form $(-,-)_\mathbb{Q} : \mathsf{E}_\mathbb{Q} \times \mathsf{E}_\mathbb{Q} \longrightarrow \mathbb{Q}$ wich, in turn, defines a positive, definite, symmetric bilinear form $(-,-)_\mathbb{R} : \mathsf{E}_\mathbb{R} \times \mathsf{E}_\mathbb{R} \longrightarrow \mathbb{R}$ on the $\mathbb{R}$-vector space $\mathsf{E}_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Q} \mathsf{E}_\mathbb{Q}$, turning it into a euclidean space. Thus, we have

$$\begin{array}{ccccc} \Phi & \subseteq & \mathsf{E}_\mathbb{Q} & \subseteq & \mathfrak{h}^* \\ & & \downarrow & & \\ & & \mathsf{E}_\mathbb{R} & & \end{array}$$

Then, Theorem II.6.1 shows that, seen as a subset of $\mathsf{E}_\mathbb{R}$, Φ is a root system of $\mathsf{E}_\mathbb{R}$.

Choose a basis Δ of Φ, and write $\Phi = \Phi^+ \sqcup \Phi^-$. Putting $\mathfrak{n}^- = \oplus_{\alpha\in\Phi^-}\mathfrak{g}_\alpha$ and $\mathfrak{n} = \mathfrak{n}^+ = \oplus_{\alpha\in\Phi^+}\mathfrak{g}_\alpha$, the Cartan-Chevalley decomposition of $\mathfrak{g}$ writes

$$\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \right) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Exercise IV.1.1 – **Root systems and isomorphisms** – Keep the above notation. Suppose, in addition, that $\mathfrak{g}'$ is a Lie algebra and $\Theta : \mathfrak{g} \longrightarrow \mathfrak{g}'$ an isomorphism of Lie algebras. Hence, $\mathfrak{g}'$ is a semisimple Lie algebra and, putting $\mathfrak{h}' = \Theta(\mathfrak{h})$, $\mathfrak{h}'$ is a maximal toral subalgebra of $\mathfrak{g}'$. Clearly, Θ induces an isomorphism of vector spaces from $\mathfrak{h}$ to $\mathfrak{h}'$ (that we still denote Θ) which, in turn, induces an isomorphism of vector spaces

$$\begin{array}{cccc} ({}^t\Theta)^{-1} : & \mathfrak{h}^* & \longrightarrow & (\mathfrak{h}')^* \\ & \lambda & \mapsto & \lambda \circ \Theta^{-1} \end{array}.$$

For all $\mu \in (\mathfrak{h}')^*$, put $\mathfrak{g}'_\mu = \{x \in \mathfrak{g}' \,|\, \forall h \in \mathfrak{h}',\ [h,x] = \mu(h)x\}$ and Φ' the subset of $(\mathfrak{h}')^*$ of those elements μ such that $\mathfrak{g}'_\mu \neq (0)$. Finally, we denote by $(\mathsf{E}'_\mathbb{R}, \Phi')$ the root system associated to the pair $(\mathfrak{g}', \mathfrak{h}')$. In addition, we denote κ' the Killing form on $\mathfrak{g}'$.

1. Sets of roots.
1.1. For all $\lambda \in \mathfrak{h}^*$, $\Theta(\mathfrak{g}_\lambda) = \mathfrak{g}'_{\lambda\circ\Theta^{-1}}$.
1.2. We have $({}^t\Theta)^{-1}(\Phi) = \Phi'$.
2. Killing forms.
2.1. For all $x \in \mathfrak{g}$, $\mathrm{ad}_{\mathfrak{g}'}(\Theta(x)) = \Theta \circ \mathrm{ad}_\mathfrak{g}(x) \circ \Theta^{-1}$.
2.2. For all $x, y \in \mathfrak{g}$, $\kappa'(\Theta(x), \Theta(y)) = \kappa(x,y)$.
2.3. Recall the isomorphism $\iota : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ and denote t_λ the element whose image under ι is $\lambda \in \mathfrak{h}^*$. Adopt a similar notation for $\mathfrak{g}'$. Then, for all $\lambda \in \mathfrak{h}^*$, $t'_{\lambda\circ\Theta^{-1}} = \Theta(t_\lambda)$.

2.4. Recall that $(-,-)$ is the symmetric bilinear form on $\mathfrak{h}^*$ correponding to κ via the identification ι. Denote $(-,-)'$ its analogue for $(\mathfrak{h}')^*$. Then, for all $\lambda, \mu \in \mathfrak{h}^*$, $(\lambda \circ \Theta^{-1}, \mu \circ \Theta^{-1})' = (\lambda, \mu)$.
3. There is an isometry between $\mathsf{E}_{\mathbb{R}}$ and $\mathsf{E}'_{\mathbb{R}}$ that sends Φ to Φ'. In particular, the root systems $(\mathsf{E}_{\mathbb{R}}, \Phi)$ and $(\mathsf{E}'_{\mathbb{R}}, \Phi')$ are isomorphic.

Exercise IV.1.2 – **Root systems and maximal toral subalgebras** – Keep the above notation. Suppose $\mathfrak{h}'$ is a maximal toral subalgebra of $\mathfrak{g}$. The root systems attached to the pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, \mathfrak{h}')$ are isomorphic (by means of an isometry of the corresponding euclidian spaces). (Hint: use Exercise IV.1.1.)

Notation IV.1.3 –
1. Denote by $A_{\mathbb{k}}$ the set of finite dimensional semisimple Lie algebras over $\mathbb{k}$ and by $\sim$ the equivalence relation given by isomorphism in $A_{\mathbb{k}}$.
2. Denote by R the set of root systems and by $\sim$ the equivalence relation given by isomorphism in R.

It follows from Exercises IV.1.1 and IV.1.2 that there is a well-defined map

$$A_{\mathbb{k}} \quad \longrightarrow \quad R/\sim$$

which, to each semisimple Lie algebra $\mathfrak{g}$, associates the isomorphism class of the root system $(\mathsf{E}_{\mathbb{R}}, \Phi)$ associated to $(\mathfrak{g}, \mathfrak{h})$ for an arbitrary choice of a maximal toral subalgebra $\mathfrak{h}$ and that this map factorises through the quotient $A_{\mathbb{k}}/\sim$ to give rise to a map

$$r \quad : \quad A_{\mathbb{k}}/\sim \quad \longrightarrow \quad R/\sim . \tag{IV.1.33}$$

We will eventually prove that this map is actually a bijection.

For the moment, we show that it sends (isomorphism classes of) simple Lie algebras to irreducible root systems.

Proposition IV.1.4 – *Keep the above notation. If $\mathfrak{g}$ is simple, then Φ is irreducible.*

Proof. Assume $\mathfrak{g}$ is a simple Lie algebra. Suppose Φ is reducible and consider a partition $\Phi = \Phi_1 \sqcup \Phi_2$ of Φ with Φ_1 and Φ_2 orthogonal to each other and nonempty. For all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, we have that $(\alpha+\beta, \alpha) \neq 0$ and $(\alpha+\beta, \beta) \neq 0$. It follows from this observation that $\alpha+\beta$ is not a root: $\mathfrak{g}_{\alpha+\beta} = 0$. Thus, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, by Lemma II.5.6.

Let $\mathfrak{l}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$, $\alpha \in \Phi_1$. An easy induction shows, using the Jacobi identity, that the above observation implies that, for all $\beta \in \Phi_2$, $[\mathfrak{l}, \mathfrak{g}_\beta] = 0$. And, since $\mathfrak{g}$ is semisimple, it has trivial center. So, we must have $\mathfrak{l} \subset \mathfrak{g}$ (strict inclusion). In addition, for all $\alpha \in \Phi_1$, we have trivially $[\mathfrak{l}, \mathfrak{g}_\alpha] \subseteq \mathfrak{l}$. But on the other hand, $[\mathfrak{h}, \mathfrak{l}] \subseteq \mathfrak{l}$. It follows that $[\mathfrak{g}, \mathfrak{l}] \subseteq \mathfrak{l}$; that is $\mathfrak{l}$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{l}$ must be nonzero, this is a contradiction since $\mathfrak{g}$ is simple. ∎

In the next statement, we will make an extensive use of the content of Remark II.6.2.

Let $\mathfrak{g}$ be a nonzero semisimple Lie algebra $\mathfrak{g}$. Recall from Theorem I.7.19 that $\mathfrak{g}$ is the direct sum of its simple ideals and that the latter pairwise commute. That is, there exists $t \in \mathbb{N}^*$ and simple, pairwise distinct,s ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_t$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_t$ and, for $1 \leq i < j \leq t$, $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$. Hence, we are in position to apply Remark II.6.2, which we do using the notation of that remark. Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$ and, for $1 \leq i \leq t$, put $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$. We have $\mathfrak{h} = \bigoplus_{1 \le i \le t} \mathfrak{h}_i$ and $\mathfrak{h}_i$ is a maximal toral subalgebra of $\mathfrak{g}_i$, $1 \le i \le t$. Recall the isometry of Remark II.6.2:

$$\delta : \mathsf{E}_{\mathbb{R}} \longrightarrow \bigoplus_{1 \le i \le t} \mathsf{F}_{i,\mathbb{R}},$$

and, for $1 \le i \le t$, let $\mathsf{E}_{i,\mathbb{R}}$ be the image of $\mathsf{F}_{i,\mathbb{R}}$ in $\mathsf{E}_{\mathbb{R}}$. So that, $\mathsf{E}_{\mathbb{R}}$ is the orthogonal direct sum of the subspaces $\mathsf{E}_{i,\mathbb{R}}$, $1 \le i \le t$.

But, for $1 \le i \le t$, $\Phi_i = \delta^{-1}(\Psi_i) \subseteq \mathsf{E}_{i,\mathbb{R}}$ and Ψ_i is the root system associated to the pair $(\mathfrak{g}_i, \mathfrak{h}_i)$ where $\mathfrak{g}_i$ is a simple Lie algebra. Hence, by Proposition IV.1.4, Ψ_i is an irreducible root system of $\mathsf{F}_{i,\mathbb{R}}$ and, therefore, Φ_i is an irreducible root system of $\mathsf{E}_{i,\mathbb{R}}$. Since $\Phi = \sqcup_{1 \le i \le t} \Phi_i$, we have proved the following statement.

Theorem IV.1.5 – *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a maximal toral subalgebra. Then, in the above notation, $\mathsf{E}_{\mathbb{R}} = \oplus_{1 \le i \le t} \mathsf{E}_{i,\mathbb{R}}$ and $\Phi = \sqcup_{1 \le i \le t} \Phi_i$ is the decomposition of Φ into irreducible components in the sense of Proposition III.7.5 (and Definition III.7.6).*

Proof. ■

Corollary IV.1.6 – *Let χ be an isomorphism class of semisimple Lie algebras. Then, its elements are simple Lie algebras if and only if the image of χ under $r : A_{\Bbbk}/\sim \longrightarrow R/\sim$ is an isomorphism class of irreducible root systems.*

Proof. This follows immediately from Theorem IV.1.5. ■

At this stage, we are in position to give a first result towards the classification of finite dimensional semisimple Lie algebras over an algebraically closed field of characteristic 0.

The first statement deals with simple Lie algebras. We do not give a proof of it as we will be able to give a more satisfactory one latter.

Theorem IV.1.7 – *Assume $\Bbbk$ is algebraically closed of characteristic 0. Suppose $\mathfrak{g}$ and $\mathfrak{g}'$ are finite dimensional simple Lie algebras and $\mathfrak{h}$ and $\mathfrak{h}'$ maximal toral subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Let Φ be the root system of the pair $(\mathfrak{g}, \mathfrak{h})$ and Φ' be the root system of the pair $(\mathfrak{g}', \mathfrak{h}')$. If $(\mathsf{E}_{\mathbb{R}}, \Phi)$ and $(\mathsf{E}'_{\mathbb{R}}, \Phi')$ are isomorphic root systems, then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic Lie algebras.*

Proof. For a more detailled statement and a complete proof, see [Humphreys; section 14.2]. ■

From Theorem IV.1.7 and the above discussion, the following statement follows.

Theorem IV.1.8 – *Assume $\Bbbk$ is algebraically closed of characteristic 0. Suppose $\mathfrak{g}$ and $\mathfrak{g}'$ are finite dimensional semisimple Lie algebras and $\mathfrak{h}$ and $\mathfrak{h}'$ maximal toral subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Let Φ be the root system of the pair $(\mathfrak{g}, \mathfrak{h})$ and Φ' be the root system of the pair $(\mathfrak{g}', \mathfrak{h}')$. If (E_R, Φ) and $(\mathsf{E}'_{\mathbb{R}}, \Phi')$ are isomorphic root systems, then $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic Lie algebras.*

Proof. Write $\mathfrak{g} = \bigoplus_{1 \le i \le t} \mathfrak{g}_i$ and $\mathfrak{g}' = \bigoplus_{1 \le i \le s} \mathfrak{g}'_i$, $t, s \in \mathbb{N}^*$, the decomposition of $\mathfrak{g}$ and $\mathfrak{g}'$ as the direct sum of their simple ideals (see Theorem I.7.19).

Let $\mathsf{E}_{\mathbb{R}}$ and $\mathsf{E}'_{\mathbb{R}}$ be the euclidean spaces attached to the pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h}')$, respectively. We adopt the notation of Theorem IV.1.5 and its proof for both the pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h}')$. So, we have a decomposition $\mathsf{E}_{\mathbb{R}} = \oplus_{1 \le i \le t} \mathsf{E}_{i,\mathbb{R}}$ into pairwise orthogonal subspaces and a partition

$\Phi = \sqcup_{1\leq i\leq t}\Phi_i$ with $\mathsf{E}_{i,\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}(\Phi_i)$ which gives the reduction of Φ into irreducible components. Similarly, we have a decomposition $\mathsf{E}'_{\mathbb{R}} = \oplus_{1\leq i\leq s}\mathsf{E}'_{i,\mathbb{R}}$ into pairwise orthogonal subspaces and a partition $\Phi' = \sqcup_{1\leq i\leq s}\Phi'_i$ with $\mathsf{E}'_{i,\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}(\Phi'_i)$ which gives the reduction of Φ into irreducible components.

Let φ be an isomorphism between the root systems $(\mathsf{E}_{\mathbb{R}}, \Phi)$ and $(\mathsf{E}'_{\mathbb{R}}, \Phi')$. By Exercise III.7.7, $t = s$ and (up to renumbering summands), for all $1 \leq i \leq t$, $\varphi(\mathsf{E}_{i,\mathbb{R}}) = \mathsf{E}'_{i,\mathbb{R}}$ and $\varphi(\Phi_i) = \Phi'_i$, so that the root systems $(\mathsf{E}_{i,\mathbb{R}}, \Phi_i)$ and $(\mathsf{E}'_{i,\mathbb{R}}, \Phi'_i)$ are isomorphic. But, on the other hand, for $1 \leq i \leq t$, $(\mathsf{E}_{i,\mathbb{R}}, \Phi_i)$ is isomorphic to the root system of $(\mathfrak{g}_i, \mathfrak{h}_i)$, while $(\mathsf{E}'_{i,\mathbb{R}}, \Phi'_i)$ is isomorphic to the root system of $(\mathfrak{g}'_i, \mathfrak{h}'_i)$ (see the proof of Theorem IV.1.5 and the notation introduced there). By Theorem IV.1.7, we deduce that, for all $1 \leq i \leq t$, $\mathfrak{g}_i$ and $\mathfrak{g}'_i$ are isomorphic Lie algebras (up to renumbering summands). The result follows, as $\mathfrak{g} = \bigoplus_{1\leq i\leq t}\mathfrak{g}_i$ and $\mathfrak{g}' = \bigoplus_{1\leq i\leq t}\mathfrak{g}'_i$. ∎

IV.2 Universal enveloping algebra.

In this section, unless otherwise specified, $\mathbb{k}$ is arbitrary.

We start by recalling the definition and basic properties of the *tensor algebra* of a $\mathbb{k}$-vector space. Extensive details may be found in [BBK-Algèbre-1-3], Chap. III, §5.

Let V be any $\mathbb{k}$-vector space. For all $i \in \mathbb{N}^*$, denote $T^i(V)$ the $\mathbb{k}$-vector space $V \otimes_{\mathbb{k}} \ldots \otimes_{\mathbb{k}} V$ (i copies). In particular, $T^1(V) = V$. Put also $T^0(V) = \mathbb{k}$ and

$$T(V) = \bigoplus_{i\in\mathbb{N}} T^i(V).$$

We will freely identify $T^i(V)$ with its image in $T(V)$, $i \in \mathbb{N}$. In particular, we have canonical injections of $\mathbb{k}$-vector spaces:

$$\mathbb{k} \stackrel{can.inj.}{\longrightarrow} T(V) \qquad \text{and} \qquad V \stackrel{can.inj.}{\longrightarrow} T(V)$$

Then, we can endow $T(V)$ with an associative algebra structure as follows (see [BBK-Algèbre-1-3; p. III.55]): there exists a unique bilinear map

$$m \;:\; T(V) \times T(V) \longrightarrow T(V)$$

such that,

(1) for all $i, j \in \mathbb{N}^*$ and $a_1, \ldots, a_{i+j} \in V$, $m(a_1 \otimes \ldots \otimes a_i, a_{i+1} \otimes \ldots \otimes a_{i+j}) = a_1 \otimes \ldots \otimes a_{i+j}$;
(2) $\forall \lambda \in \mathbb{k}$, for all $i \in \mathbb{N}^*$ and $a_1, \ldots, a_i \in V$, $m(\lambda, a_1\otimes\ldots\otimes a_i) = m(a_1\otimes\ldots\otimes a_i, \lambda) = \lambda a_1\otimes\ldots\otimes a_i$;
(3) for all $\lambda, \mu \in T^0(V)$, $m(\lambda, \mu) = \lambda\mu$.

It is not difficult to show that, equipped with the map m, the vector space $T(V)$ becomes a unital, associative $\mathbb{k}$-algebra with unit $1_{\mathbb{k}}$. Clearly, $T(V)$ is an $\mathbb{N}$-graded $\mathbb{k}$-algebra with homogeneous subspace of degree $i \in \mathbb{N}$ (the image of) $T^i(V)$. It is clear that any generating set of the $\mathbb{k}$-vector space V is a set of generators for the $\mathbb{k}$-algebra $T(V)$.

Definition IV.2.1 – *The $\mathbb{k}$-algebra $T(V)$ is called the tensor algebra of V and the map $V \stackrel{can.inj.}{\longrightarrow} T(V)$ the associated canonical injection.*

Proposition IV.2.2 – **Universal property of the tensor algebra** –
Let V be any $\mathbb{k}$-vector space, A be any associative, unital $\mathbb{k}$-algebra and $\phi : V \longrightarrow A$ be any

morphism of $\Bbbk$*-vector spaces. Then, there exists a unique morphism of associative, unital algebras* $\psi : T(V) \longrightarrow A$ *such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "inj.can."] \arrow[dr, "\phi"] & T(V) \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

Proof. Exercise. (See [BBK-Algèbre-1-3; p. III.56].) ∎

Suppose $\mathcal{B}$ is a basis of V. To any finite sequence $s = (b_1, \ldots, b_p)$, $p \in \mathbb{N}^*$, of elements of $\mathcal{B}$ we associate the pure tensor $b_s = b_1 \otimes \ldots \otimes b_p \in T^p(V)$. Further, to the empty sequence of elements of $\mathcal{B}$, denoted $\emptyset$, we associate $b_\emptyset = 1_\Bbbk \in T^0(V)$.

Proposition IV.2.3 – *Keep the notation as above. The set of all elements* b_s*, where* s *is a finite sequence of elements of* $\mathcal{B}$*, is a basis of the* $\Bbbk$*-vector space* $T(V)$*.*

Proof. See [BBK-Algèbre-1-3; p. III.62]. ∎

We now recall the definition and basic properties of the *symmetric algebra* of a $\Bbbk$-vector space. Extensive details may be found in [BBK-Algèbre-1-3], Chap. III, §6.

Retain the above notation and let I be the two-sided ideal of $T(V)$ generated by the elements $x \otimes y - y \otimes x$, $x, y \in V$.

Definition IV.2.4 – *The symmetric algebra of the* $\Bbbk$*-vector space* V *is the* $\Bbbk$*-algebra* $S(V) = T(V)/I$*.*

Since I is generated by homogeneous elements (of degree 2) of the $\mathbb{N}$-graded algebra $T(V)$, we have that

$$I = \bigoplus_{i \in \mathbb{N},\, 2 \leq i} \left(I \cap T^i(V) \right).$$

and $S(V)$ inherits a grading from that of $T(V)$. More precisely, let $\tau : T(V) \longrightarrow S(V)$ be the canonical projection. For all $i \in \mathbb{N}$, put $S^i(V) = \tau(T^i(V))$. Then, we have

$$S(V) = \bigoplus_{i \in \mathbb{N}} S^i(V)$$

and the following isomorphisms of $\Bbbk$-vector spaces

$$\Bbbk = T^0(V) \cong S^0(V), \quad T^1(V) \cong S^1(V) \quad \text{and} \quad T^i(V)/I \cap T^i(V) \cong S^i(V),\ i \geq 2,$$

all of which are induced by τ. Therefore, the canonical injection associated to $T(V)$ induces a canonical injection

$$V \stackrel{can.inj.}{\longrightarrow} T(V) \stackrel{\tau}{\longrightarrow} S(V).$$

It is then clear that the image under this canonical injection of any generating set of the $\Bbbk$-vector space V is a set of generators of the $\Bbbk$-algebra $S(V)$. From this latter fact it follows that $S(V)$ is a commutative $\Bbbk$-algebra.

Proposition IV.2.5 – **Universal property of the symmetric algebra** –
Let V be any $\mathbb{k}$-vector space, A be any commutative, associative, unital $\mathbb{k}$-algebra and $\phi : V \longrightarrow A$ be any morphism of $\mathbb{k}$-vector spaces. Then, there exists a unique morphism of associative, unital algebras $\psi : S(V) \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "inj.can."] \arrow[dr, "\phi"'] & S(V) \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

Proof. Exercise. (See [BBK-Algèbre-1-3; p. III.67].) ∎

Further, let $\mathcal{B}$ be a basis of V. Choose a total order on $\mathcal{B}$ (this is possible since, by Zermelo's Theorem, any set has a well-order, see [BBK-Ensembles; pp. III.15, III.20]). Let $b \in \mathcal{B}$. Then, b may be seen as elements of $T^1(V)$ and we consider its image under τ in $S^1(V)$, that we still denote b. Thus, following the above definition, to any finite sequence $s = (b_1 \leq \ldots \leq b_p)$, $p \in \mathbb{N}^*$ of elements of $\mathcal{B}$, we may associate the image $\tau(b_s)$ of b_s in $S(V)$ that we still denote b_s, by abuse of notation. Thus $b_s = b_1 \ldots b_p \in S(V)^p$. In particular, $b_\emptyset = 1_\mathbb{k} \in S^0(V)$.

