NdT 18/12/20

Weight diagrams for $\mathfrak{sl}_3(\mathbb{k})$

$\mathbb{k}$ is an alg. closed field of charact. 0.
We will consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(\mathbb{k}^3) \simeq \mathfrak{sl}_3(\mathbb{k})$. All the notation is taken from the note "L'algèbre de Lie $\mathfrak{sl}_n(\mathbb{k})$". The Lie subalg. $\mathfrak{h}$ of $\mathfrak{sl}_3(\mathbb{k})$ consisting of diag. matrices is a maximal toral subalgebra.
The set of roots for the pair $(\mathfrak{g}, \mathfrak{h})$ is then

$$\Phi = \{ \varphi_i - \varphi_j \;,\; 1 \leq i \neq j \leq 3 \}.$$

A basis Δ of Φ is then

$$\Delta = \{ \varphi_1 - \varphi_2 \,,\, \varphi_2 - \varphi_3 \}$$

If we put $\alpha_1 = \varphi_1 - \varphi_2$, $\alpha_2 = \varphi_2 - \varphi_3$ we then have the C.Ç. decomp.

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

where
$$\mathfrak{n}^+ = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_3}$$
$$\mathfrak{n}^- = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_3}$$

and $\alpha_3 = \alpha_1 + \alpha_2$.

Further, $\mathfrak{g}_{\alpha_1} = \mathbb{k}\, e_{12}$, $\mathfrak{g}_{\alpha_2} = \mathbb{k}\, e_{23}$, $\mathfrak{g}_{\alpha_3} = \mathbb{k}\, e_{13}$, ...

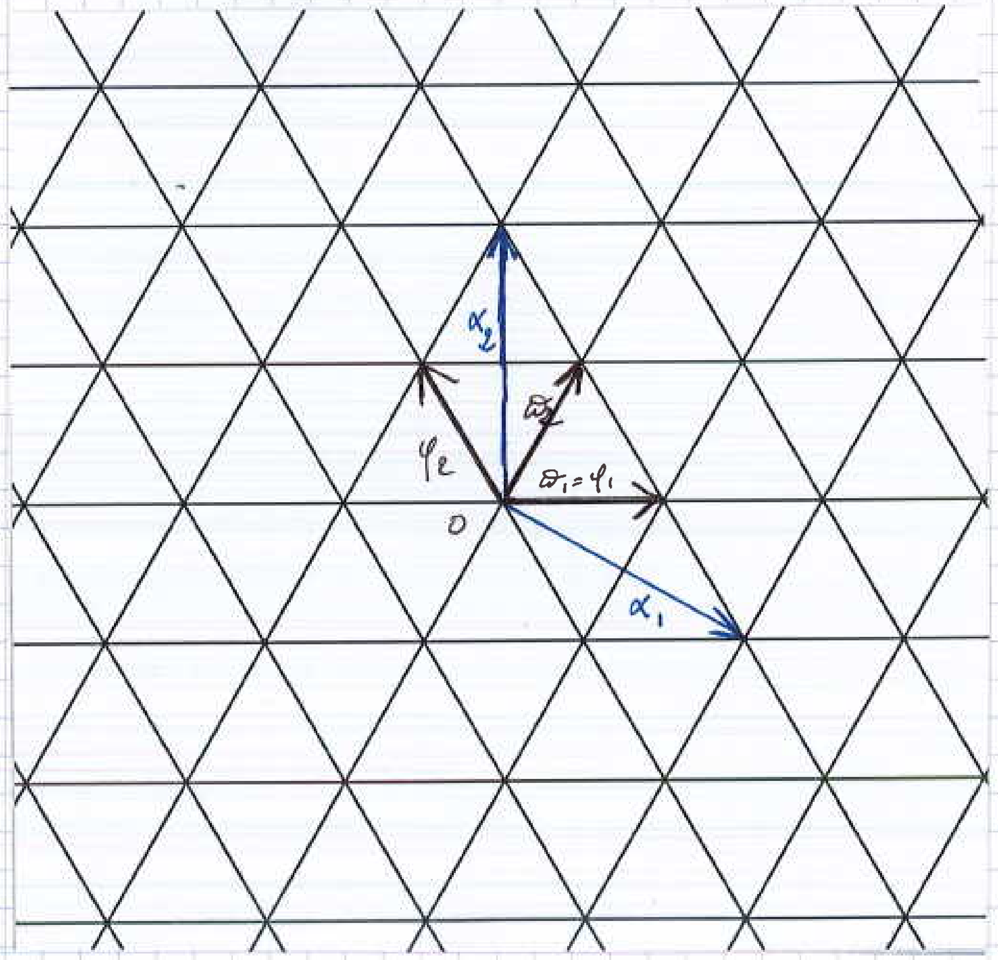
Then, the Euclidean space attached to $(\mathfrak{g}, \mathfrak{h})$ is of dimension 2 with basis α_1, α_2. The corresponding root system is of type A_2 with Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ so that (see Rmk III.9.12) the fundamental dominant weights are ϖ_1, ϖ_2

