NLT 26/11/20

Fundamental rep. of $sl_n(\mathbb{k})$

$\mathbb{k}$ is a field, $char(\mathbb{k}) = 0$.
$n \in \mathbb{N}$, $n \geq 2$

§1 Setup

We work in $gl_n(\mathbb{k})$.

Denote by e_{ij} the usual elementary matrix, $1 \leq i, j \leq n$.
Denote by e_i, $1 \leq i \leq n$, the elts of the canonical basis of $\mathbb{k}^n$.
Denote by $diag_n(\mathbb{k})$ the sub Lie alg. of $gl_n(\mathbb{k})$ of diag. matrices. Put:

$$\mathfrak{g} = sl_n(\mathbb{k}) \quad , \quad \mathfrak{h} = \mathfrak{g} \cap diag_n(\mathbb{k}).$$

Put $h_i = e_{ii}$, $1 \leq i \leq n$, so that $(h_i)_i$ is a basis of $diag_n(\mathbb{k})$ & let $(h_i^*)_i$ be the dual basis. Hence $h_i^*(h_j) = \delta_{ij}$, $1 \leq i, j \leq n$.
We have the obvious injection $\mathfrak{h} \xrightarrow{\subseteq} diag_n(\mathbb{k})$ and its transpose $(diag_n(\mathbb{k}))^* \longrightarrow \mathfrak{h}^*$ which sends a linear form on $diag_n(\mathbb{k})$ to its restriction to $\mathfrak{h}$.
Put $\varphi_i = (h_i^*)_{/\mathfrak{h}}$, $1 \leq i \leq n$ the rest. of h_i^* to $\mathfrak{h}$.
Recall the CC dec. of $\mathfrak{g}$: $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+$ w.r.t. $\mathfrak{h}$, where $\mathfrak{h} = \bigoplus_{i=1}^{n-1} \mathbb{k}(h_i - h_{i+1})$

$$n = \bigoplus_{1 \leq i < j \leq n} \mathbb{k} e_{ij} \quad , \quad n^- = \bigoplus_{1 \leq j < i \leq n} \mathbb{k} e_{ij}$$

Recall that the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ are the $\varphi_k - \varphi_\ell$ with $1 \leq k \neq \ell \leq n$ and that

$$\mathfrak{g}_{\varphi_k - \varphi_\ell} = \mathbb{k}\, e_{k\ell}$$

$$\Phi = \Phi^+ \sqcup \Phi^-$$

with $\Phi^+ = \{ \varphi_k - \varphi_\ell \,,\ 1 \leq k < \ell \leq n \}$.

Observation: It is clear that $\mathrm{diag}_n(\mathbb{k}) = \mathfrak{h} \oplus \mathbb{k} I_n$.

For a subset E of $\{1, \dots, n\}$, put

$$\varphi_E = \sum_{i \in E} \varphi_i \in \mathfrak{h}^*$$

and

$$h_E^* = \sum_{i \in E} h_i^* \in (\mathrm{diag}_n(\mathbb{k}))^*$$

Now, consider two subsets E, F of $\{1, \dots, n\}$ with the same cardinality. Then it is clear that the three following statements are equiv.

(i) $h_E^* = h_F^*$

(ii) $\varphi_E = \varphi_F$

(iii) $E = F$

Indeed, h_E^* and h_F^* take the same value on the identity matrix so that (i) $\Leftrightarrow$ (ii).

§2 The natural rep. of $\mathfrak{g}$.

The obvious map $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathbb{k}^n)$ makes $\mathbb{k}^n$ into a rep. of the Lie alg. $\mathfrak{g}$. This is called the natural rep. of $\mathfrak{g}$.
The action of $\mathfrak{g}$ on the canonical basis of $\mathbb{k}^n$ is given by:

$$\varphi_j(h_i - h_{i+1})\, e_j$$

$$* \quad (h_i - h_{i+1})(e_j) = \delta_{ij}\, e_j - \delta_{i+1\, j}\, e_j$$

$$* \quad e_{k\ell}\,(e_j) = \delta_{\ell j}\, e_k$$

Therefore, the elts of the canonical basis are all weight vectors relative to $\mathfrak{h}$. It follows from the above formulas that e_1 is a highest weight vector of weight φ_1, which generates $\mathbb{k}^n$. Hence the natural rep. is a highest weight rep. of highest weight φ_1.

Put $V = \mathbb{k}^n$. The above formulas show that

$$V = \bigoplus_{i=1}^{n} V_{\varphi_i}$$

where $V_{\varphi_i} = \mathbb{k} e_i$. Notice that, here, we use the fact that the φ_i, $1 \le i \le n$ are pairwise distinct, which follows from the obs. made in §1.