Proposition IV.2.6 – *Keep the above notation. The set of all b_s, with s a finite increasing sequence of elements of $\mathcal{B}$, is a basis of the $\mathbb{k}$-vector space $S(V)$.*

Proof. See [BBK-Algèbre-1-3; p. III.75] and [BBK-Algèbre-4-7; p. IV.2]. ∎

We now introduce the *enveloping algebra* of a Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{k}$. We may consider the tensor algebra $T(\mathfrak{g})$ of the $\mathbb{k}$-vector space $\mathfrak{g}$. Then, for all $x, y \in \mathfrak{g}$, we have the elements $x \otimes y - y \otimes x \in T^2(\mathfrak{g})$ and $[x, y] \in T^1(\mathfrak{g})$. Therefore, we may consider the elements $x \otimes y - y \otimes x - [x, y] \in T^1(\mathfrak{g}) \oplus T^2(\mathfrak{g}) \subseteq T(\mathfrak{g})$, for all $x, y \in \mathfrak{g}$, and then the two-sided ideal J of $T(\mathfrak{g})$ generated by these elements:

$$J = \langle x \otimes y - y \otimes x - [x, y],\ x, y \in \mathfrak{g} \rangle \subseteq T(\mathfrak{g}).$$

Put then

$$U(\mathfrak{g}) = T(\mathfrak{g})/J$$

and denote $\psi : T(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ the canonical projection. Together with $U(\mathfrak{g})$, we have a canonical morphism of $\mathbb{k}$-vector spaces

$$j_\mathfrak{g} : \mathfrak{g} \xrightarrow{can.inj.} T(\mathfrak{g}) \xrightarrow{\psi} U(\mathfrak{g}).$$

It is clear by definition that this map is actually a morphism of Lie algebras, where the associative algebra $U(\mathfrak{g})$ is considered as a Lie algebra in the standard way.

Since the images in $T(\mathfrak{g})$ of the elements $x \in \mathfrak{g}$ generate $T(\mathfrak{g})$ as a $\mathbb{k}$-algebra, we get that the set $\{j_\mathfrak{g}(x),\ x \in \mathfrak{g}\}$ is a generating set of the algebra $U(\mathfrak{g})$. Further, put $T^+(\mathfrak{g}) = \bigoplus_{i \in \mathbb{N}^*} T^i(\mathfrak{g})$. It is clear that $J \subseteq T^+(\mathfrak{g})$. In particular, $J \cap \mathbb{k} = \{0\}$. From this observation, it follows that the restriction to $\mathbb{k}$ of the canonical projection $T(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ is injective. Hence, $U(\mathfrak{g})$ is not the zero algebra: $0_{U(\mathfrak{g})} \neq 1_{U(\mathfrak{g})}$.

Definition IV.2.7 – *Keep the above notation. Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{k}$. The universal enveloping algebra of the Lie algebra $\mathfrak{g}$ is the $\mathbb{k}$-algebra $U(\mathfrak{g}) = T(\mathfrak{g})/J$. The map $j_\mathfrak{g} : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is called the canonical morphism (of Lie algebras) associated to it.*

Remark IV.2.8 – Keep the above notation. It is clear that if $\mathfrak{g}$ is an abelian Lie algebra, then its enveloping algebra is just its symmetric algebra. In this case, $U(\mathfrak{g})$ is thus graded (this is not true in general) and the canonical morphism is injective (this will still hold in general but is far from obvious).

Proposition IV.2.9 – **Universal property of the enveloping algebra** – *Let $\mathfrak{g}$ be any Lie algebra over $\mathbb{k}$, A be any associative, unital, $\mathbb{k}$-algebra. If $\phi : \mathfrak{g} \longrightarrow A$ is any morphism of Lie algebras (where the associative $\mathbb{k}$-algebra A is considered as a Lie algebra in the standard way). Then, there exists a unique morphism of associative, unital $\mathbb{k}$-algebras $\psi : U(\mathfrak{g}) \longrightarrow A$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "j_{\mathfrak{g}}"] \arrow[dr, "\phi"] & U(\mathfrak{g}) \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

Proof. By the universal property of the tensor algebra of $\mathfrak{g}$, we know that there exists a morphism of $\mathbb{k}$-algebras $\phi' : T(\mathfrak{g}) \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "can.inj."] \arrow[dr, "\phi"] & T(\mathfrak{g}) \arrow[d, "\phi'"] \\ & A \end{tikzcd}$$

Now, let $x, y \in \mathfrak{g}$. We have

$$\begin{array}{rcl} \phi'(x \otimes y - y \otimes x - [x,y]) & = & \phi'(xy - yx - [x,y]) \\ & = & \phi'(x)\phi'(y) - \phi'(y)\phi'(x) - \phi'([x,y]) \\ & = & \phi(x)\phi(y) - \phi(y)\phi(x) - \phi([x,y]) \\ & = & 0 \end{array}.$$

Indeed, the first equality is just the definition of the product in $T(\mathfrak{g})$, the second follows from the fact that ϕ' is a morphism of $\mathbb{k}$-algebras, the third is due to the commutativity of the later diagram and the fourth comes from the fact that, by hypothesis, ϕ is a morphism of Lie algebras. As a consequence, the ideal J is in the kernel of ϕ', from which it follows that ϕ' induces a morphism $\psi : U(\mathfrak{g}) \longrightarrow A$ of $\mathbb{k}$-algebras such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "inj.can."] \arrow[drr, "\phi"] & T(\mathfrak{g}) \arrow[r, "proj.can."] \arrow[dr, "\phi'"] & U(\mathfrak{g}) \arrow[d, "\psi"] \\ & & A \end{tikzcd}$$

This proves the existence of the morphism ψ of the statement.

The uniqueness of ψ is easy. Indeed, the commutativity of the diagram in the statement forces the image of $j_{\mathfrak{g}}(x)$, for all $x \in \mathfrak{g}$, under ψ. But, these elements generate $U(\mathfrak{g})$ as an algebra. So ψ is unique. ∎

Remark IV.2.10 – Let $\mathfrak{g}$ be a Lie algebra. It turns out that $U(\mathfrak{g})$ is determined, up to isomorphism, by its universal property. More precisely, suppose we are given an associative, unital, $\mathbb{k}$-algebra $\mathcal{A}$ together with a morphism of Lie algebras $j : \mathfrak{g} \longrightarrow \mathcal{A}$ such that, for all algebra A and all morphism $\phi : \mathfrak{g} \longrightarrow A$ of Lie algebras, there exists a unique morphism ψ of associative algebras $\psi : \mathcal{A} \longrightarrow A$ such that $\psi \circ j = \psi$, then the $\mathbb{k}$-algebras $\mathcal{A}$ and $U(\mathfrak{g})$ are isomorphic. The proof is easy and left to the reader.

Remark IV.2.11 – **Representations of $\mathfrak{g}$ versus $U(\mathfrak{g})$-modules** – Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. Suppose we are given a representation of $\mathfrak{g}$ in the vector space V with structure morphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.
1. By the universal property of the enveloping algebra of $\mathfrak{g}$, we have a commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "j_{\mathfrak{g}}"] \arrow[dr, "\rho"'] & U(\mathfrak{g}) \arrow[d, "\phi"] \\ & \mathfrak{gl}(V) \end{tikzcd}$$

where ϕ is a morphism of associative, unital, $\Bbbk$-algebras. In particular, the map ϕ induces a morphism of rings from $U(\mathfrak{g})$ to the ring $\mathrm{End}_{\mathbb{Z}}(V)$ of group homomorphisms of the abelian group V. This amounts to say that the map

$$\begin{array}{rcl} U(\mathfrak{g}) \times V & \longrightarrow & V \\ (x, v) & \mapsto & \phi(x)(v) \end{array} \tag{IV.2.34}$$

defines a left $U(\mathfrak{g})$-module structure on V. Notice, however, that since we started with the $\Bbbk$-algebras homomorphism $\phi : U(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$, in addition to the usual axioms for (left) modules, the map (IV.2.34) satisfies $\Bbbk$-bilinearity.
2. If $\mathcal{A}$ is any (associative, unital) $\Bbbk$-algebra. A linear representation of $\mathcal{A}$ is defined to be a pair (V, ϕ) where V is a $\Bbbk$-vector space, and $\phi : \mathcal{A} \longrightarrow \mathrm{End}_{\Bbbk}(V)$ a morphism of $\Bbbk$-algebras from $\mathcal{A}$ to the $\Bbbk$-algebra of endomorphisms of the $\Bbbk$-vector space V. With this vocabulary, it is equivalent to consider a representation of the Lie algebra $\mathfrak{g}$ and a linear representation of its universal enveloping algebra.

To understand better the structure of the enveloping algebra of a Lie algebra, it is convenient to link it with its symmetric algebra. This is done by filtering the enveloping algebra and considering the associated graded algebra. This is what we proceed to describe now.

First recall that, given an associative, unital $\Bbbk$-algebra $\mathcal{A}$, an $\mathbb{N}$-filtration (or simply a filtration) of $\mathcal{A}$ is a sequence of $\Bbbk$-vector subspaces $(\mathcal{A}_i)_{i\in\mathbb{N}}$ of $\mathcal{A}$ such that:
(1) $1 \in \mathcal{A}_0$;
(2) $\mathcal{A}_i \subseteq \mathcal{A}_{i+1}$, for all $i \in \mathbb{N}$;
(3) $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$, for all $i, j \in \mathbb{N}$;
(4) $\mathcal{A} = \cup_{i\in\mathbb{N}}\mathcal{A}_i$.
It will be convenient to put $\mathcal{A}_{-1} = \{0\}$ whenever we consider an $\mathbb{N}$-filtration $(\mathcal{A}_i)_{i\in\mathbb{N}}$ on the $\Bbbk$-algebra $\mathcal{A}$.

To an associative, unital $\Bbbk$-algebra $\mathcal{A}$ filtered by $\mathcal{F} = (\mathcal{A}_i)_{i\in\mathbb{N}}$, we can associate its so-called *associated graded* $\Bbbk$*-algebra* as follows.

For all $i \in \mathbb{N}$, consider the $\Bbbk$-vector space $\mathrm{gr}^i_{\mathcal{F}}(\mathcal{A}) = \mathcal{A}_i/\mathcal{A}_{i-1}$ together with the canonical projection

$$\mathrm{gr}^i_{\mathcal{F}} : \mathcal{A}_i \longrightarrow \mathcal{A}_i/\mathcal{A}_{i-1}.$$

Further, consider the $\mathbb{N}$-graded $\Bbbk$-vector space

$$\mathrm{gr}_{\mathcal{F}}(\mathcal{A}) = \bigoplus_{i\in\mathbb{N}} \mathcal{A}_i/\mathcal{A}_{i-1}.$$

It is easy to check that, for all $i, j \in \mathbb{N}$, there is a map

$$m_{i,j} : \mathcal{A}_i/\mathcal{A}_{i-1} \times \mathcal{A}_j/\mathcal{A}_{j-1} \longrightarrow \mathcal{A}_{i+j}/\mathcal{A}_{i+j-1}$$

such that, for all $a \in \mathcal{A}_i$ and all $b \in \mathcal{A}_j$

$$m_{i,j}(\mathrm{gr}^i_{\mathcal{F}}(a), \mathrm{gr}^j_{\mathcal{F}}(b)) = \mathrm{gr}^{i+j}_{\mathcal{F}}(ab).$$

These maps, in turn, give rise to a bilinear map

$$m : \mathrm{gr}_{\mathcal{F}}(\mathcal{A}) \times \mathrm{gr}_{\mathcal{F}}(\mathcal{A}) \longrightarrow \mathrm{gr}_{\mathcal{F}}(\mathcal{A})$$

such that, for all $i, j \in \mathbb{N}$, $\chi \in \mathcal{A}_i/\mathcal{A}_{i-1}$, and $\xi \in \mathcal{A}_j/\mathcal{A}_{j-1}$, $m(\chi, \xi) = m_{i,j}(\chi, \xi)$. It is then easy to check that, equipped with the map m, the $\mathbb{N}$-graded $\Bbbk$-vector space $\mathrm{gr}_{\mathcal{F}}(\mathcal{A})$ becomes an $\mathbb{N}$-graded, associative, unital $\Bbbk$-algebra whose unit is $\mathrm{gr}^0_{\mathcal{F}}(1_{\mathcal{A}})$.

Definition IV.2.12 – *Keep the above notation. Then, the $\mathbb{N}$-graded, associative, unital $\Bbbk$-algebra $\mathrm{gr}_{\mathcal{F}}(\mathcal{A})$ is called the associated graded $\Bbbk$-algebra of the filtered $\Bbbk$-algebra $(\mathcal{A}, \mathcal{F})$.*

This process does apply to the enveloping algebra of a Lie algebra.

Indeed, let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. First, notice that $T(\mathfrak{g})$ is filtered by the $\mathbb{N}$-filtration $(T(\mathfrak{g})_i)_{i\in\mathbb{N}}$ where, for all $i \in \mathbb{N}$,

$$T(\mathfrak{g})_i = \bigoplus_{0\leq k\leq i} T^k(\mathfrak{g}).$$

It follows at once that $U(\mathfrak{g})$ is filtered by the $\mathbb{N}$-filtration $\mathcal{F} = (U(\mathfrak{g})_i)_{i\in\mathbb{N}}$ where, for all $i \in \mathbb{N}$, $U(\mathfrak{g})_i$ is the canonical image of $T(\mathfrak{g})_i$ in $U(\mathfrak{g})$. We are going to link the associated graded $\Bbbk$-algebra of the filtered $\Bbbk$-algebra $(U(\mathfrak{g}), \mathcal{F})$ with $S(\mathfrak{g})$. From now on, we forget the subscript $\mathcal{F}$, to simplify notation.

For all $i \in \mathbb{N}$, we have $\Bbbk$-linear maps

$$\psi_i : T^i(\mathfrak{g}) \xrightarrow{\subseteq} T(\mathfrak{g})_i \xrightarrow{can.proj.} U(\mathfrak{g})_i \quad \text{and} \quad \varphi_i : T^i(\mathfrak{g}) \xrightarrow{\psi_i} U(\mathfrak{g})_i \xrightarrow{\mathrm{gr}^i} U(\mathfrak{g})_i/U(\mathfrak{g})_{i-1}$$

and it is easy to check that φ_i is surjective. The direct sum of the maps φ_i, $i \in \mathbb{N}$, then define a map

$$\varphi : T(\mathfrak{g}) \longrightarrow \mathrm{gr}(U(\mathfrak{g})).$$

Lemma IV.2.13 – *Keep the above notation. The map φ is an $\mathbb{N}$-graded, surjective morphism of associative, unital, $\Bbbk$-algebras and $\langle x \otimes y - y \otimes x,\, x, y \in \mathfrak{g}\rangle \subseteq \ker(\varphi)$.*

Proof. By construction, φ is a surjective morphism of $\mathbb{N}$-graded $\Bbbk$-vector spaces.

In addition, $\varphi(1_{T(\mathfrak{g})}) = \varphi_0(1_{T(\mathfrak{g})}) = \mathrm{gr}^0(1_{U(\mathfrak{g})}) = 1_{\mathrm{gr}(U(\mathfrak{g}))}$. Consider now two homogeneous elements of $T(\mathfrak{g})$: $t \in T^m(\mathfrak{g})$, $t' \in T^n(\mathfrak{g})$, $m, n \in \mathbb{N}$. Then, we have that $\varphi(t) = \varphi_m(t) = \mathrm{gr}^m(\psi_m(t))$, $\varphi(t') = \varphi_n(t) = \mathrm{gr}^n(\psi_n(t'))$ and $\varphi(tt') = \varphi_{m+n}(tt') = \mathrm{gr}^{m+n}(\psi_{m+n}(tt'))$. But, by the definition of the product in the associated graded $\Bbbk$-algebra $\mathrm{gr}(U(\mathfrak{g}))$, we also have

$$\begin{aligned}
\varphi(t)\varphi(t') &= \mathrm{gr}^m(\psi_m(t))\mathrm{gr}^n(\psi_n(t')) \\
&= \mathrm{gr}^{m+n}(\psi_m(t)\psi_n(t')) \\
&= \mathrm{gr}^{m+n}(\psi(t)\psi(t')) \\
&= \mathrm{gr}^{m+n}(\psi(tt')) \\
&= \mathrm{gr}^{m+n}(\psi_{m+n}(tt')) \\
&= \varphi(tt').
\end{aligned}$$

It is then obvious that the multiplicativity of φ extends to nonhomogeneous elements. Hence, φ is a morphism of unital algebras.

In addition, consider $x, y \in \mathfrak{g}$. We have

$$\varphi(x\otimes y-y\otimes x) = \varphi_2(x\otimes y-y\otimes x) = \mathrm{gr}^2(\psi_2(x\otimes y-y\otimes x)) = \mathrm{gr}^2(\psi(x\otimes y-y\otimes x)) = \mathrm{gr}^2(\psi([x,y])) = 0,$$

since $\psi([x,y]) \in U(\mathfrak{g})_1$. The statement is proved. ∎

Call $\tau : T(\mathfrak{g}) \longrightarrow S(\mathfrak{g})$ the canonical projection. Recall that τ is a surjective morphism of $\mathbb{N}$-graded $\Bbbk$-algebras which, then, induces morphisms of $\Bbbk$-vector spaces $\tau_n : T^n(\mathfrak{g}) \longrightarrow S^n(\mathfrak{g})$, $n \in \mathbb{N}$. By lemma IV.2.13, the $\mathbb{N}$-graded algebra morphism $\varphi : T(\mathfrak{g}) \longrightarrow \mathrm{gr}(U(\mathfrak{g}))$ factorises through τ, giving rise to an $\mathbb{N}$-graded, surjective morphism $\omega : S(\mathfrak{g}) \longrightarrow \mathrm{gr}(U(\mathfrak{g}))$ of associative $\Bbbk$-algebras. We denote the $\Bbbk$-linear maps induced by ω between homogeneous components by ω_n, $n \in \mathbb{N}$. Hence, we get the following commutative diagrams

$$\begin{tikzcd} T(\mathfrak{g}) \arrow[r, "\tau"] \arrow[dr, "\varphi"'] & S(\mathfrak{g}) \arrow[d, "\omega"] \\ & \mathrm{gr}(U(\mathfrak{g})) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T^n(\mathfrak{g}) \arrow[r, "\tau_n"] \arrow[dr, "\varphi_n"'] & S^n(\mathfrak{g}) \arrow[d, "\omega_n"] \\ & \mathrm{gr}^n(U(\mathfrak{g})) \end{tikzcd} \quad (n \in \mathbb{N}^*) \tag{IV.2.35}$$

All in all, we end up with the following commutative diagrams, for all $n \in \mathbb{N}^*$:

$$\begin{tikzcd} & U(\mathfrak{g})_n \arrow[dr, "\mathrm{gr}^n"] & \\ T^n(\mathfrak{g}) \arrow[ur, "\psi_n"] \arrow[rr, "\varphi_n"] \arrow[dr, "\tau_n"'] & & \mathrm{gr}^n(U(\mathfrak{g})) \\ & S^n(\mathfrak{g}) \arrow[ur, "\omega_n"'] & \end{tikzcd} \tag{IV.2.36}$$

To obtain the nice description of $U(\mathfrak{g})$ that we are pursuing, we need to show that, for all $n \in \mathbb{N}$, the maps ω_n are actually isomorphisms or, equivalently, that ω is an isomorphism. This result is known as the Poincaré-Birkhoff-Witt Theorem. Our next task is to establish it.

It turns out that one possible approach is to construct a representation of the Lie algebra $\mathfrak{g}$ on its symmetric algebra $S(\mathfrak{g})$. For this, fix a basis $(x_\lambda,\ \lambda \in \Lambda)$ of $\mathfrak{g}$ indexed by a set Λ. Given any finite sequence $M = (\lambda_1, \dots, \lambda_n)$ of n elements of Λ, with $n \in \mathbb{N}^*$, we consider the tensor $x_M = x_{\lambda_1} \otimes \ldots \otimes x_{\lambda_n} \in T^n(\mathfrak{g})$. (Hence, if $M = (\lambda)$ is a sequence of one element of Λ, then $x_M = x_\lambda \in T^1(\mathfrak{g})$.) Further, to the empty sequence, we associate the tensor $x_\emptyset = 1_{T(\mathfrak{g})} \in T^0(\mathfrak{g})$. We know by Proposition IV.2.3 that these tensor form a basis of $T(\mathfrak{g})$. Now, denote by z_M the canonical image of x_M in $S(\mathfrak{g})$. We have that $z_\emptyset = 1_{S(\mathfrak{g})}$ and, for all sequence $M = (\lambda_1, \dots, \lambda_n)$ of n elements of Λ, $n \in \mathbb{N}^*$, we have

$$z_M = z_{\lambda_1} \ldots z_{\lambda_n}.$$

Now, equip Λ with a total ordering. Then, by Proposition IV.2.6, the elements z_M, where M is any finite (possibly empty) *increasing* sequence of elements of Λ, form a basis of $S(\mathfrak{g})$.

Below, we will denote the total ordering on Λ by $\leq$. Further, for $\lambda \in \Lambda$ and $M = (\lambda_1, \dots, \lambda_n)$ a finite sequence of elements of Λ, we will write $\lambda \leq M$ whenever $\lambda \leq \lambda_i$, for all $1 \leq i \leq n$. By convention, for all $\lambda \in \Lambda$, $\lambda \leq \emptyset$. For $n \in \mathbb{N}$, we denote by $\mathcal{S}_n$ the set of all sequences of n elements of Λ and put $\mathcal{S}_{\leq n} = \cup_{0\leq q\leq n}\mathcal{S}_q$. We denote by $\mathcal{S}$ the set of all finite sequences of elements of Λ.

For all $p \in \mathbb{N}$, we put $S(\mathfrak{g})_p = \bigoplus_{0\leq i\leq p} S^i(\mathfrak{g})$. Hence, $(S(\mathfrak{g})_p)_{p\in\mathbb{N}}$ is a filtration of the associative algebra $S(\mathfrak{g})$.

Lemma IV.2.14 – *For all $p \in \mathbb{N}$, there exists a unique $\Bbbk$-linear map*

$$f_p \,:\, \mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})_p \longrightarrow S(\mathfrak{g})$$

satisfying the above conditions:
(A_p) for all $\lambda \in \Lambda$ and $M \in \mathcal{S}_{\leq p}$ such that $\lambda \leq M$, $f_p(x_\lambda \otimes z_M) = z_\lambda z_M$;
(B_p) for all $\lambda \in \Lambda$, for all $0 \leq q \leq p$ and $M \in \mathcal{S}_{\leq q}$, $f_p(x_\lambda \otimes z_M) - z_\lambda z_M \in S(\mathfrak{g})_q$;
(C_p) for all $\lambda, \mu \in \Lambda$ and for all $N \in \mathcal{S}_{\leq p-1}$, $f_p(x_\lambda \otimes f_p(x_\mu \otimes z_N)) = f_p(x_\mu \otimes f_p(x_\lambda \otimes z_N)) + f_p([x_\lambda, x_\mu] \otimes z_N))$.

