Exterior algebra, exterior powers and representations.

Notation: $\mathbb{k}$ is a commutative field
V is a $\mathbb{k}$-vector space

§1/ Recall the tensor algebra $T(V)$, section IV. 2 of the course. The tensor algebra $T(V)$ is graded by $T(V) = \bigoplus_{i \geq 0} T^i(V)$ where, $\forall i \in \mathbb{N}^*$, $T^i(V)$ is the i-th tensor power of V: $T^i(V) = \underbrace{V \otimes \cdots \otimes V}_{i \text{ copies}}$.

Further: $T^0(V) = \mathbb{k}$.

Let $\mathcal{I}$ be the two-sided ideal of $T(V)$ generated by the elements $v \otimes v$, $v \in V$.
Define the exterior algebra of V to be the quotient algebra:

$$\Lambda(V) = T(V)/\mathcal{I}$$

and let $\pi : T(V) \twoheadrightarrow \Lambda(V)$ be the canonical projection.

For all $n \in \mathbb{N}$, put $\mathcal{I}_n = \mathcal{I} \cap T^n(V)$.
and $\Lambda^n(V) = \pi(T^n(V))$

Since I is generated by homogeneous elts of $T(V)$, we have that :

$$I = \bigoplus_{n \geq 0} I_n .$$

In addition, the $\Bbbk$-lin. map

$$T^n(V) \hookrightarrow T(V) \xrightarrow{\pi} \Lambda(V) \quad , \; n \in \mathbb{N}$$

induces an isomorphism of $\Bbbk$-vector spaces :

$$\forall n \in \mathbb{N} : \qquad T^n(V)/_{I_n} \simeq \Lambda^n(V) .$$

<u>Notation</u>: For all $n \in \mathbb{N}$ and $v_1, \dots, v_n \in V$, we put $v_1 \wedge \dots \wedge v_n = \pi(v_1 \otimes \dots \otimes v_n)$.

<u>Rmk</u>: 1) The set of elts $v_1 \wedge \dots \wedge v_n$, $n \in \mathbb{N}^*$, $v_i \in V$ generates $\Lambda(V)$ as a $\Bbbk$-vector space, together with 1

2) For $n, m \in \mathbb{N}^*$, $\forall\, v_i \;\; 1 \leq i \leq n$, $\forall\, w_j \;\; 1 \leq j \leq m$,

$$(v_1 \wedge \dots \wedge v_n)(w_1 \wedge \dots \wedge w_m)$$
$$= v_1 \wedge \dots \wedge v_n \wedge w_1 \wedge \dots \wedge w_m .$$

3) $I_0 = I_1 = 0$ so that

$$\Bbbk \hookrightarrow T(V) \twoheadrightarrow \Lambda(V)$$
$$V \hookrightarrow T(V) \twoheadrightarrow \Lambda(V)$$

are injective. Hence, $\Bbbk \simeq \Lambda^0(V)$ and $V \simeq \Lambda^1(V)$.

<u>Rmk</u>: Let $n \in \mathbb{N}$, $n \geq 2$, $v_1, \dots, v_n \in V$.

Consider $1 \leq i \leq n-1$. Then:

$$v_1 \otimes \dots \otimes v_{i-1} \otimes (v_i + v_{i+1}) \otimes (v_i + v_{i+1}) \otimes v_{i+2} \otimes \dots \otimes v_n$$

$$= v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_i \otimes v_{i+2} \otimes \dots \otimes v_n$$

$$+ v_1 \otimes \dots \otimes v_{i-1} \otimes v_{i+1} \otimes v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n$$

$$+ v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n$$

$$+ v_1 \otimes \dots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \dots \otimes v_n.$$

Hence: $v_1 \otimes \dots \otimes v_{i-1} \otimes v_i \otimes v_{i+1} \otimes v_{i+2} \otimes \dots \otimes v_n$

$+ \ v_1 \otimes \dots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \dots \otimes v_n$

is an elt of $\mathcal{I}_n$.

It follows easily from this observation that $\mathcal{I}_n$ is the $\mathbb{k}$-span of the pure tensors $v_1 \otimes \dots \otimes v_n$ when $\exists\ 1 \leq i \neq j \leq n$ s.t. $v_i = v_j$.

Expressed in $\Lambda(V)$, the above says that:

$v_1 \wedge \dots \wedge v_n = 0$ whenever $\exists\ 1 \leq i \neq j \leq n \ / \ v_i = v_j$.