and :
$$\begin{cases} \alpha_1 = 2\varpi_1 - \varpi_2 \\ \alpha_2 = -\varpi_1 + 2\varpi_2 \end{cases}$$

An easy computation then gives

$$\begin{cases} \varpi_1 = \varphi_1 \\ \varpi_2 = \varphi_1 + \varphi_2 \end{cases}$$

Therefore, the various basis at our disposal may be represented as follows in the Euclidean space $E_{\mathbb{R}}$:



Now, we know that the set Λ^+ of dominant weights is $\Lambda^+ = \mathbb{N}\varpi_1 \oplus \mathbb{N}\varpi_2$ and that this set parametrises the iso classes of irreducible f.d. rep. of $\mathfrak{sl}_3(\mathbb{k})$. Combining this gives a bijection as follows:

$$\begin{aligned} \mathbb{N}^2 &\longrightarrow \mathrm{Irrep}(\mathfrak{g})/_\sim \\ (a,b) &\longmapsto \Gamma_{a,b} := V(a\varpi_1 + b\varpi_2) \end{aligned}$$

(or rather its iso. class.)

§1 The natural representation and its sym. powers

$V = \mathbb{k}^3$ is a rep. of $\mathfrak{g}$ in the natural way by: $\mathfrak{g} \xrightarrow{\subseteq} \mathfrak{gl}(V)$.

Now, $V = \mathbb{k}e_1 \oplus \mathbb{k}e_2 \oplus \mathbb{k}e_3$ and the action of the canonical generators of $\mathfrak{g}$ is clear. By §2 of the note "Fundamental rep. of $\mathfrak{sl}_n(\mathbb{k})$", we know that:

$$V = V_{\varphi_1} \oplus V_{\varphi_2} \oplus V_{\varphi_3} \qquad (1.1)$$

where $V_{\varphi_i} = \mathbb{k}e_i$, $1 \leq i \leq 3$.

Recall from §1, Obs., of that same note that $\varphi_1, \varphi_2, \varphi_3$ are pairwise distinct. So (1.1) is the dec. of V in weight spaces and, since $\mathfrak{n}^+ e_1 = 0$, we get that V is a highest weight rep. of weight φ_1. That is: $\Gamma_{(1,0)} \cong V$.

Let now $a \in \mathbb{N}^*$. We want to consider the rep. $S^a(V)$ of $\mathfrak{g}$. (Recall the def. of sym. power of a rep. from the note "Sym. algebras, sym. powers and representations".)

We know that the elts $e_1^u e_2^v e_3^w \in S(V)$ such that $u+v+w=a$ is a basis of the $\mathbb{k}$-vector space $S^a(V)$:

$$S^a(V) = \bigoplus_{\substack{u,v \in \mathbb{N} \\ u+v \leq a}} \mathbb{k}\, e_1^u e_2^v e_3^{a-u-v} \qquad (1.2)$$

Also, by the definition of the action of $\mathfrak{g}$ on $S^a(V)$, we have that:

$e_1^u e_2^v e_3^{a-u-v}$ is a weight vector of weight $u\varphi_1 + v\varphi_2 + (a-u-v)\varphi_3$.

$$= u\varphi_1 + v\varphi_2 + (a-u-v)(-\varphi_1 - \varphi_2)$$

$$= (2u+v-a)\varphi_1 + (u+2v-a)\varphi_2$$

(Recall that $\varphi_1 + \varphi_2 + \varphi_3 = 0$ since φ_i is the restriction to $\mathfrak{h}$ of the elt h_i^*, where $(h_i^*)_{1\leq i\leq 3}$ is the basis of $\mathrm{diag}_3(\mathbb{k})$ dual to $(h_i)_{1\leq i\leq 3}$. In other terms: $\varphi_i : \mathfrak{h} \to \mathbb{k}$ is the map that reads the i-th coordinate of elts of $\mathfrak{h}$.)