In addition, for all $p \in \mathbb{N}^*$, f_{p-1} coincides with the restriction of f_p to $\mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})_{p-1}$.

Proof. The reader is referred to [BBK-Lie-1 ; §2, no. 7]. ∎

Lemma IV.2.15 – *There exists a morphism of Lie algebras $\sigma \,:\, \mathfrak{g} \longrightarrow \mathfrak{gl}(S(\mathfrak{g}))$ such that:*
(1) for all $\lambda \in \Lambda$ and all $M \in \mathcal{S}$ such that $\lambda \leq M$, $\sigma(x_\lambda)(z_M) = z_\lambda z_M$.
(2) for all $\lambda \in \Lambda$, all $p \in \mathbb{N}$ and all $M \in \mathcal{S}_p$, $\sigma(x_\lambda)(z_M) - z_\lambda z_M \in S(\mathfrak{g})_p$.

Proof. For all $p \in \mathbb{N}$, identify the $\Bbbk$-vector space $\mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})_p$ with a subspace of $\mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})$. Then, we have an exhaustive, increasing filtration $(\mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})_p)_{p\in\mathbb{N}}$ of the $\Bbbk$-vector space $\mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})$. On the other hand, Lemma IV.2.14 provides $\Bbbk$-linear maps $f_p \,:\, \mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})_p \longrightarrow \mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})$, $p \in \mathbb{N}$, with the property that the restriction of f_p to $\mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})_{p-1}$ is f_{p-1} whenever $p \in \mathbb{N}^*$. Thus, we are in position to define a $\Bbbk$-linear map

$$f \,:\, \mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g}) \longrightarrow S(\mathfrak{g})$$

such that the image under f of any element of $\mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})$ is its image under f_p whenever $\mathfrak{g} \otimes_{\Bbbk} S(\mathfrak{g})_p$ contains that element. This map, in turn, gives rise to a $\Bbbk$-linear map

$$\sigma \,:\, \mathfrak{g} \longrightarrow \mathfrak{gl}(S(\mathfrak{g}))$$

such that, for all $x \in \mathfrak{g}$, $z \in S(\mathfrak{g})$, $\sigma(x)(z) = f(x \otimes z)$. It follows at once from conditions (C) of Lemma IV.2.14 that σ is a morphism of Lie algebras, for the standard structure of Lie algebra of the associative algebra $\mathfrak{gl}(S(\mathfrak{g}))$. In addition, conditions (1) and (2) of the statement clearly follow from conditions (A) and (B) of Lemma IV.2.14. ∎

Recall the canonical projections

$$T(\mathfrak{g}) \longrightarrow S(\mathfrak{g}) \qquad \text{and} \qquad T(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$$

with respective kernels I and J.

Lemma IV.2.16 *Let $n \in \mathbb{N}$. If $t \in T(\mathfrak{g})_n \cap J$, then its homogeneous component of degree n belongs to I.*

Proof. Denote by t_n the homogeneous component of t and write

$$t_n = \sum_{M \in \mathcal{S}_n} \alpha_M x_M$$

where $(\alpha_M)_{M \in \mathcal{S}_n}$ is a family of elements of $\Bbbk$, almost all of which are zero.

By the universal property of enveloping algebras and the universal property of tensor algebras, σ induces a morphism $\bar{\sigma} : T(\mathfrak{g}) \longrightarrow \mathfrak{gl}(S(\mathfrak{g}))$ containing J in its kernel such that the following diagram commutes

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "inj.can."] \arrow[drr, "j_{\mathfrak{g}}"] \arrow[ddrr, "\sigma"'] & & T(\mathfrak{g}) \arrow[d, "proj.can."'] \arrow[dd, bend left, "\bar{\sigma}"] \\
& & U(\mathfrak{g}) \arrow[d] \\
& & \mathfrak{gl}(S(\mathfrak{g}))
\end{tikzcd}$$

By Lemma IV.2.15, $\bar{\sigma}(t)(1_{S(\mathfrak{g})})$ is an element of $S(\mathfrak{g})_n$ whose component in $S^n(\mathfrak{g})$ is $\sum_{M \in \mathcal{S}_n} \alpha_M z_M$. On the other hand, $t \in J$, so that $\bar{\sigma}(t) = 0$. Hence, $\sum_{M \in \mathcal{S}_n} \alpha_M z_M = 0$. In other terms, $\sum_{M \in \mathcal{S}_n} \alpha_M x_M \in I$. ∎

Recall the map $\omega : S(\mathfrak{g}) \longrightarrow \mathrm{gr}(U(\mathfrak{g}))$ from (IV.2.35).

Theorem IV.2.17 – **Poincaré-Birkhoff-Witt** – *In the above notation, the map* $\omega : S(\mathfrak{g}) \longrightarrow \mathrm{gr}(U(\mathfrak{g}))$ *is an isomorphism of* $\mathbb{N}$*-graded, associative, unital* $\Bbbk$*-algebras.*

Proof. It remains to show that ω is injective and, since it is $\mathbb{N}$-graded, it is enough to show that an homogeneous element whose image is zero must be zero. By definition of ω, this amounts to showing that, if t is an homogeneous element of $T(\mathfrak{g})$ such that $\varphi(t) = 0$, then t must be in the ideal I.

Let t be such an element of $T^n(\mathfrak{g})$, for some $n \in \mathbb{N}$. Then its canonical image, $\psi(t)$, in $U(\mathfrak{g})$ must belong to $U(\mathfrak{g})_{n-1}$. Thus, there exists $t' \in T(\mathfrak{g})_{n-1}$ such that $\psi(t) = \psi(t')$, that is $t - t' \in J$. Thus, by Lemma IV.2.16, the homogeneous component of degree n of $t - t'$ belong to I. Since, clearly this homogeneous component is t, we get that $t \in I$, as requierred. ∎

The Poincaré-Birkhoff-Witt Theorem has many fundamental consequences; we now list some of them.

Corollary IV.2.18 – *Let* $n \in \mathbb{N}$ *and consider a subspace* W *of* $T^n(\mathfrak{g})$*. If the restriction of* τ_n *to* W *is an isomorphism from* W *to* $S^n(\mathfrak{g})$*, then the restriction of* ψ_n *to* W *is an isomorphism from* W *to a complement of* $U(\mathfrak{g})_{n-1}$ *in* $U(\mathfrak{g})_n$*, that is*

$$U(\mathfrak{g})_n = U(\mathfrak{g})_{n-1} \oplus \psi_n(W).$$

Proof. We use the commutative diagram (IV.2.36). By the PBW Theorem, the map ω_n is an isomorphism. Hence, the restriction of $\omega_n \circ \tau_n$ to W is an isomorphism from W to $\mathrm{gr}^n(U(\mathfrak{g}))$ and, by the commutativity of diagram (IV.2.36), the restriction of $\mathrm{gr}^n \circ \psi_n$ to W is an isomorphism from W to $\mathrm{gr}^n(U(\mathfrak{g}))$. The result follows. ∎

Suppose $(x_\lambda,\ \lambda \in \Lambda)$ is a basis of the $\Bbbk$-vector space $\mathfrak{g}$, where Λ is a totally ordered set. Abusing notation, we still denote x_λ, $\lambda \in \Lambda$, the image by $j_\mathfrak{g} : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ of x_λ. Now, to a finite sequence $M = (\lambda_1, \ldots, \lambda_n)$ of elements of Λ, $n \in \mathbb{N}^*$, we associate the product $x_M = x_{\lambda_1} \ldots x_{\lambda_n} \in U(\mathfrak{g})$. In addition, we put $x_\emptyset = 1_{U(\mathfrak{g})}$.

Corollary IV.2.19 – *Keep the above notation.*
1. Suppose $(x_\lambda,\ \lambda \in \Lambda)$ is a basis of the $\Bbbk$-vector space $\mathfrak{g}$, where Λ is a totally ordered set. The elements x_M, where M runs over the set of all finite increasing sequences of elements of Λ is a basis of the $\Bbbk$-vector space $U(\mathfrak{g})$.
2. The map $j_\mathfrak{g} : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is injective.

Proof. To start with, observe the following: for all $p, n \in \mathbb{N}$, $p \leq n$, if M is any finite sequence of p elements of Λ, x_M belongs to $U(\mathfrak{g})_n$. We are going to show, by induction on n, that the set of such elements x_M is a basis of $U(\mathfrak{g})_n$ when M runs over all finite increasing sequences of $p \leq n$ elements of Λ.

The case where $n = 0$ has already been proved (see the paragraph before Definition IV.2.7). Now, let $n \in \mathbb{N}^*$. Consider the $\Bbbk$-vector subspace W of $T^n(\mathfrak{g})$ with basis the set of pure tensors $x_{\lambda_1} \otimes \ldots \otimes x_{\lambda_n}$, where $(\lambda_1, \ldots, \lambda_n)$ is an increasing sequence of elements of Λ. Then, Corollary IV.2.18 applies to W. Therefore, we have that

$$U(\mathfrak{g})_n = U(\mathfrak{g})_{n-1} \bigoplus \left(\bigoplus_M \Bbbk x_M \right),$$

where M runs over the set of increasing sequence of n elements of Λ. So that, if the statement we want to prove holds for $U(\mathfrak{g})_{n-1}$, it also holds for $U(\mathfrak{g})_n$.

This completes the proof of Point 1. Point 2 follows since Point 1 shows that $j_\mathfrak{g}$ sends a basis of $\mathfrak{g}$ to a linearly independent familly in $U(\mathfrak{g})$. ■

The following Corollary provides a strong link between the enveloping algebra of a Lie algebra and that of any Lie subalgebra. Fix the following notation: let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. The universal property of the enveloping algebra of $\mathfrak{h}$ establishes the existence of a unique morphism of $\Bbbk$-algebras $\iota_{\mathfrak{h},\mathfrak{g}} : U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$ such that the following diagram commutes

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "j_\mathfrak{h}"] \arrow[d, "\subseteq"'] & U(\mathfrak{h}) \arrow[d, "\iota_{\mathfrak{h},\mathfrak{g}}"] \\ \mathfrak{g} \arrow[r, "j_\mathfrak{g}"] & U(\mathfrak{g}) \end{tikzcd}$$

Of course, the map $\iota_{\mathfrak{h},\mathfrak{g}} : U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$ allows to provide a left $U(\mathfrak{h})$-module structure on $U(\mathfrak{g})$, by restriction of scalars.

Corollary IV.2.20 – *Retain the above notation. The map $\iota_{\mathfrak{h},\mathfrak{g}} : U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$ is injective. Further, the left $U(\mathfrak{h})$-module $U(\mathfrak{g})$ is free.*

Proof. We may consider a basis $(x_\lambda,\ \lambda \in \Lambda)$ of $\mathfrak{g}$ where Λ is a totally ordered set with a subset Λ' such that $(x_\lambda,\ \lambda \in \Lambda')$ is a basis of $\mathfrak{h}$ and any element of Λ' is less than or equal to any element of $\Lambda \setminus \Lambda'$. (Use Zermelo's Theorem for the indexing set of any basis of $\mathfrak{h}$ and for the indexing set of any complement of this basis to a basis of $\mathfrak{g}$ and then glue conveniently the two total orderings obtained in that way.)

Consider the basis of $U(\mathfrak{h})$ attached to the ordered set Λ' by Corollary IV.2.19 and the basis of $U(\mathfrak{g})$ attached to the ordered set Λ in the same manner. By abuse of notation, we denote by x_M the elements of these two bases, with M a finite increasing sequence of elements of Λ' or Λ according to which basis we work with.

It is clear that, for all finite increasing sequence M of elements of Λ', $\iota_{\mathfrak{h},\mathfrak{g}}(x_M) = x_M$. Hence, $\iota_{\mathfrak{h},\mathfrak{g}}$ must be injective.

It is clear also that the set of all x_M, with M a finite increasing sequence of elements of $\Lambda \setminus \Lambda'$ is a basis of $U(\mathfrak{g})$, considered as a left $U(\mathfrak{h})$-module. ■

Exercise IV.2.21 – Assume $\mathbb{k}$ is algebraically closed of characteristic zero. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{k}$ and $\mathfrak{h}$ a maximal toral Lie subalgebra. Denote by Φ the root system associated to the pair $(\mathfrak{g}, \mathfrak{h})$, Δ a base of Φ and let

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n} \quad \text{and} \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$$

be the asssociated Cartan-Chevalley decomposition (see the end of Section III.3). By Corollary IV.2.20, $U(\mathfrak{n}^-)$ and $U(\mathfrak{b})$ identify with subalgebras of the $\mathbb{k}$-algebra $U(\mathfrak{g})$.
1. There exists an isomorphism of $\mathbb{k}$-vector spaces as follows:

$$\begin{array}{rcl} U(\mathfrak{n}^-) \otimes_{\mathbb{k}} U(\mathfrak{b}) & \longrightarrow & U(\mathfrak{g}) \\ x \otimes y & \mapsto & xy \end{array}.$$

2. The above isomorphism is actually an morphism of left $U(\mathfrak{n}^-)$-modules and of right $U(\mathfrak{b})$-modules (where the $(U(\mathfrak{n}^-), U(\mathfrak{b}))$-bimodule structure of $U(\mathfrak{g})$ is given by the multiplication in $U(\mathfrak{g})$).

We conclude this section by introducing the notion of *free Lie algebra over a set*. Its construction is quite straightforward; however, its universal property depends on the Poincaré-Birkhoff-Witt Theorem (or rather, the injectivity of the canonical morphism from a Lie algebra to its enveloping algebra).

Let X be a set. We denote by $\mathbb{k}^{(X)}$ the $\mathbb{k}$-vector space of maps from X to $\mathbb{k}$ whose support is finite. For all $x \in X$, we let ϵ_x be the map

$$\epsilon_x \,:\, X \longrightarrow \mathbb{k}$$

that sends x to $1_{\mathbb{k}}$ and all other elements of X to $0_{\mathbb{k}}$. Clearly, the set $(\epsilon_x,\, x \in X)$ is a basis of $\mathbb{k}^{(X)}$. In particular, we have an injective map $X \longrightarrow \mathbb{k}^{(X)}$, $x \mapsto \epsilon_x$.

Then, we consider the tensor algebra $T\left(\mathbb{k}^{(X)}\right)$ of the $\mathbb{k}$-vector space $\mathbb{k}^{(X)}$ together with the canonical injection $\mathbb{k}^{(X)} \longrightarrow T\left(\mathbb{k}^{(X)}\right)$ (we will freely identify an element in $\mathbb{k}^{(X)}$ with its canonical image in $T\left(\mathbb{k}^{(X)}\right)$).

Definition IV.2.22 – *Retain the above notation. The free Lie algebra on the set X, denoted $L(X)$, is the Lie subalgebra of $T\left(\mathbb{k}^{(X)}\right)$ generated by the elements ϵ_x, $x \in X$. The natural map $X \longrightarrow L(X)$, $x \mapsto \epsilon_x$ is called the canonical injection.*

The free Lie algebra over a set has the following universal property.

Proposition IV.2.23 – *Let X be a set and let $L(X)$ be the free Lie algebra over X. If $\mathfrak{g}$ is a Lie algebra and $\phi : X \longrightarrow \mathfrak{g}$ any map, then there exits a unique morphism $\psi : L(X) \longrightarrow \mathfrak{g}$ of Lie algebras such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "can.inj."] \arrow[dr, "\phi"'] & L(X) \arrow[d, "\psi"] \\ & \mathfrak{g} \end{tikzcd}$$

Proof. Suppose such a morphism ψ exists, then we must have: for all $x \in X$, $\psi(\epsilon_x) = \phi(x)$. Since the elements ϵ_x, $x \in X$ generate $L(X)$ as a Lie algebra, ψ must be unique.

We now prove the existence. We have a commutative diagram as follows

$$\begin{tikzcd} X \arrow[r, "can.inj."] \arrow[dr, "\phi"'] & \Bbbk^{(X)} \arrow[r, "can.inj."] \arrow[d, "\phi'"'] & T\left(\Bbbk^{(X)}\right) \arrow[d, "\phi''"'] & L(X) \arrow[l, "\supseteq"'] \arrow[dl, "\psi"] \\ & \mathfrak{g} \arrow[r, "j_{\mathfrak{g}}"'] & U(\mathfrak{g}) & \end{tikzcd}$$

where ϕ' is a morphism of $\Bbbk$-vector spaces whose existence is due to the universal property of $\Bbbk^{(X)}$ applied to the map $\phi : X \longrightarrow \mathfrak{g}$, ϕ'' is a morphism of $\Bbbk$-algebras whose existence is due to the universal property of $T\left(\Bbbk^{(X)}\right)$ applied to the map $j_{\mathfrak{g}} \circ \phi' : \Bbbk^{(X)} \longrightarrow U(\mathfrak{g})$, and ψ is the morphism of Lie algebras obtained by restriction of ϕ'' to $L(X)$.

Now, clearly, for all $x \in X$, $\psi(\epsilon_x) = \phi''(\epsilon_x) = j_{\mathfrak{g}}(\phi'(\epsilon_x))$. As these elements generate the Lie algebra $L(X)$ and $j_{\mathfrak{g}}(\mathfrak{g})$ is a Lie subalgebra of $U(\mathfrak{g})$, it follows that $\psi(L(X)) \subseteq j_{\mathfrak{g}}(\mathfrak{g})$.

But, by the PBW Theorem, the Lie algebra $\mathfrak{g}$ is isomorphic to its image in $U(\mathfrak{g})$ under $j_{\mathfrak{g}}$. So, identifying them, we may view ψ as a morphism of Lie algebras from $L(X)$ to $\mathfrak{g}$.

The proof is complete. ∎

IV.3 Generators and relations for semisimple Lie algebras.

In this section, we assume that $\Bbbk$ is algebraically closed and of characteristic 0.

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, let $\mathfrak{h}$ be a maximal toral subalgebra (equivalently a Cartan subalgebra). Let Φ be the set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$ and let Δ be a base of the root system $(\mathsf{E}_{\mathbb{R}}, \Phi)$. Then, Proposition III.4.2 establishes that the set $\sum_{\alpha\in\Delta} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$ generates $\mathfrak{g}$ as a Lie algebra.

More precisely, put $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ and, forall $1 \le i \le \ell$, $h_i = h_{\alpha_i} = \dfrac{2}{\kappa_{\mathfrak{g}}(t_{\alpha_i}, t_{\alpha_i})} t_{\alpha_i}$. We may consider elements $x_i \in \mathfrak{g}_{\alpha_i}$ and $y_i \in \mathfrak{g}_{-\alpha_i}$ such that

$$\mathfrak{g}_{\alpha_i} = \Bbbk x_i, \qquad \mathfrak{g}_{-\alpha_i} = \Bbbk y_i \qquad \text{and} \qquad [x_i, y_i] = h_i,$$

(cf. Theorem II.5.13 and Proposition II.5.15). In addition, the set $\{x_i, y_i, h_i, 1 \le i \le \ell\}$ generates the Lie algebra $\mathfrak{g}$. The next statement establishes relations among these generators.

Proposition IV.3.1 – *Retain the above notation. Then, for $1 \le i, j \le \ell$, the following relations hold:*
(1) $[h_i, h_j] = 0$;

(2) $[x_i, y_j] = \delta_{ij} h_i$*;*
(3) $[h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j$ *and* $[h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j$*;*
(4) $(\mathrm{ad}_{\mathfrak{g}}(x_i))^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0$*, for* $i \neq j$*;*
(5) $(\mathrm{ad}_{\mathfrak{g}}(y_i))^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0$*, for* $i \neq j$*.*

Proof. (1) This follows from the fact that $\mathfrak{h}$ is abelian, by Lemma II.5.2.
(2) The case where $i = j$ is trivial. Suppose $i \neq j$. Then, $\alpha_i - \alpha_j$ is not a root (since α_i and α_j belong to the same base) so that $\mathfrak{g}_{\alpha_i - \alpha_j} = 0$. But, on the other hand, Lemma II.5.6 shows that $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_j}] \subseteq \mathfrak{g}_{\alpha_i - \alpha_j}$. Hence the result.
(3) This is clear by definition of the root spaces.
(4) We have that $\alpha_j - \alpha_i$ is not a root. So, the α_i-string through α_j is $\alpha_j, \alpha_j + \alpha_i, \ldots, \alpha_j + q\alpha_i$, with $-q = \langle \alpha_j, \alpha_i \rangle$, see Proposition III.2.15. In particular, α_j is $\alpha_j + (1 - \langle \alpha_j, \alpha_i \rangle)\alpha_i$ is not a root. But, by Lemma II.5.6, the right hand side of relation (4) belongs to $\mathfrak{g}_{\alpha_j + (1 - \langle \alpha_j, \alpha_i \rangle)\alpha_i}$. So, it must be zero.
(5) The same argument as for relation (4) works. ∎

Remark IV.3.2 – In Proposition IV.3.1, relations (1), (2), (3) are called *Weyl's relations*, while relations (4) and (5) are called *Serre's relations*.

The relations listed in Proposition IV.3.1 only depend on the Cartan integers, that is on the root system of $(\mathfrak{g}, \mathfrak{h})$. Hence, starting from a root system, we may construct a Lie algebra by considering the free Lie algebra on a set with 3ℓ elements, and factoring it out by the ideal generated by the above relations. This is what we do now.

Fix a root system Φ in a euclidean space E of dimension ℓ, $\ell \in \mathbb{N}^*$, a base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of Φ and put

$$c_{i,j} = \langle \alpha_i, \alpha_j \rangle, \quad 1 \leq i, j \leq \ell.$$

Consider a set $\{X_i^\circ, Y_i^\circ, H_i^\circ, 1 \leq i \leq \ell\}$ and denote by L° the free Lie algebra on this set. Further, consider the Lie ideal K° of L° generated by the elements

$$[H_i^\circ, H_j^\circ], \quad [X_i^\circ, Y_j^\circ] - \delta_{ij} H_i^\circ, \quad [H_i^\circ, X_j^\circ] - c_{ji} X_j^\circ \quad \text{and} \quad [H_i^\circ, Y_j^\circ] + c_{ji} Y_j^\circ, \ 1 \leq i, j \leq \ell.$$

We denote by L que Lie algebra L° / K° and by X_i, Y_i, H_i the respective images of $X_i^\circ, Y_i^\circ, H_i^\circ$, $1 \leq i \leq \ell$, under the canonical projection

$$L^\circ \longrightarrow L .$$

Observe that the vector subspace generated by the H_i, $1 \leq i \leq \ell$, is an abelian Lie subalgebra of L.