Actually, the rep. $\mathbb{k}^n$ above is irreducible. This can be shown by playing with the formulas giving the action on basis elts of the $e_{k\ell}$, $1 \leq k \neq \ell \leq n$. But it is more direct and elegant to use the fact that weight spaces are all one dimensional. Indeed, supp. $W \subseteq \mathbb{k}^n$ is a subrepresentat° Then, W must be the direct sum of its weight spaces (Theo. V.2.6, 4). Therefore, if $W \neq (0)$, it must contain an elt elt e_i for some $i \in \{1, -, n\}$. It is then easy to see that W must contain all the e_j, $1 \leq j \leq n$, by letting n^+ and n^- act on this e_i. Hence $W = V$.

<u>Concl.:</u> the natural rep. of $sl_n(\mathbb{k})$ is irreducible and a highest weight rep. of weight φ_1, generated by the highest weight vector e_1 of weight φ_1.

§3 Exterior powers of the natural representation.

Fix $p \in \mathbb{N}$, $1 \leq p \leq n$. Recall that $V = \mathbb{k}^n$ is the natural rep. of $\mathfrak{g}$. We consider the rep. of $\mathfrak{g}$ in $\Lambda^p(V)$.

For all subset E of cardinality p of $\{1, \dots, n\}$, we put $e_E = e_{i_1} \wedge \dots \wedge e_{i_p}$ where $E = \{e_{i_1}, \dots, e_{i_p}\}$ and $i_1 < \dots < i_p$. Then, $\{e_E, E \subseteq \{1, \dots, n\}, |E| = p\}$ is a $\mathbb{k}$-basis of the vector space $\Lambda^p(V)$.

Recall that, $\forall x \in \mathfrak{g}$, $\forall v_1, \dots, v_p \in V$, then

$$x \cdot v_1 \wedge \dots \wedge v_p = \sum_{i=1}^{p} v_1 \wedge \dots \wedge v_{i-1} \wedge x.v_i \wedge v_{i+1} \wedge \dots \wedge v_p .$$

Recall also that, $\forall v_1, \dots, v_p \in V$, $v_1 \wedge \dots \wedge v_p = 0$ whenever $\exists\, 1 \leq i \neq j \leq p$ such that $v_i = v_j$ and that $\dots \wedge v_i \wedge \dots \wedge v_j \wedge \dots = \pm \dots \wedge v_j \wedge \dots \wedge v_i \wedge \dots$.

From all this, it follow by explicit computations on the basis of the formulas for the action of $\mathfrak{g}$ on V that $\overset{\forall 1 \leq i \leq n-1}{,\vee,}$ $\forall\, 1 \leq k \neq l \leq n$, $\forall E \subseteq \{1, \dots, n\}$, $|E| = p$: with $E = \{i_1, \dots, i_p\}$, $i_1 < \dots < i_p$:

$$e_{kl}(e_E) = \begin{cases} 0 & \text{if } l \notin I \\ \pm e_{E \setminus \{l\} \sqcup \{k\}} & \text{if } l \in I, k \notin I \\ 0 & l \in I, k \in I \end{cases}$$

$$(h_i - h_{i+1})(e_E) = \left(\sum_{j \in E} q_j\right)(h_i - h_{i+1})\, e_E$$

Using the observation in §1, we get that the elements $\varphi_E = \sum_{j \in E} \varphi_j$, E a subset of $\{1, \dots, n\}$ with p elts, are pairwise distinct. Therefore

$$\Lambda^p(V) = \bigoplus_E V_{\varphi_E}$$

and $V_{\varphi_E} = \mathbb{k}\, e_E$.

In addition, a clever use of the expression of $e_{kl}(e_E)$ above shows that V is generated by $e_1 \wedge \dots \wedge e_p$ as a representation since any basis elt e_E may be reached from $e_1 \wedge \dots \wedge e_p$ by applying convenient elts $e_{kl} \in \mathfrak{n}^+$. Further $e_1 \wedge \dots \wedge e_p$ is a h.w.v. of weight $\varphi_1 + \dots + \varphi_p$.

Finally, as in §2, Theorem V.2.6 4) apply to show that $\Lambda^p V$ is irreducible (since weight spaces are one-dimensional).

<u>Concl.:</u> for $1 \leq p \leq n$, $\Lambda^p(V)$ is irreducible and a highest weight representation of weight $\varphi_1 + \dots + \varphi_p$, generated by the highest weight vector $e_1 \wedge \dots \wedge e_p$ of weight $\varphi_1 + \dots + \varphi_p$.

<u>Rmk:</u> For $1 \leq p \leq n-1$, the rep. $\Lambda^p(V)$ is called fundamental since, as one can easily show (cf. Rmk III.9.12, 2)), the associated weights are the fundamental weights in the abstract sense.