<u>Thm</u>: Let $\mathcal{B} = (v_i)_{i \in I}$ be a basis of V indexed by the nonempty set I and suppose a total order on I is given. Then, the set consisting of 1 together with the elts $v_{i_1} \wedge \dots \wedge v_{i_s}$, $s \in \mathbb{N}^*$, $i_1 < \dots < i_s \in I$ and $v_{i_j} \in V$ is a $\mathbb{k}$-basis of $\Lambda(V)$.

<u>Proof</u>: (See [BBK-Algèbre-1-3], Chap. III, §7, no. 8).

Rmk If V is finite dimensional and $d = \dim(V)$, then, $\Lambda(V)$ is finite dimensional. More precisely, if $\{v_1, \dots, v_d\}$ is a basis of V, then for all $1 \leq p \leq n$, the elts $v_{i_1} \wedge \dots \wedge v_{i_p}$ with $1 \leq i_1 < \dots < i_p \leq d$ form a basis of $\Lambda^p(V)$; so that

$$\dim \Lambda^p(V) = \binom{n}{p} = \frac{n!}{p!\,\overline{n-p}!}$$

Further
$$\Lambda^0(V) = \mathbb{k}.1$$
$$\Lambda^p(V) = 0 \qquad \forall\, p > n.$$

The above is a far from exhaustive treatment. For a detailled account on the ext. alg. see [BBK-Algebre-1-3], Chap 3, §7.

§ 2 Exterior powers and representations.

<u>Notation:</u> $\mathbb{k}$ a field

V a $\mathbb{k}$-vectorspace, $\mathfrak{g}$ a Lie algebra over $\mathbb{k}$

(V, ρ) a rep. of $\mathfrak{g}$ in V.

<u>2.1</u>: Let $n \in \mathbb{N}$, $n \geq 2$. It is not difficult to show that the map:

$$\begin{aligned} \tau_n : \mathfrak{g} &\longrightarrow \mathfrak{gl}(T^n(V)) \\ x &\longmapsto \sum_{i=1}^{n} \underbrace{id_V \otimes \dots \otimes id_V}_{i-1} \otimes \rho(x) \otimes \underbrace{id_V \otimes \dots \otimes id_V}_{n-i} \end{aligned}$$

is a morphism of Lie algebras. Put $\tau_0 : \mathfrak{g} \to \mathfrak{gl}(T^0(V))$ the triv. rep.

This generalises Ex. I.2.6 1. of the course in the special case where $V = V'$, $\rho = \rho'$.

Taking the direct sum of these representations gives a representation of $\mathfrak{g}$ in $T(V)$:

$$\begin{aligned} \tau : \mathfrak{g} &\longrightarrow \mathfrak{gl}(T(V)) \\ x &\longmapsto \bigoplus_{n \geq 0} \tau_n(x) \end{aligned}$$

<u>Obs.</u>: It is not difficult to prove that, $\forall t, t' \in T(V)$, $\forall x \in \mathfrak{g}$ $\tau(x)(t t') = \tau(x)(t) . t' + t\, \tau(x)(t')$. That is, $\forall x \in \mathfrak{g}$ $\tau(x)$ is a derivation of $T(V)$. (Check first the above identity on pure tensors and then

extend it to all the elts.)

<u>2.2</u> Recall the ideal $\mathcal{I}$ introduced earlier:

$$\mathcal{I} = \langle v \otimes v,\ v \in V \rangle \trianglelefteq T(V).$$

<u>Rmk</u> Let $x \in \mathfrak{g}$, $v \in V$. Expanding $\left(\tau(x)(v) + v\right) \otimes \left(\tau(x)(v) + v\right)$, it is easy to show that

$$\tau(x)(v \otimes v) \in \mathcal{I}_2$$

It follows, using the fact that $\tau(x)$ is a derivation, that $\tau(x)(\mathcal{I}) \subseteq \mathcal{I}$.
Therefore, we get a representation of $\mathfrak{g}$ on $\Lambda(V) := T(V)/_{\mathcal{I}}$. It is defined by:

$$\begin{aligned} \varepsilon : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\Lambda(V)) \\ x &\longmapsto \varepsilon(x) \end{aligned}$$

where, for all $p \geq 1$ and all $v_1, \dots, v_p \in V$,

$$\varepsilon(x)(v_1 \wedge \dots \wedge v_p) = \sum_{1 \leq i \leq p} v_1 \wedge \dots \wedge v_{i-1} \wedge \rho(x)(v_i) \wedge v_{i+1} \wedge \dots \wedge v_p$$

Clearly, $\forall p \in \mathbb{N}$, $\Lambda^p(V)$ is a subrepresentation of $\Lambda(V)$.