Now, let V be a $\Bbbk$-vector space with basis $\{v_1, \ldots, v_\ell\}$. We consider the tensor algebra $T(V)$ of V. Recall that the set consisting of 1 together with the products $v_{i_1} \ldots v_{i_t}$, $t \in \mathbb{N}^*$, $1 \leq i_1, \ldots, i_t \leq \ell$ is a basis of the $\Bbbk$-vector space $T(V)$. We now consider the following 3ℓ endomorphisms of $T(V)$. Let $1 \leq j \leq \ell$. Define

$$\begin{array}{rcrcl} \mathcal{H}_j & : & T(V) & \longrightarrow & T(V) \\ & & 1 & \mapsto & 0 \\ & & v_{i_1} \ldots v_{i_t} & \mapsto & -(c_{i_1,j} + \ldots + c_{i_t,j}) v_{i_1} \ldots v_{i_t} \end{array} .$$

Let $\mathcal{Y}_j$ be left multiplication by v_j in $T(V)$:

$$\begin{array}{rcrcl} \mathcal{Y}_j & : & T(V) & \longrightarrow & T(V) \\ & & a & \mapsto & v_j a \end{array} .$$

And finally, define $\mathcal{X}_j$ by induction on t as follows

$$\begin{array}{rcrcl} \mathcal{X}_j & : & T(V) & \longrightarrow & T(V) \\ & & 1 & \mapsto & 0 \\ & & v_i & \mapsto & 0 \\ & & v_{i_1}\dots v_{i_t} & \mapsto & \left(\mathcal{Y}_{i_1}\mathcal{X}_j + \delta_{i_1,j}\mathcal{H}_j\right)(v_{i_2}\dots v_{i_t}) \end{array}.$$

By the universal property of L°, there is a Lie algebra homomorphism as follows:

$$\begin{array}{rcrcl} \phi^\circ & : & L^\circ & \longrightarrow & \mathfrak{gl}(T(V)) \\ & & H_i^\circ & \mapsto & \mathcal{H}_i \\ & & X_i^\circ & \mapsto & \mathcal{X}_i \\ & & Y_i^\circ & \mapsto & \mathcal{Y}_i \end{array}$$

Remark IV.3.3 –
1. Let $1 \le i \le \ell$. Clearly, the endomorphism $\mathcal{H}_i$ stabilises the homogeneous components of $T(V)$, while $\mathcal{Y}_i$ sends $T^d(V)$ into $T^{d+1}(V)$, for all $d \in \mathbb{N}$. From this it follows, by an obvious induction, that $\mathcal{X}_i$ sends $T^d(V)$ into $T^{d-1}(V)$, for all $d \in \mathbb{N}$ (with the convention that $T^{-1}(V) = \{0\}$).
2. Let $1 \le i, j \le \ell$. An obvious calculation shows that

$$\mathcal{X}_i(v_j v_j) = -\delta_{i,j} c_{j,i} v_j.$$

Lemma IV.3.4 – *In the above notation, $K^\circ \subseteq \ker(\phi^\circ)$.*

Proof. By definition, the endomorphisms $\mathcal{H}_i$, $1 \le i \le \ell$, are simultaneously diagonalisable, hence they pairwise commute:

$$\text{for all} \quad 1 \le i, j \le \ell, \quad \mathcal{H}_i\mathcal{H}_j - \mathcal{H}_j\mathcal{H}_i = 0. \tag{IV.3.37}$$

We then show that:

$$\text{for all} \quad 1 \le i, j \le \ell, \quad \mathcal{X}_i\mathcal{Y}_j - \mathcal{Y}_j\mathcal{X}_i = \delta_{i,j}\mathcal{H}_i. \tag{IV.3.38}$$

Clearly, the left and right hand side of this equality send 1 to zero. Now, let $t \in \mathbb{N}$, $t \ge 2$ and $1 \le i_1, \dots, i_t \le \ell$, by definition of $\mathcal{X}_i$, we have that $\mathcal{X}_i\mathcal{Y}_{i_1}(v_{i_2}\dots v_{i_t}) = \left(\mathcal{Y}_{i_1}\mathcal{X}_i + \delta_{i_1,i}\mathcal{H}_i\right)(v_{i_2}\dots v_{i_t})$. So, the left and right hand side of the above equality take the same value on any pure tensor of non zero degree in the elements v_k, $1 \le k \le \ell$. So (IV.3.38) holds.
It is straightforward to show that

$$\text{for all} \quad 1 \le i, j \le \ell, \quad \mathcal{H}_i\mathcal{Y}_j - \mathcal{Y}_j\mathcal{H}_i = -c_{ji}\mathcal{Y}_j. \tag{IV.3.39}$$

We now comme to the following observation, for all $1 \le i, j, k \le \ell$:

$$\begin{array}{rcl} 0 & = & [\mathcal{H}_i, [\mathcal{X}_j, \mathcal{Y}_k]] \\ & = & [[\mathcal{H}_i, \mathcal{X}_j], \mathcal{Y}_k]] + [\mathcal{X}_j, [\mathcal{H}_i, \mathcal{Y}_k]] \\ & = & [[\mathcal{H}_i, \mathcal{X}_j], \mathcal{Y}_k]] - c_{k,i}[\mathcal{X}_j, \mathcal{Y}_k] \\ & = & [[\mathcal{H}_i, \mathcal{X}_j], \mathcal{Y}_k]] - c_{j,i}[\mathcal{X}_j, \mathcal{Y}_k] \\ & = & [[\mathcal{H}_i, \mathcal{X}_j] - c_{j,i}\mathcal{X}_j, \mathcal{Y}_k] \end{array}. \tag{IV.3.40}$$

Indeed, the first equality follows from relations (IV.3.37) and (IV.3.38), the second is the Jacobi identity in $\mathfrak{gl}(T(V))$, the third follows from relation (IV.3.39) and the fourth from relation (IV.3.38).

Now, let $1 \le i, j \le \ell$, and put $\Delta = [\mathcal{H}_i, \mathcal{X}_j] - c_{j,i}\mathcal{X}_j$. It is obvious that $\Delta(1) = 0$. But, relation (IV.3.40) tells us that Δ commutes with any $\mathcal{Y}_k$, $1 \le k \le \ell$. Hence, by an immediate induction on t, we get that $\Delta(v_{i_1} \dots v_{i_t}) = 0$ for all $t \in \mathbb{N}^*$ and all $1 \le i_1, \dots, i_t \le \ell$. Thus, we have shown that

$$\text{for all} \quad 1 \le i, j \le \ell, \quad \mathcal{H}_i\mathcal{X}_j - \mathcal{X}_j\mathcal{H}_i = c_{ji}\mathcal{X}_j. \tag{IV.3.41}$$

The proof is complete. ■

It follows from Lemma IV.3.4 that the representation of L° on $T(V)$ gives rise to a representation of L on $T(V)$ by means of the Lie algebra homomorphism ϕ such that the following diagram commutes:

$$\begin{tikzcd} L^\circ \arrow[r, "\phi^\circ"] \arrow[d] & \mathfrak{gl}(T(v)) \\ L \arrow[ur, "\phi"] & \end{tikzcd}$$

Theorem IV.3.5 *Keep the above notation. Let H be the $\mathbb{k}$-subspace of L generated by the elements H_i, $1 \le i \le \ell$. Let X be the Lie subalgebra of L generated by the elements X_i, $1 \le i \le \ell$, and Y be the Lie subalgebra of L generated by the elements Y_i, $1 \le i \le \ell$. Then,*
1. the family $\{Y_i, H_i, X_i,\ 1 \le i \le \ell\}$ is linearly independant;
2. the family $\{H_i,\ 1 \le i \le \ell\}$ is a basis of H;
3. $L = Y \oplus H \oplus X$.

Proof. 1. Clearly, it is enough to show that the family of endomorphisms $\{\mathcal{Y}_i, \mathcal{H}_i, \mathcal{X}_i,\ 1 \le i \le \ell\}$ is linearly independant. In addition, by the first point in Remark IV.3.3, it is enough to show that the three families $\{\mathcal{H}_i,\ 1 \le i \le \ell\}$, $\{\mathcal{Y}_i,\ 1 \le i \le \ell\}$ and $\{\mathcal{X}_i,\ 1 \le i \le \ell\}$ are linearly independant. This is what we do know.

Suppose $a_1, \dots, a_\ell \in \mathbb{k}$ satisfy $\sum_{1 \le i \le \ell} a_i\mathcal{H}_i = 0$ in $\mathfrak{gl}(T(V))$. Applying the last equality to v_j, $1 \le j \le \ell$, it follows that

$$\forall\, 1 \le j \le \ell, \qquad \sum_{1 \le i \le \ell} a_i c_{j,i} = 0.$$

But, by Remark III.10.4, the Cartan matrix is invertible. So, we must have $a_i = 0$, $1 \le i \le \ell$. Hence, the set $\{H_i,\ 1 \le i \le \ell\}$ is linearly independant.

Next, suppose $a_1, \dots, a_\ell \in \mathbb{k}$ satisfy $\sum_{1 \le i \le \ell} a_i\mathcal{Y}_i = 0$ in $\mathfrak{gl}(T(V))$. Now applying this identity to 1 provides $\sum_{1 \le i \le \ell} a_i v_i = 0$. So, we must have $a_i = 0$, $1 \le i \le \ell$. Hence, the set $\{Y_i,\ 1 \le i \le \ell\}$ is linearly independant.

Now, suppose $a_1, \dots, a_\ell \in \mathbb{k}$ satisfy $\sum_{1 \le i \le \ell} a_i\mathcal{X}_i = 0$ in $\mathfrak{gl}(T(V))$. Applying this identity to $v_j v_j$ for any $1 \le j \le \ell$ provides $a_j = 0$ by the second point of Remark IV.3.3. Hence, the set $\{X_i,\ 1 \le i \le \ell\}$ is linearly independant.
2. This is a particular case of 1.
3. For the rest of this proof, we consider the restriction $\mathrm{ad} : H \longrightarrow \mathfrak{gl}(L)$ of the adjoint action of L to the abelian Lie subalgebra H of L. Of course, the adjoint action of H on L stabilises H. Now, let $(i_1, \dots, i_t)$, $t \in \mathbb{N}^*$ be a finite sequence of elements of $\{1, \dots, \ell\}$. Recall the multibracket $[X_{i_1}, \dots, X_{i_t}]$ associated to this sequence and recall that the set of all such multibrackets generates X. (Necessary details on multibrackets that we will use come from Exercise I.2.9.) Then, the following relations hold

$$\forall\, 1 \le j \le \ell, \qquad [H_j, [X_{i_1}, \dots, X_{i_t}]] = (c_{i_1,j} + \dots + c_{i_t,j})[X_{i_1}, \dots, X_{i_t}]. \tag{IV.3.42}$$

Indeed, when $t = 1$, the above relation holds by definition of L. The general result then follows by an obvious induction on t, using the Jacobi identity. Similarly, we get the following relation:

$$\forall\, 1 \leq j \leq \ell, \qquad [H_j, [Y_{i_1}, \ldots, Y_{i_t}]] = -(c_{i_1,j} + \ldots + c_{i_t,j})[Y_{i_1}, \ldots, Y_{i_t}]. \tag{IV.3.43}$$

In particular, (IV.3.42) and (IV.3.43) show that the adjoint action of H stabilises both X and Y.

Let us now introduce useful linear maps on H. First, denote by $(H_1^*, \ldots, H_\ell^*)$ the dual basis of the basis $(H_1, \ldots, H_\ell)$ of H. Then, for $1 \leq i \leq \ell$, consider the linear form

$$\begin{array}{cccc} \nu_i & : & H & \longrightarrow & \Bbbk \\ & & H_j & \mapsto & c_{i,j} \end{array};$$

in other terms,

$$\nu_i = \sum_{1 \leq k \leq \ell} c_{i,k} H_k^*.$$

The matrix that expresses the coordinates of the family $(\nu_1, \ldots, \nu_\ell)$ in terms of the basis $(H_1^*, \ldots, H_\ell^*)$ is invertible, since it is the transpose of the Cartan matrix. It follows that $(\nu_1, \ldots, \nu_\ell)$ is a basis of H^*. Introduce now the following subspace of L attached to $\lambda \in H^*$:

$$L_\lambda = \{x \in L,\, [h, x] = \lambda(h)x,\, \forall\, h \in H\}$$

and recall that the sum of these subspaces is direct:

$$\sum_{\lambda \in H^*} L_\lambda = \bigoplus_{\lambda \in H^*} L_\lambda \subseteq L. \tag{IV.3.44}$$

In the above notation, (IV.3.42) and (IV.3.43) show that

$$[X_{i_1}, \ldots, X_{i_t}] \in L_{\nu_{i_1} + \ldots + \nu_{i_t}} \qquad \text{and} \qquad [Y_{i_1}, \ldots, Y_{i_t}] \in L_{-(\nu_{i_1} + \ldots + \nu_{i_t})}. \tag{IV.3.45}$$

From this it follows that

$$X \subseteq \bigoplus_{\lambda \in (\sum_{1 \leq i \leq \ell} \mathbb{N}\nu_i) \setminus \{0\}} L_\lambda \qquad \text{and} \qquad Y \subseteq \bigoplus_{\lambda \in (\sum_{1 \leq i \leq \ell} (-\mathbb{N})\nu_i) \setminus \{0\}} L_\lambda \tag{IV.3.46}$$

since multibrackets in the X_i (resp. Y_i) generate X (resp. Y) as vector spaces. In addition, clearly, $H \subseteq L_0$. Hence, from (IV.3.44), we get that

$$Y + H + X = Y \oplus H \oplus X.$$

To conclude the proof of Point 3, it is now enough to prove that $Y + H + X$ is actually a Lie subalgebra of L, since this subspace contains a family of generators of L. And, using Point 3 in Exercise I.2.9 this reduces to showing that $Y + H + X$ is stable under $\mathrm{ad}_L(X_i)$, $\mathrm{ad}_L(Y_i)$ and $\mathrm{ad}_L(H_i)$, $1 \leq i \leq \ell$. The case of $\mathrm{ad}_L(H_i)$, $1 \leq i \leq \ell$, follows from (IV.3.42) and (IV.3.43). But, on the other hand, we have that, for $t \in \mathbb{N}$, $t \geq 2$,

$$[Y_j, [X_{i_1}, \ldots, X_{i_t}]] \in X \qquad \text{and} \qquad [X_j, [Y_{i_1}, \ldots, Y_{i_t}]] \in Y.$$

Indeed, this follows from an easy induction on t using the Jacobi identity, the defining relations of L and relations (IV.3.42) and (IV.3.43).

The proof is now complete. ∎

Remark IV.3.6 – We are now in position to strengthen easily the results obtained in Theorem IV.3.5 and its proof. For this, retain the notation of that theorem and its proof.
1. It follows easily from inclusions (IV.3.46) and Point 3 of Theorem IV.3.5 that

$$H = L_0\,, \qquad X = \bigoplus_{\lambda\in(\sum_{1\leq i\leq \ell}\mathbb{N}\nu_i)\setminus\{0\}} L_\lambda \qquad \text{and} \qquad Y = \bigoplus_{\lambda\in(\sum_{1\leq i\leq \ell}(-\mathbb{N})\nu_i)\setminus\{0\}} L_\lambda. \tag{IV.3.47}$$

2. Let now $\lambda \in \left(\sum_{1\leq i\leq \ell}\mathbb{N}\nu_i\right)\setminus\{0\}$ and write $\lambda = \sum_{1\leq i\leq\ell} n_i\nu_i$, $n_i\in\mathbb{N}$, $1\leq i\leq\ell$ (not all of them being 0). Consider a nonzero multibracket $[X_{j_1},\dots,X_{j_t}]$, $t\in\mathbb{N}^*$, $1\leq j_k\leq\ell$, $1\leq k\leq t$. Then, by (IV.3.45) and Point 1, if this multibracket belongs to L_λ, then, the number of occurences of i, $1\leq i\leq\ell$, in the sequence $(j_1,\dots,j_t)$ must equal n_i. It follows at once that there are finitely many such multibrackets. We have proved that,

$$\forall\lambda\in\left(\sum_{1\leq i\leq\ell}\mathbb{N}\nu_i\right)\setminus\{0\}, \quad \dim_{\Bbbk}(L_\lambda)<\infty.$$

Clearly, the same reasoning applies to any $\lambda\in\left(\sum_{1\leq i\leq\ell}(-\mathbb{N})\nu_i\right)\setminus\{0\}$.

Remark IV.3.7 – Keep the above notation. Fix an integer i, $1\leq i\leq\ell$. Recall the canonical basis (x,y,h) of $\mathfrak{sl}_2(\Bbbk)$. We can define a linear map from $\mathfrak{sl}_2(\Bbbk)$ to L as follows:

$$\begin{array}{rcl} \mathfrak{sl}_2(\Bbbk) & \longrightarrow & L \\ x & \mapsto & X_i \\ y & \mapsto & Y_i \\ h & \mapsto & H_i \end{array}.$$

Since, by definition of L, we have $[X_i,Y_i]=H_i$, $[H_i,X_i]=2X_i$ and $[H_i,Y_i]=-2Y_i$, it follows from Exercise I.1.22 that the above map is actually a morphism of Lie algebras. Moreover, it is an isomorphism of vector spaces by Point 1 of Theorem IV.3.5. Hence, the Lie subalgebra of L generated by X_i, Y_i and H_i (which is just $\Bbbk Y_i\oplus\Bbbk H_i\oplus\Bbbk X_i$) is isomorphic to $\mathfrak{sl}_2(\Bbbk)$.

We now come to the last step of the construction. It consists in factoring out L by the ideal generated by Serre's relation.

Denote by I (resp J) the Lie ideal of the Lie algebra X (resp. Y) generated by the elements $X_{i,j}$, $1\leq i\neq j\leq\ell$, (resp. $Y_{i,j}$, $1\leq i\neq j\leq\ell$), where

$$\forall\, 1\leq i\neq j\leq\ell, \qquad X_{i,j}=(\mathrm{ad}_L(X_i))^{-c_{j,i}+1}(X_j) \qquad \text{and} \qquad Y_{i,j}=(\mathrm{ad}_L(Y_i))^{-c_{j,i}+1}(Y_j).$$

In addition, denote by K the Lie ideal of L generated by $X_{i,j}$ and $Y_{i,j}$, $1\leq i\neq j\leq\ell$.

We consider the Lie algebra $\mathfrak{g}=L/K$, the canonical projection $\pi\,:\,L\longrightarrow\mathfrak{g}$ and put $\mathfrak{h}=\pi(H)$, $\mathfrak{n}^-=\pi(Y)$ and $\mathfrak{n}^+=\pi(X)$. Therefore, since $L=Y+H+X$, we have that $\mathfrak{g}=\mathfrak{n}^-+\mathfrak{h}+\mathfrak{n}^+$. In addition, for $1\leq i\leq\ell$, we put $x_i=\pi(X_i)$, $y_i=\pi(Y_i)$ and $h_i=\pi(h_i)$.

Lemma IV.3.8 – *For all $1\leq k\leq\ell$ and all $1\leq i\neq j\leq\ell$, we have the identity $\mathrm{ad}_L(X_k)(Y_{i,j})=0$ in L.*

Proof. Suppose first that $k \neq i$. In that case, $[X_k, Y_i] = 0$, so that $\mathrm{ad}_L(X_k)$ and $\mathrm{ad}_L(Y_i)$ commute. Hence

$$\mathrm{ad}_L(X_k)(Y_{i,j}) = \mathrm{ad}_L(X_k) \circ \mathrm{ad}_L(Y_i)^{-c_{j,i}+1}(Y_j) = \mathrm{ad}_L(Y_i)^{-c_{j,i}+1} \circ \mathrm{ad}_L(X_k)(Y_j).$$

Now, if $k \neq j$, $[X_k, Y_j] = 0$ and the above displayed element is 0. And, if $k = j$, $[X_k, Y_j] = H_j$. Thus

$$\mathrm{ad}_L(X_k)(Y_{i,j}) = \mathrm{ad}_L(X_k) \circ \mathrm{ad}_L(Y_i)^{-c_{j,i}+1}(Y_j) = \mathrm{ad}_L(Y_i)^{-c_{j,i}+1} \circ \mathrm{ad}_L(X_k)(Y_j) = \mathrm{ad}_L(Y_i)^{-c_{j,i}+1}(H_j).$$

Now, $c_{i,j} \leq 0$ since $i \neq j$. So, either $c_{i,j} = 0$ (and thus $c_{j,i} = 0$) and the above element is zero since $[Y_i, H_j] = c_{i,j} Y_i$. Or, $-c_{i,j} + 1 \geq 2$ and the above element is zero, again because of the identity $[Y_i, H_j] = c_{i,j} Y_i$.

Suppose now that $k = i$. Recall Remark IV.3.7; it asserts that the subalgebra of L generated by X_i, Y_i, H_i is isomorphic to $\mathfrak{sl}_2(\mathbb{k})$. Now, restricting ad_L to this subalgebra, we get an action of $\mathfrak{sl}_2(\mathbb{k})$ on L to which we may apply Lemma II.4.4. Now, $[H_i, Y_j] = -c_{j,i} Y_j$ and $[X_i, Y_j] = 0$. So, we may apply Lemma II.4.4 (with $v = Y_j$ and $\lambda = -c_{j,i}$) which shows that

$$\mathrm{ad}_L(X_i) \circ \mathrm{ad}_L(Y_i)^{-c_{j,i}+1}(Y_j) = 0,$$

as desired. ∎

Lemma IV.3.9 – *In the above notation, the following holds:*
1. I and J are ideals of L;
2. $K = I + J$;
3. the restriction, $\pi_{|H} : H \longrightarrow \mathfrak{h}$, is an isomorphism of Lie algebras.

Proof. 1. We deal with the case of J; the case of I is similar. We will make use of the results of Exercise I.2.9. By that exercise, it is enough to show that, for all $1 \leq k \leq \ell$,

$$[X_k, J] \subseteq J, \quad [H_k, J] \subseteq J \quad \text{and} \qquad [Y_k, J] \subseteq J.$$

Of course, the last inclusion is clear since, by definition, J is an ideal of Y.

Consider now the elements of the form

$$[a_1, \dots, a_t], \quad t \in \mathbb{N}^*, \ a_1, \dots, a_t \in Y, a_t = Y_{i,j}, \text{ for some } 1 \leq i \neq j \leq \ell. \tag{IV.3.48}$$

By Exercise I.2.9, we know that J is the span of these elements. So, in order to show that J is left stable by any $\mathrm{ad}_L(H_k)$, $1 \leq k \leq \ell$, it is enough to show that, for all $1 \leq k \leq \ell$, $\mathrm{ad}_L(H_k)$ sends an element as in (IV.3.48) into J. We prove this by induction on t. First observe that, for all $1 \leq i \neq j \leq \ell$, $Y_{i,j}$ is a multibracket of the type investigated in relation (IV.3.43). Hence, this relation shows it is an eigenvector for $\mathrm{ad}_L(H_k)$, $1 \leq k \leq \ell$. This gives the result when $t = 1$. Now, consider an element as in (IV.3.48) for $t > 1$. Then, by the Jacobi identity, we have

$$[H_k, [a_1, \dots, a_t]] = [H_k, [a_1, [a_2, \dots, a_t]]] = -[[a_2, \dots, a_t], [H_k, a_1]] - [a_1, [[a_2, \dots, a_t], H_k].$$

Now, by the induction hypothesis, together with the fact that $\mathrm{ad}_L(H_k)$ stabilises Y, the left hand side of the previous identity belongs to J. This finishes the induction, proving the desired result.

Finally, to show that J is left stable by any $\mathrm{ad}_L(X_k)$, $1 \leq k \leq \ell$, we proceed in the same manner. The case $t = 1$ of the induction is provided by Lemma IV.3.8. Further, as above, for $t > 1$, we have

$$[X_k, [a_1, \dots, a_t]] = [X_k, [a_1, [a_2, \dots, a_t]]] = -[[a_2, \dots, a_t], [X_k, a_1]] - [a_1, [[a_2, \dots, a_t], X_k].$$

By the induction hypothesis, the second term in the right hand side of the above equation is in J. As to the first term, it must also be in J since $[X_k, Y] \subseteq Y + H$ (by the proof of Point 3 of Theorem IV.3.5). At this stage, Point 1 is proved.
2. Now, clearly, $I + J \subseteq K$. But, by Point 1, $I + J$ is an ideal of L which contains all the generators of K. Hence, $I + J = K$.
3. This follows at once from Point 2; indeed: $L = X \oplus H \oplus Y$ and $K = I + J \subseteq X \oplus Y$. ■

In view of the third point of Lemma IV.3.9, the restriction of π to H allows to identify the Lie subalgebra H of L with the Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. As a consequence, the linear forms ν_i, $1 \leq i \leq \ell$, introduced above induce similar linear forms on $\mathfrak{h}$. To avoid accumulating notation, we still denote them ν_i. So, we have now a basis of $\mathfrak{h}^*$ consisting of the following linear forms, $1 \leq i \leq \ell$:

$$\begin{array}{cccc} \nu_i & : & \mathfrak{h} & \longrightarrow & \Bbbk \\ & & h_j & \mapsto & c_{i,j} \end{array} . \qquad \text{(IV.3.49)}$$

For $\lambda \in \mathfrak{h}^*$, put

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \,|\, [h, x] = \lambda(h)x\}.$$

Lemma IV.3.10 – *In the above notation, the following holds:*
1. $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, $\mathfrak{h} = \mathfrak{g}_0$, $\mathfrak{n}^+ = \bigoplus_{\lambda \in (\sum_{1 \leq i \leq \ell} \mathbb{N}\nu_i) \setminus \{0\}} \mathfrak{g}_\lambda$, $\mathfrak{n}^- = \bigoplus_{\lambda \in (\sum_{1 \leq i \leq \ell} (-\mathbb{N})\nu_i) \setminus \{0\}} \mathfrak{g}_\lambda$*;*
2. the family $\{x_i, y_i, h_i,\, 1 \leq i \leq \ell\}$ *is linearly independent;*
3. for all $1 \leq i \leq \ell$*,* $\mathrm{ad}_\mathfrak{g}(x_i)$ *and* $\mathrm{ad}_\mathfrak{g}(y_i)$ *are locally nilpotent derivations of* $\mathfrak{g}$*.*

Proof. 1. Recall first that the sum of the subspaces $\mathfrak{g}_\lambda$, $\lambda \in \mathfrak{g}^*$ is direct (cf. Exercise I.2.10). Notice in addition that, for all $\lambda \in H^* \cong \mathfrak{h}^*$, $\pi(L_\lambda) \subseteq \mathfrak{g}_\lambda$. Therefore, the first point follows from (IV.3.46). (And actually, for all $\lambda \in H^* \cong \mathfrak{h}^*$, $\pi(L_\lambda) = \mathfrak{g}_\lambda$.)
2. Using the same reasoning as in the second point of Remark IV.3.6, we get that $L_{\nu_i} = \mathrm{Span}(X_i)$ and $L_{-\nu_i} = \mathrm{Span}(Y_i)$, $1 \leq i \leq \ell$. By the argument used in the proof of the first point, it follows that $\mathfrak{g}_{\nu_i} = \mathrm{Span}(x_i)$ and $\mathfrak{g}_{-\nu_i} = \mathrm{Span}(y_i)$. Point 2 then follows by Point 1, the third statement in Lemma IV.3.9 and the second point of Theorem IV.3.5.
3. Fix $1 \leq i \leq \ell$ and let V denote the subspace of $\mathfrak{g}$ of those elements which are sent to zero by a sufficiently high power of $\mathrm{ad}_\mathfrak{g}(x_i)$:

$$V = \{x \in \mathfrak{g},\, \mathrm{ad}_\mathfrak{g}^s(x_i)(x) = 0,\ s \gg 0\}.$$

By Exercise I.3.3 applied to $\mathrm{ad}_\mathfrak{g}(x_i)$, V is a Lie subalgebra of $\mathfrak{g}$. But, on the other hand, the relations holding in $\mathfrak{g}$ by construction show that $x_i, y_i \in V$, for all $1 \leq i \leq \ell$. Hence, $V = \mathfrak{g}$. That is, $\mathrm{ad}_\mathfrak{g}(x_i)$ is locally nilpotent. Clearly, the same holds for $\mathrm{ad}_\mathfrak{g}(y_i)$, $1 \leq i \leq \ell$. ■

In view of Exercise I.3.4 and the third point of Lemma IV.3.10, for all $1 \leq i \leq \ell$, we may introduce the following automorphism of the Lie algebra $\mathfrak{g}$:

$$\Theta_i = \exp(\mathrm{ad}_\mathfrak{g}(x_i)) \circ \exp(\mathrm{ad}_\mathfrak{g}(-y_i)) \circ \exp(\mathrm{ad}_\mathfrak{g}(x_i)).$$

Lemma IV.3.11 – *In the above notation, the following holds.*
1. For all $1 \leq i, j \leq \ell$*,* $\Theta_i(h_j) = h_j - c_{i,j}h_i$*. In particular, for* $1 \leq i \leq \ell$*,* Θ_i *induces an involutive Lie automorphism of* $\mathfrak{h}$*, which we denote* θ_i*.*
2. For all $1 \leq i, j \leq \ell$*,* $\nu_i \circ \theta_j = \nu_i - c_{i,j}\nu_j \in \mathfrak{h}^*$*.*

Proof. Using Weyl's relations in $\mathfrak{g}$, we get that $\mathrm{ad}_{\mathfrak{g}}(x_i)(h_j) = -c_{i,j}x_i$ and thus $\mathrm{ad}_{\mathfrak{g}}(x_i)^2(h_j) = 0$. Thus,

$$\exp(\mathrm{ad}_{\mathfrak{g}}(x_i))(h_j) = h_j - c_{i,j}x_i.$$

Similarly, we get that $\mathrm{ad}(-y_i)(h_j - c_{i,j}x_i) = -c_{i,j}(y_i + h_i)$, then that $\mathrm{ad}(-y_i)^2(h_j - c_{i,j}x_i) = 2c_{i,j}y_i$ and finally $\mathrm{ad}(-y_i)^3(h_j - c_{i,j}x_i) = 0$, so that

$$\exp(\mathrm{ad}_{\mathfrak{g}}(-y_i)) \circ \exp(\mathrm{ad}_{\mathfrak{g}}(x_i))(h_j) = h_j - c_{i,j}h_i - c_{i,j}x_i.$$

Further, $\mathrm{ad}_{\mathfrak{g}}(x_i)(h_j - c_{i,j}h_i - c_{i,j}x_i) = c_{i,j}x_i$ and thus $\mathrm{ad}_{\mathfrak{g}}(x_i)^2(h_j - c_{i,j}h_i - c_{i,j}x_i) = 0$. So that

$$\exp(\mathrm{ad}_{\mathfrak{g}}(x_i)) \circ \exp(\mathrm{ad}_{\mathfrak{g}}(-y_i)) \circ \exp(\mathrm{ad}_{\mathfrak{g}}(x_i))(h_j) = h_j - c_{i,j}h_i.$$

This establishes Point 1. Point 2 follows at once. ∎

In order to study the spaces $\mathfrak{g}_\lambda$ occuring in the decomposition of $\mathfrak{g}$ given by Lemma IV.3.10, we need to transfer geometric properties of the root system Φ of the euclidean space E. Actually, we will also have to consider weights associated to (E, Φ) as in Section III.9. For this, we introduce the following notation.

Recall that $(\nu_1, \dots, \nu_\ell)$ is a basis of the $\mathbb{k}$-vector space $\mathfrak{h}^*$. Put

$$\mathfrak{h}^*_{\mathbb{Q}} = \mathrm{Span}_{\mathbb{Q}}\{\nu_1, \dots, \nu_\ell\} \qquad \text{and} \qquad \mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{h}^*_{\mathbb{Q}}.$$

We have an obvious $\mathbb{R}$-linear isomorphism

$$\begin{array}{cccc} \iota & : & \mathsf{E} & \longrightarrow & \mathfrak{h}^*_{\mathbb{R}} \\ & & \alpha_i & \mapsto & \nu_i \end{array}. \tag{IV.3.50}$$

This isomorphism allows to transfer the euclidean structure from E to $\mathfrak{h}^*_{\mathbb{R}}$. On both side, we denote by $(-,-)$ the corresponding scalar product. Recall from Section III.9 the set of weights, that we denote by Λ, associated to Φ. At this stage, it is worth pointing out that all the linear forms on $\mathfrak{h}$ occuring in the decomposition given by Lemma IV.3.10 belong to $\iota(\Lambda)$ (actually to the image under ι of the root lattice associated to Φ).

Now, let $\mathcal{W}$ be the Weyl group of the root system (E, Φ). Conjugating with ι, we get an action of $\mathcal{W}$ on $\mathfrak{h}^*_{\mathbb{R}}$. This action of $\mathcal{W}$ on $\mathfrak{h}^*_{\mathbb{R}}$ is thus given by the following formula:

$$\forall\, w \in \mathcal{W},\ \forall \lambda \in \mathfrak{h}^*_{\mathbb{R}}, \quad w.\lambda = \iota \circ w \circ \iota^{-1}(\lambda).$$

In particular, using Lemma IV.3.11, we have the following identity:

$$\forall\, 1 \le i, j \le \ell, \quad s_{\alpha_i}.\nu_j = \iota \circ s_{\alpha_i} \circ \iota^{-1}(\nu_j) = \iota \circ s_{\alpha_i}(\alpha_j) = \iota(\alpha_j - c_{j,i}\alpha_i) = \nu_j - c_{j,i}\nu_i = \nu_j \circ \theta_i,$$

from which we get that,

$$\forall\, 1 \le i \le \ell,\ \forall \lambda \in \bigoplus_{1 \le i \le \ell} \mathbb{Z}\nu_i, \quad s_{\alpha_i}.\lambda = \lambda \circ \theta_i \tag{IV.3.51}$$

We are now in position to study the spaces occuring in the decomposition of $\mathfrak{g}$ given by Lemma IV.3.10.

Lemma IV.3.12 – *In the above notation, the following holds.*
1. For all $1 \leq i \leq \ell$ *and for all* $\lambda \in \bigoplus_{1\leq i\leq \ell} \mathbb{Z}\nu_i$, $\Theta_i(\mathfrak{g}_\lambda) = \mathfrak{g}_{\lambda\circ\theta_i} = \mathfrak{g}_{s_{\alpha_i}.\lambda}$.
2. Let $\lambda, \mu \in \bigoplus_{1\leq i\leq \ell} \mathbb{Z}\nu_i$ *with* $\mu \in \mathcal{W}.\lambda$, *then* $\dim_{\Bbbk}(\mathfrak{g}_\lambda) = \dim_{\Bbbk}(\mathfrak{g}_\mu)$.
3. For all $\lambda \in \iota(\Phi)$, *then* $\dim_{\Bbbk}(\mathfrak{g}_\lambda) = 1$ *and, for all* $k \in \mathbb{Z} \setminus \{-1,0,1\}$, $\dim_{\Bbbk}(\mathfrak{g}_{k\lambda}) = 0$.
4. For all $\lambda \in \left(\sum_{1\leq i\leq \ell} \mathbb{N}\nu_i\right) \cup \left(\sum_{1\leq i\leq \ell}(-\mathbb{N})\nu_i\right)$, *then* $\mathfrak{g}_\lambda \neq (0)$ *implies that* $\lambda \in \iota(\Phi) \cup \{0\}$.
5. We have that

$$\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\lambda\in\iota(\Phi)} \mathfrak{g}_\lambda \right).$$

and $\dim_{\Bbbk}(\mathfrak{g}) = \ell + |\Phi|$.

Proof. 1. Consider $x \in \mathfrak{g}_\lambda$. For all $h \in \mathfrak{h}$, we have

$$[h, \Theta_i(x)] = \Theta_i([\Theta_i^{-1}(h), x]) = \Theta_i([\theta_i^{-1}(h), x]) = \lambda \circ \theta_i^{-1}(h)\Theta_i(x) = \lambda \circ \theta_i(h)\Theta_i(x).$$

(The first equality follows from the fact that Θ_i is a morphism of Lie algebras, the third holds because $x \in \mathfrak{g}_\lambda$ and the fourth becaus θ_i is involutive.) Thus, we have that

$$\Theta_i(\mathfrak{g}_\lambda) \subseteq \mathfrak{g}_{\lambda\circ\theta_i}.$$

Now, let $y \in \mathfrak{g}_{\lambda\circ\theta_i}$ and put $x = \Theta_i^{-1}(y)$. Then, for all $h \in \mathfrak{h}$,

$$[h, x] = [h, \Theta_i^{-1}(y)] = \Theta_i^{-1}([\theta_i(h), y]) = \Theta_i^{-1}([\theta_i(h), y]) = \lambda(h)\Theta_i^{-1}(y) = \lambda(h)x.$$

Hence,

$$\Theta_i(\mathfrak{g}_\lambda) \supseteq \mathfrak{g}_{\lambda\circ\theta_i}.$$

This establishes Point 1.
2. The Weyl group is generated by the reflections (cf. Theorem III.6.4) s_{α_i}, $1 \leq i \leq \ell$. So the result follows from Point 1 since the Θ_i, $1 \leq i \leq \ell$, are automorphisms.
3. We have already proved that $\dim_{\Bbbk}(\mathfrak{g}_{\nu_i}) = 1$ (see the proof of Lemma IV.3.10). A similar argument actually shows that, for all $k \in \mathbb{Z} \setminus \{-1,0,1\}$, $\dim_{\Bbbk}(\mathfrak{g}_{k\nu_i}) = 0$. The result then follows by Point 2, since any element of $\iota(\Phi)$ is in the $\mathcal{W}$-orbit of some ν_i, $1 \leq i \leq \ell$ (cf. Point 3 of Theorem III.6.4).
4. Let $\lambda \in \left(\sum_{1\leq i\leq \ell} \mathbb{N}\nu_i\right) \cup \left(\sum_{1\leq i\leq \ell}(-\mathbb{N})\nu_i\right)$ and suppose $\mathfrak{g}_\lambda \neq (0)$. We are in position to apply Exercise III.6.11 to $\iota^{-1}(\lambda)$. The decomposition of $\mathfrak{g}$ given by Lemma IV.3.10 and Point 2 above lead to $\lambda = \xi\iota(\beta)$ for some $\xi \in \mathbb{R}$ and some $\beta \in \Phi$. Now, β is conjugate under $\mathcal{W}$ to some root in Δ: there existe $w \in \mathcal{W}$ and $1 \leq i \leq \ell$ such that $w(\beta) = \alpha_i$. Now, we get that $w.\lambda = \xi\nu_i$. On the other hand, by Point 2 above, we have that $\dim_{\Bbbk}(\mathfrak{g}_{\xi\nu_i}) = \dim_{\Bbbk}(\mathfrak{g}_\lambda) \neq 0$. It follows at once that $\xi \in \mathbb{Z}$ and by Point 3 above that $\xi \in \{-1,0,1\}$. All in all, we have shown that $\lambda \in \iota(\Phi) \cup \{0\}$.
5. Point 4 and Lemma IV.3.10 give that

$$\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\lambda\in\iota(\Phi)} \mathfrak{g}_\lambda \right).$$

Further, by Point 3, each summand in the sum $\bigoplus_{\lambda\in\iota(\Phi)} \mathfrak{g}_\lambda$ is one dimensional. ∎

Remark IV.3.13 – Keep the above notation.
1. Fix an integer i, $1 \leq i \leq \ell$. We already mentioned that $\mathfrak{g}_{\nu_i} = \Bbbk x_i$ and $\mathfrak{g}_{-\nu_i} = \Bbbk y_i$ (cf. the proof

of Lemma IV.3.10). In addition, arguing as in Remark IV.3.7 by means of Point 2 in Lemma IV.3.10, we have an isomorphism of Lie algebras

$$\begin{array}{ccl} \mathfrak{sl}_2(\Bbbk) & \longrightarrow & \mathrm{Span}\{y_i, h_i, x_i\} \subseteq \mathfrak{g} \\ x & \mapsto & x_i \\ y & \mapsto & y_i \\ h & \mapsto & h_i \end{array}.$$

2. Let now λ be any element of $\iota(\Phi)$. We know that there exists $w \in \mathcal{W}$ and $1 \leq i \leq \ell$ such that $w(\iota^{-1}(\lambda)) = \alpha_i$ and that w is a product of reflections associated to simple roots (cf. Theorem III.6.4). By Point 1 of Lemma IV.3.12, it follows that there exists and automorphism Θ of $\mathfrak{g}$ such that $\Theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{\nu_i}$ and $\Theta(\mathfrak{g}_{-\lambda}) = \mathfrak{g}_{-\nu_i}$. If in addition we put $\Theta^{-1}(h_i) = h_\lambda$, we get that

$$\Theta^{-1}(\mathrm{Span}\{y_i, h_i, x_i\}) = \mathfrak{g}_{-\lambda} \oplus \Bbbk h_\lambda \oplus \mathfrak{g}_\lambda.$$

Hence, $\mathfrak{g}_{-\lambda} \oplus \Bbbk h_\lambda \oplus \mathfrak{g}_\lambda$ is a Lie subalgebra of $\mathfrak{g}$ which, by Point 1, is isomorphic to $\mathfrak{sl}_2(\Bbbk)$.

Theorem IV.3.14 – *In the above notation, $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{h}$ a maximal toral subalgebra of $\mathfrak{g}$ and $\iota(\Phi)$ is the set of roots associated to $(\mathfrak{g}, \mathfrak{h})$.*

Proof. We first show the semisimplicity of $\mathfrak{g}$. By Lemma I.7.9, it suffices to prove that the only abelian ideal of $\mathfrak{g}$ is (0). Let $\mathfrak{i}$ be an abelian ideal of $\mathfrak{g}$. By Lemma IV.3.12, we have that

$$\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\lambda \in \iota(\Phi)} \mathfrak{g}_\lambda \right). \tag{IV.3.52}$$

Of course,

$$\mathfrak{i} \supseteq (\mathfrak{h} \cap \mathfrak{i}) \bigoplus \left(\bigoplus_{\lambda \in \iota(\Phi)} (\mathfrak{g}_\lambda \cap \mathfrak{i}) \right).$$

We first show that this inclusion is an equality. Denote by E the set of elements of $\mathfrak{i}$ whose components in the decomposition (IV.3.52) are not all in $\mathfrak{i}$. Suppose E is not empty and choose in E an element x with a minimal number of nonzero components in the decomposition (IV.3.52). Write $x = \sum_{\mu \in \iota(\Phi) \cup \{0\}} x_\mu$, where $x_\mu \in \mathfrak{g}_\mu$, for all $\mu \in \iota(\Phi) \cup \{0\}$. Clearly, $x \neq 0$. Therefore, there exists $\lambda \in \iota(\Phi) \cup \{0\}$ such that $x_\lambda \neq 0$. Then, we have that, for all $h \in \mathfrak{h}$, the element

$$[h, x] - \lambda(h)x = \sum_{\mu \in \iota(\Phi) \cup \{0\}} (\mu(h) - \lambda(h)) x_\mu$$

is in $\mathfrak{i}$ and, by the minimality hypothesis on x, it follows that all its components in the decomposition (IV.3.52) lie in $\mathfrak{i}$. That is,

$$\forall\ h \in \mathfrak{h},\ \forall \mu \in \iota(\Phi) \cup \{0\},\ (\mu(h) - \lambda(h)) x_\mu \in \mathfrak{i}.$$

But, for all $\mu \in \iota(\Phi) \cup \{0\}$, if $\mu \neq \lambda$, there exists $h \in \mathfrak{h}$ such that $\mu(h) \neq \lambda(h)$. Hence, we get that

$$\forall \mu \in \iota(\Phi) \cup \{0\},\ \mu \neq \lambda,\ x_\mu \in \mathfrak{i}.$$

From which it follows that $x_\lambda = x - \sum_{\mu \in \iota(\Phi) \cup \{0\},\, \mu \neq \lambda} x_\mu \in \mathfrak{i}$. This shows that all the components of x in the decomposition (IV.3.52) lie in $\mathfrak{i}$; a contradiction. Therefore, E is empty and we have proved that

$$\mathfrak{i} = (\mathfrak{h} \cap \mathfrak{i}) \bigoplus \left(\bigoplus_{\lambda \in \iota(\Phi)} (\mathfrak{g}_\lambda \cap \mathfrak{i}) \right).$$

Now, let $\lambda \in \iota(\Phi)$. By Remark IV.3.13 (and using its notation), $\mathfrak{g}_{-\lambda} \oplus \Bbbk h_\lambda \oplus \mathfrak{g}_\lambda$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\Bbbk)$. Since $\mathfrak{i}$ is abelian, if follows that its intersection with $\mathfrak{i}$ must be trivial. In particular, $\mathfrak{i} \cap \mathfrak{g}_\lambda = (0)$. Therefore, we have shown that

$$\mathfrak{i} \subseteq \mathfrak{h}.$$

Now, let $h \in \mathfrak{i}$. For all $1 \leq i \leq \ell$, we have $[h, x_i] = \nu_i(h)x_i \in \mathfrak{i}$ and hence $\nu_i(h) = 0$. Since the elements ν_i, $1 \leq i \leq \ell$, form a basis of $\mathfrak{h}^*$, it follows that $h = 0$. Therefore $\mathfrak{i} = 0$. We have proved the semisimplicity of $\mathfrak{g}$.

We now prove that $\mathfrak{h}$ is a maximal toral subalgebra of $\mathfrak{g}$. It is clear from the decomposition (IV.3.52) that $\mathfrak{h}$ is toral. The same decomposition shows that $\mathfrak{h}$ is its own normalizer: $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Suppose now that $\mathfrak{h} \subseteq \mathfrak{h}'$, where $\mathfrak{h}'$ is a toral subalgebra of $\mathfrak{g}$. Then, being toral, $\mathfrak{h}'$ must be abelian, so that it is included in the normaliser of $\mathfrak{h}$ that is, by the above, in $\mathfrak{h}$. We have shown that $\mathfrak{h}$ is toral maximal.

The fact that $\iota(\Phi)$ is the set of roots associated to the pair $(\mathfrak{g}, \mathfrak{h})$ is also clear from decomposition (IV.3.52). ■

Notation IV.3.15 – Let (E, Φ) be a root system and Δ a base of Φ. It will be convenient to have a specific notation for the Lie algebras constructed above. So, from now on, the algebras $\mathfrak{g}$ and $\mathfrak{h}$ will be denoted $\mathfrak{g}_{\Phi,\Delta}$ and $\mathfrak{h}_{\Phi,\Delta}$, respectively.

Remark IV.3.16 – **Independence of $\mathfrak{g}_{\Phi,\Delta}$ with respect to Δ** – Let (E, Φ) be a root system. Suppose that Δ and Δ' are bases of Φ. Then, we may consider the two pairs $(\mathfrak{g}_{\Phi,\Delta}, \mathfrak{h}_{\Phi,\Delta})$ and $(\mathfrak{g}_{\Phi,\Delta'}, \mathfrak{h}_{\Phi,\Delta'})$, consisting of a semisimple Lie algebra and a maximal toral subalgebra. It follows immediately from Remark III.10.4 and the universal property of free Lie algebras that there exists an isomorphism from $\mathfrak{g}_{\Phi,\Delta}$ to $\mathfrak{g}_{\Phi,\Delta'}$ that sends $\mathfrak{h}_{\Phi,\Delta}$ onto $\mathfrak{h}_{\Phi,\Delta'}$.

Using Notation IV.1.3, it follows from Remark IV.3.16 that there is a well defined map

$$R \quad \longrightarrow \quad A_\Bbbk/\sim$$

that associates to any root system (E, Φ) the isomorphism class of the semisimple Lie algebra $\mathfrak{g}_{\Phi,\Delta}$ obtained from an arbitrary choice of a base Δ of Φ. Further, it readily follows from the definition of isomorphism of root systems that this map factorises through $R/\sim$ to give rise to a map

$$a_\Bbbk \quad : \quad R/\sim \quad \longrightarrow \quad A_\Bbbk/\sim \tag{IV.3.53}$$

Theorem IV.3.17 – *The maps $r : A_\Bbbk/\!\sim \longrightarrow R/\!\sim$, cf. (IV.1.33), and $a_\Bbbk : R/\!\sim \longrightarrow A_\Bbbk/\!\sim$, cf. (IV.3.53) are bijective and inverse to each other.*

Proof. We first show that $a_\Bbbk \circ r = \mathrm{id}$. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Choose an arbitrary maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. To the pair $(\mathfrak{g}, \mathfrak{h})$ associate the root system $(\mathsf{E}_\mathbb{R}, \Phi)$ and choose an arbitrary base Δ of $(\mathsf{E}_\mathbb{R}, \Phi)$. Then, by the universal property of free Lie algebras, Proposition IV.3.1 and Proposition III.4.2, we have a surjective morphism of Lie algebras $\mathfrak{g}_{\Phi,\Delta} \longrightarrow \mathfrak{g}$. On the other hand, we have $\dim_\Bbbk(\mathfrak{g}_{\Phi,\Delta}) = \dim_\Bbbk(\mathfrak{g})$, by Lemma IV.3.12 and the discussion in Sections II.5 and II.6, so that the above morphism is actually an isomorphism. This indeed proves that $a_\Bbbk \circ r = \mathrm{id}$.

We now show that $r \circ a_\Bbbk = \mathrm{id}$. Let (E, Φ) be a root system and Δ an arbitrary base of (E, Φ). To this data, we associate the Lie algebra $\mathfrak{g}_{\Phi,\Delta}$, which is semisimple and finite dimensional, its maximal toral subalgebra $\mathfrak{h}_{\Phi,\Delta}$ and the set of roots $\iota(\Delta)$ associated to the pair $(\mathfrak{g}_{\Phi,\Delta}, \mathfrak{h}_{\Phi,\Delta})$; cf.

Theorem IV.3.14 and (IV.3.50). To simplify notation, put $\mathfrak{g} = \mathfrak{g}_{\Phi,\Delta}$ and $\mathfrak{h} = \mathfrak{h}_{\Phi,\Delta}$. Therefore, we may apply the process described in Section II.6 to the pair $(\mathfrak{g}, \mathfrak{h})$. Let κ be the Killing form of $\mathfrak{g}$. We have the associated euclidean space $\mathsf{E}_{\mathbb{R}}$ as constructed in Section II.6, whose scalar product we denote $(-,-)_{\mathsf{E}_{\mathbb{R}}}$ to avoid ambiguity with the scalar product $(-,-)$ of E. Correspondingly, we will use the notation $\langle -,-\rangle_{\mathsf{E}_{\mathbb{R}}}$. To each linear form $\nu_i \in \mathfrak{h}^*$, $1 \leq i \leq \ell$ (cf. (IV.3.49)) associate $t_{\nu_i} \in \mathfrak{h}$ as in Notation II.5.11. By the second point of Theorem II.5.13 and Remark IV.3.13, we have:

$$\forall\, 1 \leq i \leq \ell, \quad h_i = 2\frac{t_{\nu_i}}{\kappa(t_{\nu_i}, t_{\nu_i})}.$$

So, for all $1 \leq i, j \leq \ell$,

$$\langle \nu_i, \nu_j \rangle_{\mathsf{E}_{\mathbb{R}}} = 2\frac{(\nu_i, \nu_j)_{\mathsf{E}_{\mathbb{R}}}}{(\nu_j, \nu_j)_{\mathsf{E}_{\mathbb{R}}}} = 2\frac{\kappa(t_{\nu_i}, t_{\nu_j})}{\kappa(t_{\nu_j}, t_{\nu_j})} = \kappa(t_{\nu_i}, h_j) = \nu_i(h_j) = c_{i,j} = \langle \alpha_i, \alpha_j \rangle_{\mathsf{E}}.$$

This shows that the isomorphism

$$\iota \;:\; \mathsf{E} \longrightarrow \mathsf{E}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}}$$

of (IV.3.50) is an isomorphism of root systems from (E, Φ) to $(\mathsf{E}_{\mathbb{R}}, \iota(\Phi))$, by Proposition III.10.5. We have proved that $r \circ a_{\mathbb{k}} = \mathrm{id}$. ∎

Part V

Representations.

V.1 Weight spaces.

In this section, we assume that $\Bbbk$ is algebraically closed and of characteristic 0.

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, $\mathfrak{h}$ a maximal toral subalgebra of $\mathfrak{g}$ and Φ the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$. Thus we have the Cartan-Chevalley decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right). \tag{V.1.54}$$

In addition, let V be a representation of $\mathfrak{g}$.

Definition V.1.1 – *For all $\lambda \in \mathfrak{h}^*$, put $V_\lambda = \{v \in V \mid \forall h \in \mathfrak{h},\, h.v = \lambda(h)v\}$. If $V_\lambda \neq \{0\}$, we say that λ is a weight of the representation V and we call V_λ the weight subspace of V of weight λ.*

The following statement is basic.

Proposition V.1.2 – *Keep the above notation.*
1. For $\alpha \in \Phi$ and $\lambda \in \mathfrak{h}^*$, $\mathfrak{g}_\alpha . V_\lambda \subseteq V_{\alpha+\lambda}$.
2. The sum $\sum_{\lambda \in \mathfrak{h}^*} V_\lambda$ is direct and is a subrepresentation of V.
3. If V is finite dimensional, then $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$.

Proof. 1. Let $x \in \mathfrak{g}_\alpha$, $h \in \mathfrak{h}$ and $v \in V_\lambda$. Then, $h.(x.v) = x.(h.v) + [h,x].v = x.(\lambda(h)v) + (\alpha(h)x).v = (\lambda(h) + \alpha(h))x.v$. This proves Point 1.
2. It follows at once from Point 1 and (V.1.54) that $\sum_{\lambda \in \mathfrak{h}^*} V_\lambda$ is a subrepresentation of V.

To show that this sum is direct, we prove the following statement, by induction on $p \in \mathbb{N}^*$: consider $p \in \mathbb{N}^*$, λ_i, $1 \leq i \leq p$, pairwise distinct elements of $\mathfrak{h}^*$ and $v_i \in V_{\lambda_i}$, $1 \leq i \leq p$, if $v_1 + \ldots + v_p = 0$, then $v_1 = \ldots = v_p = 0$. The statement is obvious for $p = 1$. Suppose it holds for some $p \in \mathbb{N}^*$ and consider λ_i, $1 \leq i \leq p+1$, pairwise distinct elements of $\mathfrak{h}^*$ and $v_i \in V_{\lambda_i}$, $1 \leq i \leq p+1$. Suppose $v_1 + \ldots + v_{p+1} = 0$. Then, for all $h \in \mathfrak{h}$,

$$0 = h. \left(\sum_{1 \leq i \leq p+1} v_i \right) - \lambda_{p+1}(h) \left(\sum_{1 \leq i \leq p+1} v_i \right) = \sum_{1 \leq i \leq p} (\lambda_i(h) - \lambda_{p+1}(h)) v_i.$$

Hence, by the induction hypothesis, for all $h \in \mathfrak{h}$, and for all $1 \leq i \leq p$, $(\lambda_i(h) - \lambda_{p+1}(h))v_i = 0$. But, for all $1 \leq i \leq p$, $\lambda_i \neq \lambda_{p+1}$, hence there exists $h \in \mathfrak{h}$ such that $\lambda_i(h) \neq \lambda_{p+1}(h)$. Therefore, we get that, for all $1 \leq i \leq p$, $v_i = 0$, from whitch it follows at once that, in addition, $v_{p+1} = 0$. This finishes the induction.
3. Suppose now that V is finite dimensional. We are in position to apply Theorem II.3.8. It asserts that all the elements of $\mathfrak{h}$ act by diagonalisable endomorphisms. Since in addition $\mathfrak{h}$ is abelian and finite dimensional, then the elements of $\mathfrak{h}$ act by simultaneously diagonalisable endomorphisms: there exists a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of V such that any vector in $\mathcal{B}$ is an eigenvector for the action of all $h \in \mathfrak{h}$. Now, for $1 \leq i \leq n$ and $h \in \mathfrak{h}$, put

$$h.b_i = \lambda_i(h) b_i,$$

$\lambda_i(h) \in \Bbbk$. It is then obvious that, for all $1 \leq i \leq n$, $\lambda_i : \mathfrak{h} \longrightarrow \Bbbk$, $h \mapsto \lambda_i(h)$ is a linear form on $\mathfrak{h}$. Point 3 follows. ∎

Example V.1.3 – If we consider the adjoint action of $\mathfrak{g}$ on itself, we see that the Cartan-Chevalley decomposition is nothing but the decomposition of $\mathfrak{g}$ into weight spaces. In other terms, the weights of $(\mathfrak{g}, \mathrm{ad}_{\mathfrak{g}})$ are the roots of $(\mathfrak{g}, \mathfrak{h})$ together with 0.

Example V.1.4 – Suppose $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$ and $\mathfrak{h} = \mathbb{k}h$ (cf. notation of Section II.4).

Fix $m \in \mathbb{N}$. Recall from Section II.4 the (simple) representation $(\mathbb{k}^{m+1}, \rho_m)$, and recall that $\rho_m(h)$ is diagonalisable with eigenvalues $m - 2i$, $0 \le i \le m$. Clearly, we have an isomorphism of vector spaces

$$\begin{array}{cccc} \iota & : & \mathfrak{h}^* & \longrightarrow & \mathbb{k} \\ & & \lambda & \mapsto & \lambda(h) \end{array}.$$

Then, the weights of $(\mathbb{k}^{m+1}, \rho_m)$ are the elements of the set $\{\iota^{-1}(m - 2i),\ 0 \le i \le m\}$.

V.2 Highest weight representations.

In this section, we assume that $\mathbb{k}$ is algebraically closed and of characteristic 0.

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra $\mathfrak{h}$ a maximal toral subalgebra of $\mathfrak{g}$ and Φ the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$. In addition, let Δ be a base of the root system Φ. Then Φ is the disjoint union of the set Φ^+ of positives roots and the set Φ^- of negative roots: $\Phi = \Phi^- \sqcup \Phi^+$. The Cartan-Chevalley decomposition of $\mathfrak{g}$ then reads

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^- = \oplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$ and $\mathfrak{n} = \mathfrak{n}^+ = \oplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ are nilpotent Lie subalgebras of $\mathfrak{g}$ and where $\mathfrak{b} = \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ is a solvable Lie subalgebra of $\mathfrak{g}$. Finally, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ (cf. Example III.3.18).

Definition V.2.1 – *Let V be a representation of $\mathfrak{g}$.*
1. Let $\lambda \in \mathfrak{h}^*$. An element $v \in V$ is called a highest weight vector of weight λ if it is a nonzero element of V_λ such that $\mathfrak{n}.v = 0$.
2. An element $v \in V$ is called a highest weight vector if it is a highest weight vector of weight λ for some $\lambda \in \mathfrak{h}^*$.

Remark V.2.2 – In Humphreys' book, a highest weight vector (of weight λ) is called a maximal vector (of weight λ).

Remark V.2.3 – **Highest weight vectors and finite dimensional representations** – Let (V, ρ) be a nonzero finite dimensional representation of $\mathfrak{g}$. Since $\mathfrak{b}$ is a solvable Lie subalgebra of $\mathfrak{g}$, $\rho(\mathfrak{b})$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Lie's Theorem then ensures there is a full flag $\mathcal{F}$ of V such that $\rho(\mathfrak{b}) \subseteq \mathfrak{b}_{\mathcal{F}}(V)$. Moreover, $\rho(\mathfrak{n}) = [\rho(\mathfrak{b}), \rho(\mathfrak{b})] \subseteq [\mathfrak{b}_{\mathcal{F}}(V), \mathfrak{b}_{\mathcal{F}}(V)] = \mathfrak{n}_{\mathcal{F}}(V)$. Hence, there exists a nonzero vector $v \in V$ which is a common eigenvector of all the elements of $\mathfrak{b}$ and satisfies $\mathfrak{n}.v = 0$; such a vector is a highest weight vector of V. We have shown that every finite dimensional representation of $\mathfrak{g}$ has a highest weight vector.

The study of finite dimensional representations of $\mathfrak{g}$ relies on that of the larger class of representations generated by a highest weight vector.

Definition V.2.4 – **Highest weight representations.**
1. Let $\lambda \in \mathfrak{h}^*$. A representation of $\mathfrak{g}$ is called a highest weight representation of weight λ if it is generated (as a representation) by a highest weight vector of weight λ.
2. A representation of $\mathfrak{g}$ is called a highest weight representation if it is a highest weight representation of weight λ for some $\lambda \in \mathfrak{h}^*$.

Remark V.2.5 – In Humphreys' book, a highest weight representation (of weight λ) is called a standard cyclic representation (of weight λ).

The structure of highest weight representations is not difficult to describe.

For all $\alpha \in \Phi^+$, fix $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ and consider $h_\alpha \in \mathfrak{h}$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ as in Theorem II.5.13. We may therefore consider the basis of $\mathfrak{g}$ consisting of $\{y_\alpha,\, \alpha \in \Phi^+\}$, $\{x_\alpha,\, \alpha \in \Phi^+\}$ and any basis of $\mathfrak{h}$ and we may totally order it in such a way that each element y_α be less than each element of the chosen basis of $\mathfrak{h}$ and that each element of the chosen basis of $\mathfrak{h}$ be less than each element x_α. Now, using (a right-hand side version of) Corollary IV.2.20 and its proof we get that $U(\mathfrak{g})$ is a free right $U(\mathfrak{b})$-module with basis the set $\mathcal{B}$ consisting of the ordered products of the (images in $U(\mathfrak{g})$ of the elements) y_α.

Finally, write $\mathbb{N}\Delta$ (resp. $\mathbb{N}\Phi^+$) for the set of linear combinations with coefficients in $\mathbb{N}$ of the elements of Δ (resp. Φ^+). Hence, $\mathbb{N}\Delta = \mathbb{N}\Phi^+$.

Theorem V.2.6 – *Let V be a representation of $\mathfrak{g}$ and v a highest weight vector of weight $\lambda \in \mathfrak{h}^*$ of V that generates V as a representation (hence, V is a highest weight representation of weight λ).*
1. The elements of the set $\mathcal{B}.v$ are weight vectors and generate V as a vector space. In particular, V is the direct sum of its weight spaces.
2. The weights of V are all of the form $\lambda - \mu$ with $\mu \in \mathbb{N}\Delta$.
3. All the weight spaces of V are finite dimensional and $\dim_{\Bbbk}(V_\lambda) = 1$.
4. Any subrepresentation of V is the direct sum of its weight spaces.
5. The representation V is indecomposable; it has a unique maximal strict subrepresentation and a unique irreducible quotient.
6. Any nonzero quotient representation of V is a highest weight representation of weight λ.

Proof. By definition of a highest weight vector, it is clear that $U(\mathfrak{b}).v = \Bbbk.v$ and, by the P.B.W. Theorem, it follows that $V = U(\mathfrak{g}).v = U(\mathfrak{n}^-).v$. The elements of $\mathcal{B}.v$ are weight vectors of weight $\lambda - \mu$, $\mu \in \mathbb{N}\Phi^+$, by the first point of Proposition V.1.2 and they generate $U(\mathfrak{n}^-).v$ as a vector space (since the elements of $\mathcal{B}$ generate $U(\mathfrak{n}^-)$ as a $\Bbbk$-vector space). This proves points 1 and 2.

We now show Point 3. Put $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. By Point 2, any weight of V belongs to $\lambda - \mathbb{N}\Delta$. Let $\lambda - \sum_{1 \le i \le \ell} q_i\alpha_i$ be such a weight, $q_i \in \mathbb{N}$, $1 \le i \le \ell$. Any element of $\mathcal{B}$ is of the form $b = \prod_{\alpha \in \Phi^+} y_\alpha^{p_\alpha}$, $p_\alpha \in \mathbb{N}$ for $\alpha \in \Phi^+$ and $b.v$ is then of weight $\lambda - \sum_{\alpha \in \Phi^+} p_\alpha \alpha$. Therefore, the weight space of weight $\lambda - \sum_{1 \le i \le \ell} q_i\alpha_i$ is generated as a vector space by the elements $b.v$ with $b = \prod_{\alpha \in \Phi^+} y_\alpha^{p_\alpha}$, $p_\alpha \in \mathbb{N}$ for $\alpha \in \Phi^+$, and

$$\sum_{\alpha \in \Phi^+} p_\alpha \alpha = \sum_{1 \le i \le \ell} q_i \alpha_i$$

(cf. Proposition V.1.2, Point 2). But, it is not difficult to see that the above equality implies that, for all $\alpha \in \Phi^+$, $p_\alpha \le \max\{q_i,\, 1 \le i \le \ell\}$. This entails that the set of families (p_α) satisfying the above identity is finite and that it reduces to the trivial family in case $q_i = 0$ for all $1 \le i \le \ell$. Point 3 follows.
4. By Point 1, $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$. Let now W be a subrepresentation of V. We must show that W is the direct sum of its weight spaces: $W = \bigoplus_{\mu \in \mathfrak{h}^*} W \cap V_\mu$. This amounts to showing that, for all $w \in W$, the components of w in the decomposition of V as the sum of its weight spaces are all in W. Suppose, to the contrary, that there exist elements in W that do not satisfy this condition

and choose one, denoted w, with a minimal number of nonzero components. Clearly, w is not a weight vector. Therefore, $w = w_1 + \ldots + w_n$, with $1 < n$, where the w_i are weight vectors of weight $\mu_i \in \mathfrak{h}^*$, the μ_i, $1 \leq i \leq n$, being pairwise distinct. Let $h \in \mathfrak{h}$ such that $\mu_1(h) \neq \mu_2(h)$. Then $h.w - \mu_1(h)w = (\mu_2(h) - \mu_1(h))w_2 + \ldots + (\mu_n(h) - \mu_1(h))w_n \in W$. The minimality hypothesis made on w forces, on the one hand, $w_2 \notin W$ and, on the other hand, $(\mu_2(h) - \mu_1(h))w_2 \in W$, which is absurd. Point 4 is proved.
5. It follows from the above that any strict subrepresentation of V is contained in $V' = \bigoplus_{\mu \neq \lambda} V_\mu$. Hence, the sum of all the strict subrepresentations of V is a strict subrepresentation and, clearly, it is maximum, for the inclusion, among strict subrepresentation. Obviously, this forces V to be indecomposable. The rest of Point 5 clearly follows.
6. Any such quotient is the quotient of V by a strict subrepresentation. Therefore, by (the proof of) Point 5, the image of v in it is a nonzero vector which, clearly, is a highest weight vector of weight λ, which generates that quotient. ■

Corollary V.2.7 – *Let V be a highest weight representation of $\mathfrak{g}$. If V is irreductible, then two highest weight vectors of V are linearly dependent; in particular, all the highest weight vectors have the same weight.*

Proof. Suppose V is a highest weight representation of weight λ and let $v \in V_\lambda$ be a highest weight vector of weight λ. Let w be a highest weight vector of weight $\mu \in \mathfrak{h}^*$. By definition, w is nonzero and, by the irreducibility of V, it must generate V. Hence, V is also a highest weight representation of weight μ. But then, by Point 2 of Theorem V.2.6, $\lambda - \mu$ and $\mu - \lambda$ both belong to $\mathbb{N}\Delta$. As Δ is a basis of $\mathfrak{h}^*$, this implies that $\lambda = \mu$. It remains to apply Point 3 of Theorem V.2.6 to conclude. ■

We now come to the problem of the existence and unicity of highest weight representations of a given weight. Unicity is not difficult.

Theorem V.2.8 – *Let $\lambda \in \mathfrak{h}^*$. Two irreducible highest weight representations of weight λ are isomorphic.*

Proof. Let V and W be irreducible highest weight modules of weight λ and $v \in V$, $w \in W$ highest weight vectors of weight λ such that $V = U(\mathfrak{g})v$ and $W = U(\mathfrak{g})w$. We consider the $U(\mathfrak{g})$-module $V \oplus W$. Clearly, the element (v, w) is a highest weight vector of weight λ of $V \oplus W$. Consider the submodule $S = U(\mathfrak{g})(v, w)$ of $V \oplus W$ that (v, w) generates in $V \oplus W$. We consider now the restrictions to S of the natural projections of $V \oplus W$ on V and W:

$$p\,:\,S \overset{can.inj.}{\longrightarrow} V \oplus W \overset{can.proj.}{\longrightarrow} V \quad \text{and} \quad q\,:\,S \overset{can.inj.}{\longrightarrow} V \oplus W \overset{can.proj.}{\longrightarrow} W.$$

Clearly, p and q are morphisms of $U(\mathfrak{g})$-modules and are surjective. Therefore, each of V and W is isomorphic to a quotient of S. But, S is a highest weight module. So, by Theorem V.2.6, it has a unique irreducible quotient. Now, since V and W are irreducible and isomorphic to a quotient of S, both of them must be isomorphic to the unique irreducible quotient of S and, in particular, they must be isomorphic between them. ■

We now consider the problem of the existence of a highest weight representation of a given weight.

Let $\lambda \in \mathfrak{h}^*$. We consider the following representation $(\rho_\lambda, D_\lambda)$ of the solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. By definition, $D_\lambda = \mathbb{k}$ and the structure morphism $\rho_\lambda\,:\,\mathfrak{b} \longrightarrow \mathfrak{gl}(\mathbb{k}) \cong \mathbb{k}$ maps any element

$h+\sum_{\alpha\in\Phi^+} x_\alpha \in \mathfrak{b} = \mathfrak{h}\oplus(\oplus_{\alpha\in\Phi^+}\mathfrak{g}_\alpha)$ to $\lambda(h)$. The Lie bracket of any two elements of $\mathfrak{b}$ is in $\mathfrak{n}$ and, hence, acts by 0. Since $\mathfrak{gl}(\Bbbk)$ is commutative, it follows that ρ_λ is a morphism of Lie algebras. Therefore, D_λ is a linear representation of the $\Bbbk$-algebra $U(\mathfrak{b})$ (see Remark IV.2.11) and we may consider the left $U(\mathfrak{g})$-module

$$Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} D_\lambda$$

and put $v_\lambda = 1 \otimes 1 \in Z(\lambda)$.

Remark V.2.9 – Retain the above notation and fix $\lambda \in \mathfrak{h}^*$.
1. By definition, $Z(\lambda)$ is a left $U(\mathfrak{g})$-module, hence in particular a $\Bbbk$-vector space. More precisely, $U(\mathfrak{g})$ acts on $Z(\lambda)$ by $\Bbbk$-linear endomorphisms and $Z(\lambda)$ is endowed with the structure of a linear representation of the $\Bbbk$-algebra $U(\mathfrak{g})$ (see Remark IV.2.11). Therefore, we have a Lie algebra homomorphism

$$\mathfrak{g} \xrightarrow{j_{\mathfrak{g}}} U(\mathfrak{g}) \longrightarrow \mathfrak{gl}(Z(\lambda))$$

which endows $Z(\lambda)$ with the structure of a representation of the Lie algebra $\mathfrak{g}$.
2. Recall from Exercise IV.2.21 that multiplication in $U(\mathfrak{g})$ defines an isomorphism

$$U(\mathfrak{n}^-) \otimes_{\Bbbk} U(\mathfrak{b}) \longrightarrow U(\mathfrak{g})$$

as left $U(\mathfrak{n}^-)$-modules and right $U(\mathfrak{b})$-modules. Therefore, using standard results on tensor products, we get an isomorphism of left $U(\mathfrak{n}^-)$-modules as follows:

$$\begin{array}{rcl} Z(\lambda) & = & U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} D_\lambda \\ & \cong & (U(\mathfrak{n}^-) \otimes_{\Bbbk} U(\mathfrak{b})) \otimes_{U(\mathfrak{b})} D_\lambda \\ & \cong & U(\mathfrak{n}^-) \otimes_{\Bbbk} (U(\mathfrak{b})) \otimes_{U(\mathfrak{b})} D_\lambda \\ & \cong & U(\mathfrak{n}^-) \otimes_{\Bbbk} D_\lambda \\ & \cong & U(\mathfrak{n}^-) \otimes_{\Bbbk} \Bbbk \\ & \cong & U(\mathfrak{n}^-); \end{array}$$

which, for all $y \in U(\mathfrak{n}^-)$, sends $y \otimes 1 \in Z(\lambda)$ to y.

Definition V.2.10 – *Let $\lambda \in \mathfrak{h}^*$. Then, the representation $Z(\lambda)$ of $\mathfrak{g}$ is called the Verma module associated to λ.*

Lemma V.2.11 – *Let $\lambda \in \mathfrak{h}^*$. The representation $Z(\lambda)$ of $\mathfrak{g}$ is a highest weight representation of weight λ of $\mathfrak{g}$ and v_λ is a highest weight vector of $Z(\lambda)$ of weight λ which generates $Z(\lambda)$.*

Proof. By definition of $Z(\lambda)$, v_λ is a weight element of weight λ of $Z(\lambda)$ and it is in the kernel of the action of $\mathfrak{n}$. In addition, in the isomorphism of Remark V.2.9, v_λ maps to a nonzero element, hence is itself nonzero. Since, in addition, v_λ obviously generates $Z(\lambda)$ as a $U(\mathfrak{g})$-module, the result is established. ∎

Notation V.2.12 – Let $\lambda \in \mathfrak{h}^*$. By Theorem V.2.6 and Lemma V.2.11, the highest weight representation $Z(\lambda)$ has a unique irreducible quotient. We will denote this quotient by $V(\lambda)$.

Corollary V.2.13 – *For all $\lambda \in \mathfrak{h}^*$, there exists an irreducible highest weight representation of weight λ of $\mathfrak{g}$.*

Proof. This follows from Lemma V.2.11 and Theorem V.2.6. ∎

The following Remark summarizes the results obtained so far.

Remark V.2.14 –
1. For all $\lambda \in \mathfrak{h}^*$, $V(\lambda)$ is an irreducible highest weight representation of weight λ of $\mathfrak{g}$ (cf. Notation V.2.12 and Corollary V.2.13). In addition, it follows easily from Corollary V.2.7 that, if λ and μ are distinct elements of $\mathfrak{h}^*$, then $V(\lambda)$ and $V(\mu)$ are not isomorphic.
2. Let V be a finite dimensional irreducible representation of $\mathfrak{g}$. Since V is finite dimensional, by Remark V.2.3, it must have a highest weight vector. Since, in addition, it is irreducible, this highest weight vector must generate V as a representation. Hence, there exists $\lambda \in \mathfrak{h}^*$ such that V is a highest weight representation of weight λ. Now, by Theorem V.2.8, we deduce that V is isomorphic as a representation to $V(\lambda)$.
3. Therefore, the exhaustive list of irreducible finite dimensional representations of $\mathfrak{g}$ (up to isomorphism) coincides with the list of representations $V(\lambda)$ which are finite dimensional over $\mathbb{k}$.

Example V.2.15 – **Verma modules for $\mathfrak{sl}_2(\mathbb{k})$** – Put $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$. Recall the canonical generators x, h, y of $\mathfrak{g}$ from Section II.4. Then, the Cartan-Chevalley decomposition of $\mathfrak{g}$ reads $\mathfrak{sl}_2(\mathbb{k}) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}^- = \mathbb{k}y$, $\mathfrak{h} = \mathbb{k}h$ and $\mathfrak{n} = \mathbb{k}x$. Of course, $\mathfrak{h}^*$ identifies to $\mathbb{k}$ by means of the isomorphism $\mathfrak{h}^* \longrightarrow \mathbb{k}$, $\lambda \mapsto \lambda(h)$.

Since $\mathfrak{n}^-$ is a one-dimensional Lie algebra, its enveloping algebra is just its symmetric algebra. More precisely, $U(\mathfrak{n}^-)$ is generated, as a $\mathbb{k}$-algebra, by (the canonical image of) y and has the set $\{y^i,\, i \in \mathbb{N}\}$ as a $\mathbb{k}$-basis.
1. Description of the Verma module $Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} D_\lambda$, $\lambda \in \mathfrak{h}^*$.

For all $i \in \mathbb{N}$, put $e_i = y^i \otimes 1 \in Z(\lambda)$ (and $e_{-1} = 0$, for convenience). By Remark V.2.9, the set $\{e_i,\, i \in \mathbb{N}\}$ is a basis of the $\mathbb{k}$-vector space $Z(\lambda)$. In order to decribe the representation $Z(\lambda)$ of $\mathfrak{g}$, we give explicit expressions for the actions of x, h and y on the elements of this basis. First, recall that (the images in $U(\mathfrak{g})$ of) x, h, y generate $U(\mathfrak{g})$ as a $\mathbb{k}$-algebra and that the following relations hold in $U(\mathfrak{g})$: $xy - yx = h$, $hx - xh = 2x$ and $hy - yh = -2y$. It is then easy to establish the following identities in $U(\mathfrak{g})$:

$$\forall\, i \in \mathbb{N}, \qquad xy^i - y^i x = iy^{i-1}(h - (i-1)) \quad \text{and} \quad hy^i - y^i h = -2iy^i.$$

From these relations, we get the following identities describing the action of x, h and y on the above basis of $Z(\lambda)$, for all $i \in \mathbb{N}$:

$$x.e_i = i\,(\lambda(h) - (i-1))\, e_{i-1}; \tag{V.2.55}$$

$$h.e_i = (\lambda(h) - 2i)e_i; \tag{V.2.56}$$

$$y.e_i = e_{i+1}. \tag{V.2.57}$$

We notice that the weight spaces of $Z(\lambda)$ are the lines $\mathbb{k}e_i$, $i \in \mathbb{N}$ (and they are nothink but the eigenspaces of the action of h).
2. Simplicity of $Z(\lambda)$.
2.1. If $\lambda(h) \notin \mathbb{N}$, $Z(\lambda)$ is irreducible. Details of the proof of this statement are left as an easy and very interesting exercise. Here is a sketch of a proof. Consider a nonzero element v of $Z(\lambda)$. By the hypothesis on $\lambda(h)$, the coefficients appearing in relations (V.2.55) for $i \in \mathbb{N}^*$ are not zero. From this it follows that, if j is the greatest integer such that the coefficient of e_j in the expression of v in the above basis is nonzero, then $x^j.v$ is a nonzero scalar multiple of e_0. Then, it obviously follows that, for all $k \in \mathbb{N}$, $y^k.(x^j.v)$ is a nonzero scalar multiple of e_k. This shows that any nonzero element of $Z(\lambda)$ generates $Z(\lambda)$ as a representation. That is, $Z(\lambda)$ is irreducible.
2.2. Suppose $\lambda(h) = m \in \mathbb{N}$. Put

$$M(\lambda) = \mathrm{Span}_{\mathbb{k}}(e_i,\, i \in \mathbb{N},\, i \geq m+1).$$

The identities (V.2.55), (V.2.56) and (V.2.57) show that $M(\lambda)$ is a subrepresentation of $Z(\lambda)$. It is not difficult to show that the corresponding (finite dimensional) quotient representation is irreducible, using the same strategy as in 2.1. But, actually, much more is true. Let Y be any strict subrepresentation of $Z(\lambda)$. As $Z(\lambda)$ is a highest weight representation, Theorem V.2.6, Point 4, applies and Y must be the direct sum of its weight spaces. That is, Y must be the direct sum of its intersections with the (weight spaces of $Z(\lambda)$, that is the) lines $\Bbbk e_i$, $i \in \mathbb{N}$. But, this forces $Y \cap \Bbbk e_i = (0)$, whenever $0 \leq i \leq m$ for, otherwise, Y would contain a basis vector e_j for some $0 \leq j \leq m$ and therefore e_0 itself, which would contradict the fact that Y is a strict subrepresentation. All in all, we have proved that Y must be included in $M(\lambda)$. Hence, $M(\lambda)$ is a maximum strict subrepresentation of $Z(\lambda)$ and the corresponding quotient is therefore the simple representation $V(\lambda)$.
2.3. To sum up the above, according to whether $\lambda(h)$ belongs to $\mathbb{N}$ or not, the simple representation $V(\lambda)$ of $\mathfrak{g}$ is finite dimensional or infinite dimensional. If $\lambda(h) = m \in \mathbb{N}$, then $V(\lambda)$ is a finite dimensional representation of dimension $m+1$ and it is easy to prove that it is isomorphic to the representation $(\rho_m, \Bbbk^{m+1})$ discussed in Section II.4. If $\lambda(h) \notin \mathbb{N}$, $V(\lambda) = Z(\lambda)$.

By Remark V.2.14 the question as to whether a representation $V(\lambda)$, $\lambda \in \mathfrak{h}^*$, is finite dimensional is crucial. This is the point we now investigate.

Theorem V.2.16 – *Let $\lambda \in \mathfrak{h}^*$. If $V(\lambda)$ is finite dimensional then, for all $\alpha \in \Delta$, $\lambda(h_\alpha) \in \mathbb{N}$.*

Proof. Let $\alpha \in \Delta$. Choose $x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ and $y_\alpha \in \mathfrak{g}_{-\alpha} \setminus \{0\}$ as in Theorem II.5.13. Let S_α be the Lie subalgebra of $\mathfrak{g}$ generated by x_α and y_α; S_α is isomorphic to $\mathfrak{sl}_2(\Bbbk)$ via an isomorphism sending h_α to h and $V(\lambda)$ becomes in that way a representation of $\mathfrak{sl}_2(\Bbbk)$.

Now let $v \in V(\lambda)$ be a highest weight vector of weight λ of the representation $V(\lambda)$ of $\mathfrak{g}$. Then, clearly, v is an eigenvector of eigenvalue $\lambda(h_\alpha)$ for the action of h_α and an element of the kernel of the action of x_α. Now, by Weyl's Theorem, the representation $V(\lambda)$ of $\mathfrak{sl}_2(\Bbbk)$ decomposes as a sum of finite dimensional irreducible representations of $\mathfrak{sl}_2(\Bbbk)$. The description of the finite dimensional irreducible representations of $\mathfrak{sl}_2(\Bbbk)$ then show that we must have $\lambda(h_\alpha) \in \mathbb{N}$, as requierred. ■

Our aim now is to prove that the necessary condition of Theorem V.2.16 for $V(\lambda)$ to be finite dimensional is actually sufficient. For this, preparatory results will be convenient.

Remark V.2.17 – **Abstract weights versus concrete weights** – At this stage, a remark is in order to avoid ambiguities between various meanings of the word *weight*.
1. Recall the context. We are given a semisimple Lie algebra $\mathfrak{g}$ and a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. There is a set of roots $\Phi \subseteq \mathfrak{h}^*$ attached to the pair $(\mathfrak{g}, \mathfrak{h})$ and a corresponding root system $\Phi \subseteq \mathsf{E}_{\mathbb{R}}$ in the euclidean space $\mathsf{E}_{\mathbb{R}}$. Fix, in addition, a base $\Delta = \{\alpha_1, \dots, \alpha_n\}$ of the root system $(\mathsf{E}_{\mathbb{R}}, \Phi)$.
2. We have a notion of weight attached to the root system $(\mathsf{E}_{\mathbb{R}}, \Phi)$, in the sense of section III.9. These are elements $\lambda \in \mathsf{E}_{\mathbb{R}}$ such that, for all $\alpha \in \Phi$, $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ (cf. Definition III.9.1); the set of weights is denoted Λ_Φ. By Lemma III.9.3, we have

$$\Lambda_\Phi = \{\lambda \in \mathsf{E}_{\mathbb{R}} \,|\, \forall \alpha \in \Delta,\ \langle \lambda, \alpha \rangle \in \mathbb{Z}\}.$$

Attached to Δ, we have the fundamental weights ϖ_i, $1 \leq i \leq n$ (cf. Definition III.9.9). By definition, $\varpi_i \in \Lambda_\Phi$ and, actually, the set $\{\varpi_i,\, 1 \leq i \leq n\}$ is a $\mathbb{Z}$-basis of the (free) abelian group Λ_Φ (Lemma III.9.10). But, on the other hand, by Remark III.9.12, the fundamental weights are elements of $\mathsf{E}_{\mathbb{Q}} = \mathrm{Span}_{\mathbb{Q}}(\alpha_i,\, 1 \leq i \leq n)$, because the Cartan matrix is invertible, with coefficients in $\mathbb{Z}$. All in all, we get that

$$\Lambda_\Phi \subseteq \mathsf{E}_{\mathbb{Q}}.$$

The same arguments actually show that $\{\varpi_i,\, 1 \leq i \leq n\}$ form a $\mathbb{Q}$-basis of $\mathsf{E}_{\mathbb{Q}}$ and a $\mathbb{k}$-basis of $\mathfrak{h}^*$ (cf. (II.6.12)).
3. Let now $\lambda \in \mathfrak{h}^*$.

Recall that the Killing form, $\kappa_{\mathfrak{g}}$, on $\mathfrak{h}$ is nondegenerate and therefore allows the identification $\iota : \mathfrak{h} \longrightarrow \mathfrak{h}^*$, $h \mapsto \kappa_{\mathfrak{g}}(h, -)$. By transfer of structure via ι, we get a nondegenerate bilinear form $(-,-) : \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow \mathbb{k}$. Recall also that, for all $\alpha \in \Phi$, $(\alpha, \alpha) \neq 0$ (Proposition II.5.12).

Suppose that:

$$\forall\, 1 \leq i \leq n, \quad 2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}. \tag{V.2.58}$$

We know that $\{\varpi_i,\, 1 \leq i \leq n\}$ is a $\mathbb{k}$-basis of $\mathfrak{h}^*$. So, there exists $c_i \in \mathbb{k}$, $1 \leq i \leq n$, such that $\lambda = \sum_{1 \leq i \leq n} c_i \varpi_i$. But then, by definition of the fundamental weights, for $1 \leq i \leq n$,

$$c_i = 2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}.$$

So, by Point 2, we have that $\lambda \in \mathsf{E}_{\mathbb{Q}} \subseteq \mathsf{E}_{\mathbb{R}}$ and, by hypothesis on λ, we actually have $\lambda \in \Lambda_\Phi$.
4. Point 3 above applies in particular to the following context. Let V be a finite dimensional representation of $\mathfrak{g}$. Let $\lambda \in \mathfrak{h}^*$ be a weight of the representation V (so, in the sense of Definition V.1.1). Consider $1 \leq i \leq n$, and elements $x_i \in \mathfrak{g}_{\alpha_i}$, $y_i \in \mathfrak{g}_{-\alpha_i}$ and $h_i \in \mathfrak{h}$ as in Theorem II.5.13: the subalgebra of $\mathfrak{g}$ generated by x_i and y_i is isomorphic to $\mathfrak{sl}_2(\mathbb{k})$ and any element of the weight space V_λ is an eigenvector for the action of $h \in \mathfrak{sl}_2(\mathbb{k})$, whose eigenvalue is $\lambda(h_i)$. Since V is finite dimensional, seen as a representation of $\mathfrak{sl}_2(\mathbb{k})$ it is the direct sum of finite dimensional irreducible representations of $\mathfrak{sl}_2(\mathbb{k})$, so that we must have $\lambda(h_i) \in \mathbb{Z}$. But,

$$\lambda(h_i) = \frac{2}{\kappa_{\mathfrak{g}}(t_{\alpha_i}, t_{\alpha_i})}\lambda(t_{\alpha_i}) = \frac{2}{\kappa_{\mathfrak{g}}(t_{\alpha_i}, t_{\alpha_i})}\kappa_{\mathfrak{g}}(t_\lambda, t_{\alpha_i}) = \frac{2}{(\alpha_i, \alpha_i)}(\lambda, \alpha_i).$$

We have shown that the weights of any finite dimensional representation of $\mathfrak{g}$ satisfies the hypothesis (V.2.58). By Point 3, such a weight is a weight in the abstract sense, that is, belongs to $\Lambda_\Phi \subseteq \mathsf{E}_{\mathbb{R}}$.

Lemma V.2.18 – *For $1 \leq i \leq n$, consider elements $x_i \in \mathfrak{g}_{\alpha_i}$, $y_i \in \mathfrak{g}_{-\alpha_i}$ and $h_i \in \mathfrak{h}$ as in Theorem II.5.13. Then, for $1 \leq i, j \leq n$ and $k \in \mathbb{N}$, the following relations hold in $U(\mathfrak{g})$:*
1. $[x_j, y_i^{k+1}] = 0$, if $i \neq j$;
2. $[h_j, y_i^{k+1}] = -(k+1)\alpha_i(h_j)y_i^{k+1}$;
3. $[x_i, y_i^{k+1}] = -(k+1)y_i^k(k.1 - h_i)$.

Proof. Recall the (injective) morphism of Lie algebras $j_{\mathfrak{g}} : \mathfrak{g} \longrightarrow U(\mathfrak{g})$.

When $k = 0$, the relations in the statement hold in $\mathfrak{g}$ and, therefore, in $U(\mathfrak{g})$. Indeed, the first one follows from the fact that $\alpha_j - \alpha_i$ is not a root (see Lemma II.5.6) and the two others are true by definition. The result now follows by a straightforward induction on k. ∎

Theorem V.2.19 – *Let $\lambda \in \mathfrak{h}^*$. Let V be an irreducible highest weight representation of $\mathfrak{g}$, of weight λ. Suppose that $\lambda(h_i) \in \mathbb{N}$ for all $1 \leq i \leq n$, then the following holds:*
1. $\Pi(V) \subseteq \Lambda_\Phi \subseteq \mathsf{E}_{\mathbb{Q}}$ and $\Pi(V)$ (seen as a subset of $\mathsf{E}_{\mathbb{R}}$) is stable under the action of W_Φ;
2. V is finite dimensional.

Proof. We let v be a highest weight vector of weight λ that generates V and denote by $\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ the Lie algebra morphism that defines the representation of $\mathfrak{g}$ in V. For $1 \leq i \leq n$, put $m_i = \lambda(h_i) \in \mathbb{N}$.
Step 1. Fix $1 \leq i \leq n$ and recall the elements $x_i, h_i, y_i \in \mathfrak{g}$ and the subalgebra S_i that they generate. We first investigate the action of S_i in V. It follows from the relations in Lemma V.2.18 that the element $y_i^{m_i+1}.v$ is a weight vector of weight $\lambda - (m_i+1)\alpha_i$ annihilated by $\mathfrak{n}$. But, V being irreducible, Corollary V.2.7 applies to show that $y_i^{m_i+1}.v = 0$, since $\lambda \neq \lambda - (m_i+1)\alpha_i$. Now, put

$$F_i = \mathrm{Span}_{\Bbbk}(y_i^k.v,\, 0 \leq k \leq m_i\}.$$

Using Lemma V.2.18 again as well as the above, we get that F_i is a nonzero finite dimensional subrepresentation of V seen as a representation of S_i.

Now, let F be any finite dimensional subrepresentation of V seen as a representation of S_i. Put

$$\mathfrak{g}.F = \mathrm{Span}_{\Bbbk}(g.w,\, g \in \mathfrak{g},\, w \in F).$$

Since $\mathfrak{g}$ and F are finite dimensional, $\mathfrak{g}.F$ is also finite dimensional. Further, since

$$\forall\, z \in S_i,\, g \in \mathfrak{g},\, w \in F, \quad z.(g.w) = g.(z.w) + [z,g].w,$$

$\mathfrak{g}.F$ is a subrepresentation of V seen as a representation of S_i. Therefore, if we let $\mathcal{T}_i$ be the set of all finite dimensional subspaces of V stable under the action of S_i, the subspace $\sum_{F \in \mathcal{T}_i} F$ is stable under the action of $\mathfrak{g}$ (and nonzero since it contains F_i and therefore v). As V is irreducible, we deduce that

$$V = \sum_{F \in \mathcal{T}_i} F.$$

That is: seen as a representation of S_i, V is the sum of finite dimensional subrepresentations.

Step 2. Since V is a highest weight representation of $\mathfrak{g}$ of weight λ, by Theorem V.2.6, it is the sum of its weight spaces and all its weights belong to the set $\lambda - \mathbb{N}\Delta$. Therefore, the hypothesis on λ implies that, for all weight μ of V, we have that $\mu(h_i) \in \mathbb{Z}$, for all $1 \leq i \leq n$. This means that we are in the hypothesis of Point 3 in Remark V.2.17 and we deduce (see Point 2 of the same Remark) that all the weights of V actually lie in $\Lambda_\Phi \subseteq \mathsf{E}_{\mathbb{Q}}$ and may therefore be seen as weights in the abstract sense. In particular, we are in position to use the action of the Weyl group on $\mathsf{E}_{\mathbb{R}}$ (which restrict to Λ_Φ).

Consider a weight μ of V and $w \in V_\mu$. Fix $1 \leq i \leq n$. By step 1, there exists a finite dimensional subrepresentation E of V seen as a representation of S_i that contains w. By the results of Section II.4, the representation E of S_i may be written

$$E = \bigoplus_{1 \leq s \leq r} E_s, \tag{V.2.59}$$

where $r \in \mathbb{N}^*$ and, for all $1 \leq s \leq r$, E_s is an irreducible finite dimensional representation of S_i, which dimension we denote $d_i \in \mathbb{N}^*$. Write

$$w = \sum_{1 \leq s \leq r} w_s,$$

where, for $1 \leq s \leq r$, $w_s \in E_s$. Since $w \neq 0$, there exists $1 \leq t \leq r$ such that $w_t \neq 0$. Clearly, for all $1 \leq s \leq r$, we have $\phi(h_i)(w_s) = \mu(h_i)w_s$, so that w_s is an eigenvector of $\phi(h_i)$ in the irreducible representation E_s of S_i.

Suppose first that $\mu(h_i) \in \mathbb{N}$. The nonzero vector w_t as eigenvalue $\mu(h_i)$ w.r.t. the endomorphism $\phi(h_i)$, and, we may consider the vector

$$\phi(y_i)^{\mu(h_i)}(w_t).$$

By the structure of the irreducible representation E_t of S_i (as discribed in Section II.4), this vector is therefore an eigenvector of eigenvalue $\mu(h_i) - 2\mu(h_i) = -\mu(h_i)$ for $\phi(h_i)$ and is therefore nonzero. From this, we deduce immediately that $\phi(y_i)^{\mu(h_i)}(w)$ is a nonzero element of E. But, $w \in V_\mu$. So, applying the first statement of Proposition V.1.2, we get that $\phi(y_i)^{\mu(h_i)}(w)$ is a nonzero element of V, of weight $\mu - \mu(h_i)\alpha_i$, since $y_i \in \mathfrak{g}_{-\alpha_i}$. But

$$\mu - \mu(h_i)\alpha_i = \mu - \langle \mu, \alpha_i \rangle \alpha_i = s_{\alpha_i}(\mu).$$

Therefore, we have shown that, if $\mu(h_i) \in \mathbb{N}$, then $s_{\alpha_i}(\mu)$ is a weight of V.

A similar argument deals with the case where $-\mu(h_i) \in \mathbb{N}$. It is enough to consider the element $\phi(x_i)^{-\mu(h_i)}(w)$; it turns out that it is a nonzero vector of weight $s_{\alpha_i}(\mu)$.

Hence, we have proved that, whenever μ is a weight of V, then for all $1 \leq i \leq n$, its image under the simple reflection s_{α_i} is also a weight of V.

Step 3. As the Weyl group is generated by the simple reflections, it follows from the previous step that W_Φ stabilises the set of weights of V. It only remains to establish that V is finite dimensional.

Notice first that the hypothesis on λ means that λ is dominant, in the sense of section III.9. As pointed out above, $\Pi(V)$ is a union of W_Φ-orbits. Let $\mathcal{O}$ be such an orbit. By Proposition III.9.14, $\mathcal{O}$ contains exactly one dominant weight μ and, as $\mu \in \Pi(V)$, we must have that $\mu \preceq \lambda$, by Theorem V.2.6. But λ is dominant, so there are finitely many dominant weights ν such that $\nu \preceq \lambda$, as Lemma III.9.16 establishes. All together, we have shown that $\Pi(V)$ is the disjoint union of finitely many orbits, all of whitch are finite since W_Φ is finite. Therefore, $\Pi(V)$ is finite. Now, Theorem V.2.6 shows that each weight space of V must be finite dimensional. Therefore, V is finite dimensional, as the sum of finitely many finite dimensional subspaces. ■

Theorem V.2.19 establishes that, if V is an irreducible highest weight representation of $\mathfrak{g}$, of weight λ such that $\lambda(h_i) \in \mathbb{N}$ for all $1 \leq i \leq n$, then the set $\Pi(V)$ of weights of V is a union of W_Φ-orbits. However, it does not clearly link the weight spaces attached to weights belonging to the same orbit. Such a link is the goal that we pursue now.

We start with the following rather technical, but crucial, result.

Exercise V.2.20 – In this exercise, we only assume that $\mathbb{k}$ is of characteristic 0.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, (V, ϕ) a finite dimensional representation of $\mathfrak{g}$ and $a \in \mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{g}}(a) : \mathfrak{g} \longrightarrow \mathfrak{g}$ and $\phi(a) : V \longrightarrow V$ are nilpotent endomorphisms. Let $b \in \mathfrak{g}$.
1. The endomorphism $\mathrm{ad}_{\mathfrak{gl}(V)}(\phi(a)) : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V)$ is nilpotent (cf. Lemma I.4.7).
1.1. For all $n \in \mathbb{N}$, $\phi((\mathrm{ad}_{\mathfrak{g}}(a))^n(b)) = (\mathrm{ad}_{\mathfrak{gl}(V)}(\phi(a)))^n(\phi(b))$.
1.2. The following equality holds in $\mathfrak{gl}(V)$:

$$\exp\left(\mathrm{ad}_{\mathfrak{gl}(V)}(\phi(a))\right)(\phi(b)) = \phi\left(\exp(\mathrm{ad}_{\mathfrak{g}}(a))(b)\right).$$

2. As $\phi(a)$ is nilpotent, we may consider the endomorphism $\exp(\phi(a))\phi(b)\exp(-\phi(a)) : V \longrightarrow V$.
2.1. Let $p \in \mathbb{N}$ be such that $\phi(a)^n = 0$ for all integer $n > p$. Then

$$\exp(\phi(a))\phi(b)\exp(-\phi(a)) = \sum_{0 \leq i,j \leq p} \frac{1}{i!}\frac{1}{j!}\phi(a)^i\phi(b)\phi(-a)^j.$$

2.2. Let $q \in \mathbb{N}$ be such that $\left(\mathrm{ad}_{\mathfrak{gl}(V)}(\phi(a))\right)^n = 0$ for all integer $n > q$. Then

$$\exp\left(\mathrm{ad}_{\mathfrak{gl}(V)}(\phi(a))\right)(\phi(b)) = \sum_{0 \leq k \leq q} \frac{1}{k!}\left(\mathrm{ad}_{\mathfrak{gl}(V)}(\phi(a))\right)^k(\phi(b)).$$

From which it follows that

$$\exp\left(\mathrm{ad}_{\mathfrak{gl}(V)}(\phi(a))\right)(\phi(b)) = \sum_{0 \leq k \leq q} \sum_{s,t \in \mathbb{N},\, s+t=k} \frac{1}{s!}\frac{1}{t!}\phi(a)^s\phi(b)\phi(-a)^t.$$

Hint: to deduce the last equality from the previous one, one can use the two commuting endomorphisms $L_{\phi(a)} : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V)$, $f \mapsto \phi(a)f$ (left multiplication by $\phi(a)$) and $R_{\phi(-a)} : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}(V)$, $f \mapsto f\phi(-a)$ (right multiplication by $\phi(-a)$) and the binomial expansion formula together with the identity $\mathrm{ad}_{\mathfrak{gl}(V)}(\phi(a)) = L_{\phi(a)} + R_{\phi(-a)}$.
2.3. The following identity holds in $\mathfrak{gl}(V)$:

$$\exp\left(\mathrm{ad}_{\mathfrak{gl}(V)}(\phi(a))\right)(\phi(b)) = \exp(\phi(a))\phi(b)\exp(-\phi(a)).$$

3. The following identity holds in $\mathfrak{gl}(V)$:

$$\phi\left(\exp(\mathrm{ad}_{\mathfrak{g}}(a))(b)\right) = \exp(\phi(a))\phi(b)\exp(-\phi(a)).$$

Remark V.2.21 – **Some automorphisms of $\mathfrak{g}$ associated to simple roots** –
Put $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. For $1 \leq i \leq n$, consider elements $x_i \in \mathfrak{g}_{\alpha_i}$, $y_i \in \mathfrak{g}_{-\alpha_i}$ and $h_i \in \mathfrak{h}$ as in Theorem II.5.13.
1. The endomorphisms $\mathrm{ad}_{\mathfrak{g}}(x_i)$ and $\mathrm{ad}_{\mathfrak{g}}(y_i)$ are nilpotent derivations of $\mathfrak{g}$ (cf. Lemma II.5.6). Therefore, we may consider the following automorphism of the Lie algebra $\mathfrak{g}$:

$$\Theta_i = \exp(\mathrm{ad}_{\mathfrak{g}}(x_i))\exp(\mathrm{ad}_{\mathfrak{g}}(-y_i))\exp(\mathrm{ad}_{\mathfrak{g}}(x_i)).$$

2. For $1 \leq i,j \leq n$, let $c_{i,j} = \langle \alpha_i, \alpha_j \rangle$. It is not difficult to show, using Proposition IV.3.1, that the following relations hold:

$$\forall\, 1 \leq i,j \leq n, \quad \Theta_i(h_j) = h_j - c_{i,j}h_i.$$

From which it follows that Θ_i induces an involutive automorphism of Lie algebra of $\mathfrak{h}$, that we still denote Θ_i.
3. It follows that we have the relations,

$$\forall\, 1 \leq i,j \leq n, \quad \alpha_i \circ \Theta_j = \alpha_i - c_{i,j}\alpha_j$$

(since, for $1 \leq i \leq n$, α_i is the linear form on $\mathfrak{h}$ that sends h_j to $c_{i,j}$). Hence, the above equalities read

$$\forall\, 1 \leq i,j \leq n, \quad {}^t\Theta_j(\alpha_i) = \alpha_i - c_{i,j}\alpha_j.$$

In particular, the endomorphism ${}^t\Theta_j$ of $\mathfrak{h}^*$ actually stabilises $\mathsf{E}_{\mathbb{Q}}$.
On the other hand, let s_i be the simple reflection of $\mathsf{E}_{\mathbb{R}}$ associated to α_i. We have that, for all $1 \leq i,j \leq n$, $s_j(\alpha_i) = \alpha_i - c_{i,j}\alpha_j$. Therefore, the simple reflections stabilise $\mathsf{E}_{\mathbb{Q}}$ and

$$\forall\, 1 \leq i \leq n, \quad \left({}^t\Theta_i\right)_{|\mathsf{E}_{\mathbb{Q}}} = (s_i)_{|\mathsf{E}_{\mathbb{Q}}}.$$

4. Suppose now that we are given a finite dimensional representation (V, ϕ) of $\mathfrak{g}$.

Let $1 \leq i \leq n$. By the representation theory of $\mathfrak{sl}_2(\Bbbk)$ (applied to V considered as a representation of the Lie subalgebra of $\mathfrak{g}$ generated by x_i, y_i and h_i), we get that $\phi(x_i)$ and $\phi(y_i)$ are nilpotent endomorphisms of V. Thus, we are in position to consider the automorphism of V defined by

$$f_i = \exp(\phi(x_i)) \exp(\phi(-y_i)) \exp(\phi(x_i)).$$

By Point 3 in Exercise V.2.20, we get that the following relations hold:

$$\forall\, g \in \mathfrak{g}, \quad \phi(\Theta_i(g)) = f_i \circ \phi(g) \circ f_i^{-1}.$$

Notice that, since the restriction of Θ_i to $\mathfrak{h}$ is an involution, we actually have:

$$\forall\, h \in \mathfrak{h}, \quad \phi(\Theta_i(h)) = f_i \circ \phi(h) \circ f_i^{-1} = f_i^{-1} \circ \phi(h) \circ f_i. \tag{V.2.60}$$

Theorem V.2.22 – *Let V be an irreducible highest weight representation of $\mathfrak{g}$, of weight λ. Suppose that $\lambda(h_i) \in \mathbb{N}$ for all $1 \leq i \leq n$. Then, for all weight μ of V and all $w \in W_\Phi$,* $\dim_\Bbbk(V_{w(\mu)}) = \dim_\Bbbk(V_\mu)$.

Proof. We let v be a highest weight vector of weight λ that generates V and denote by $\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ the Lie algebra morphism that defines the representation of $\mathfrak{g}$ in V.

Recall from Theorem V.2.19 that the set $\Pi(V)$ of weights of V satisfies $\Pi(V) \subseteq \Lambda_\Phi \subseteq \mathsf{E}_\mathbb{Q}$ and is stable under the action of W_Φ and that V is finite dimensional.

Consider a weight μ of V and $w \in V_\mu$. For all $1 \leq i \leq n$, we have that

$$\forall\, h \in \mathfrak{h}, \quad \phi(h)(f_i^{-1}(w)) = f_i^{-1} \circ (\phi(\Theta_i(h))(w)) = \mu(\Theta_i(h)) f_i^{-1}(w) = (s_i(\mu)(h)) f_i^{-1}(w).$$

Indeed, the first equality above follows from the first equality in (V.2.60), the second holds because $w \in V_\mu$, and the third is Point 3 of Remark V.2.60. This shows that

$$f_i^{-1}(V_\mu) \subseteq V_{s_i(\mu)}. \tag{V.2.61}$$

Now, using the second equality in (V.2.60), rather than the first, the same argument leads to

$$f_i(V_\nu) \subseteq V_{s_i(\nu)} \tag{V.2.62}$$

for all weight ν of V. But, the inclusion (V.2.61) shows that is μ is a weight, then $s_i(\mu)$ is also a weight and therefore, the inclusion (V.2.62) may be applied to $s_i(\mu)$, leading to

$$f_i^{-1}(V_\mu) = V_{s_i(\mu)}. \tag{V.2.63}$$

As the Weyl group is generated by the simple reflections, the result follows. ∎

We are now in position to classify finite dimensional representations of $\mathfrak{g}$. Let $\mathrm{Irrep}(\mathfrak{g})$ denote the set of finite dimensional irreducible representations of $\mathfrak{g}$ and by $\sim$ the equivalence relation on this set defined by isomorphism of representations. By Theorem V.2.19, we have a map as follows:

$$\begin{array}{ccc} \{\lambda \in \mathfrak{h}^* \mid \forall\, 1 \leq i \leq n,\ \lambda(h_i) \in \mathbb{N}\} & \longrightarrow & \mathrm{Irrep}(\mathfrak{g})/\sim \\ \lambda & \mapsto & \mathrm{cl}(V(\lambda)) \end{array}, \tag{V.2.64}$$

where, for $\lambda \in \mathfrak{h}^*$, $\mathrm{cl}(V(\lambda))$ stands for the isomorphism class of the representation $V(\lambda)$.

Corollary V.2.23 – **Classification of finite dim. irreducible representations** – *The map (V.2.64) is a bijection.*

Proof. The injectivity of this map follows from Point 1 in Remark V.2.14; its surjectivity follows from Theorem V.2.16 together with Point 2 in Remark V.2.14. ∎

Remark V.2.24 – Recall Remark V.2.17. It asserts that we have the following equalities between subsets of $\mathsf{E}_{\mathbb{Q}}$:

$$\Lambda_\Phi = \{\lambda \in \mathfrak{h}^* \mid \forall\, 1 \leq i \leq n,\, \lambda(h_i) \in \mathbb{Z}\} \quad \text{and} \quad \Lambda_\Phi^+ = \{\lambda \in \mathfrak{h}^* \mid \forall\, 1 \leq i \leq n,\, \lambda(h_i) \in \mathbb{N}\}.$$

Therefore, Corollary V.2.23 actually gives a bijection:

$$\begin{array}{ccc} \Lambda_\Phi^+ & \longrightarrow & \mathrm{Irrep}(\mathfrak{g})/\sim \\ \lambda & \mapsto & \mathrm{cl}(V(\lambda)) \end{array}.$$

We conclude this section by showing that the set of weights of $V(\lambda)$, $\lambda \in \Lambda_\Phi^+$, is actually a saturated set of weights, in the sense of Section III.9.

Consider $\lambda \in \Lambda_\Phi^+$, $V(\lambda)$ the associated irreducible finite dimensional representation of $\mathfrak{g}$ and denote by $\Pi(\lambda)$ the set of weights of $V(\lambda)$. As already noticed, we know that

$$\Pi(\lambda) \subseteq \Lambda_\Phi.$$

Recall, on the other hand, the notion of saturated set of weights, as introduced in Section III.9.

We begin with a preparatory Lemma which, for all $\mu \in \Pi(\lambda)$ and all $\alpha \in \Phi$, describes the set of weights of the form $\mu + i\alpha$ which are also in $\Pi(\lambda)$. Its statement and proof are parallel with those of Proposition III.2.15 about strings of roots.

Lemma V.2.25 – *Retain the above notation. For all $\mu \in \Pi(\lambda)$ and all $\alpha \in \Phi$, there exist $r, q \in \mathbb{N}$ such that*

$$\Pi(\lambda) \cap \{\mu + i\alpha,\, i \in \mathbb{Z}\} = \{\mu + i\alpha,\, i \in \mathbb{Z},\, -r \leq i \leq q\}.$$

In addition, $r = q + \langle \mu, \alpha \rangle$.

Proof. Put $V = V(\lambda)$, to simplify notation. Consider $\mu \in \Pi(\lambda)$.

For $\alpha \in \Phi$, we consider elements $x_\alpha, h_\alpha, y_\alpha$ as in Theorem II.5.13 and denote by S_α the Lie subalgebra of $\mathfrak{g}$ that they generate (which is therefore isomorphic to $\mathfrak{sl}_2(\mathbb{k})$). Of course, $V(\lambda)$ may be considered as a representation of S_α. Let $W = \bigoplus_{i\in\mathbb{Z}} V_{\mu+i\alpha}$. By Proposition V.1.2, W is a subrepresentation of the representation $V(\lambda)$ of S_α. Of course, W is finite dimensional since $V(\lambda)$ is. Therefore, we may consider $r, q \in \mathbb{N}$ such that $-r = \min\{i \in \mathbb{Z} \mid V_{\mu+i\alpha} \neq (0)$ and $q = \max\{i \in \mathbb{Z} \mid V_{\mu+i\alpha} \neq (0)$. Hence,

$$W = \bigoplus_{-r\leq i\leq q} V_{\mu+i\alpha}.$$

In addition, for all $i \in \mathbb{Z}$, h_α acts on $V_{\mu+i\alpha}$ by scalar multiplication by $\mu(h_\alpha) + 2i$. All in all, h_α acts diagonally on W, and the set of its eigenvalues is

$$\{\mu(h_\alpha) + 2i,\, i \in \mathbb{Z},\, -r \leq i \leq q,\, V_{\mu+i\alpha} \neq (0)\}.$$

On the other hand, we may consider W as a representation of S_α and, as such, it decomposes as the sum of finitely many irreducible representations of S_α, by Weyl's Theorem. But, S_α is isomorphic to $\mathfrak{sl}_2(\mathbb{k})$ as a Lie algebra, by an isomorphism which sends h_α to h. So, we are in position to use the structure of the finite dimensional irreducible representations of $\mathfrak{sl}_2(\Bbbk)$ (and in particular the eigenvalues of the action of h in these representations) to conclude that, for all $i \in \mathbb{Z}$, $-r \leq i \leq q$, we must have $V_{\mu+i\alpha} \neq 0$. That is,

$$\Pi(\lambda) \cap \{\mu + i\alpha,\, i \in \mathbb{Z}\} = \{\mu + i\alpha,\, i \in \mathbb{Z},\, -r \leq i \leq q\}.$$

On the other hand, we know that the Weyl group stabilises $\Pi(\lambda)$ (cf. Theorem V.2.19). In particular, the reflection σ_α associated to the root α stabilises $\Pi(\lambda)$ and, clearly, it stabilises the set $\{\mu + i\alpha,\, i \in \mathbb{Z}\}$. As, for all $i \in \mathbb{Z}$, $\sigma_\alpha(\mu + i\alpha) = \mu + (-i - \langle \mu, \alpha \rangle)\alpha$, it follows that the map $\mathbb{Z} \longrightarrow \mathbb{Z}$, $i \mapsto -i - \langle \mu, \alpha \rangle$ induces a bijection from $\{i \in \mathbb{Z},\, -r \leq i \leq q\}$ to itself which, clearly is decreasing. Hence, this induced bijection must exchange q and $-r$. From this, we get that $r = q + \langle \mu, \alpha \rangle$. ∎

Theorem V.2.26 – *In the above notation, we have that:*
1. $\Pi(\lambda)$ *is a saturated set of weights of the root system* $(\mathsf{E}_\mathbb{R}, \Phi)$ *of* $\mathfrak{g}$*;*
2. an element $\mu \in \Lambda_\Phi$ *belongs to* $\Pi(\lambda)$ *if, and only if, all the elements* ν *of its* W_Φ*-orbit satisfy* $\nu \preceq \lambda$*.*

Proof. The first point is an immediate consequence of Lemma V.2.25. Therefore, $\Pi(\lambda)$ is a saturated set of weights with highest weight λ, in the sense of Definition III.9.20. The second point then follows using Proposition III.9.14 and Remark III.9.25. ∎