In particular: $e_1^a \in S^a(V)$ is a weight elt of weight $a\varphi_1$.

Moreover, it is easy to see that $\mathfrak{n}^+ . e_1^a = 0$. (6)

Indeed, $\forall x \in \mathfrak{n}^+$, $x . e_1 = \sum_{i,j} \underbrace{e_1 \ldots e_1}_{i} (x.e_1) \underbrace{e_1 \ldots e_1}_{j}$

$= 0$

So, e_1^a is a highest weight vector of weight $a\varphi_1$ of $S^a(V)$.

We now show that $S^a(V)$ is actually irreducible. First, notice that the map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $(u,v) \longrightarrow (2u+v-a, u+2v-a)$ is an affine map whose linear part is invertible. So this map is a bijection. Hence, the weights $(2u+v-a)\varphi_1 + (u+2v-a)\varphi_2$, $(u,v) \in \mathbb{N}^2$, are pairwise distinct. Therefore, (1.2) gives the weight decomposition of $S^a(V)$; namely

$$S^a(V) = \bigoplus_\mu S^a(V)_\mu$$

where μ runs over the set $\{(2u+v-a)\varphi_1 + (u+2v-a)\varphi_2,$ $u,v \in \mathbb{N}, u+v \le a\}$ and, for $\mu = (2u+v-a)\varphi_1 + (u+2v-a)\varphi_2$, $S^a(V)_\mu = \mathbb{k} . e_1^u e_2^v e_3^{a-u-v}$.

Proceeding as in the note "Fundamental rep. of $sl_n(\mathbb{k})$", §2, and using Theorem V.2.6, 4), we deduce that any nonzero subrepresentation of $S^a(V)$ must contain an elt of the form $e_1^u e_2^v e_3^{a-u-v}$.

At this stage, we know that if W is a subrep. of $S^a(V)$, $W \neq 0$, then there exist integers u, v, w s.t. $e_1^u e_2^v e_3^w \in W$ and $u+v+w=a$.

Then, by definition of the action of $\mathfrak{g}$ on $S^a(V)$:

$$e_{12} \cdot e_1^u e_2^v e_3^w = v.\, e_1^{u+1} e_2^{v-1} e_3^w$$

Therefore,

$$e_{12}^v \cdot e_1^u e_2^v e_3^w = v!\; e_1^{u+v} e_3^w$$

Similarly,

$$e_{13}^w \cdot e_1^{u+v} e_3^w = v!\, w!\; e_1^{u+v+w} = v!\, w!\; e_1^a.$$

So: successive applications of e_{12} and e_{13} sends $e_1^u e_2^v e_3^w$ to (a multiple of) e_1^a.
Therefore $e_1^a \in W$ and $S^a(V) = W$.
We have proved that V is irreducible.

<u>Concl.</u>: $S^a(V)$ is an irreducible rep. of $\mathfrak{g}$ of highest weight $a\varphi_1$; that is:

$$\Gamma_{(a,0)} \cong S^a(V)$$

(since $\varphi_1 = \varpi_1$).

§ 2 The dual of the nat. rep. and its sym. powers

We want to consider the rep. V^*, where $V = k^3$ is the natural representation. Let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis of the canon. basis of V. Then, for $1 \le i \le n$, since e_i has weight φ_i, e_i^* has weight $-\varphi_i$ (see the note "Dual representation and weight"). Moreover, by definition of the dual of a representation, we have:

$$e_{ij} \cdot e_k^* = - \delta_{ik} e_j^* \qquad (2.1)$$

In particular $e_{12} \cdot e_3^* = e_{23} e_3^* = 0$. So $n^+ e_3^* = 0$. That is: e_3^* is a highest weight vector of weight $-\varphi_3 = \varphi_1 + \varphi_2 = \varpi_2$.

At this stage, we know that V^* has three distinct weights, all of which has weight space of dim. 1. So, again, a nonzero sub. rep. of V^* must contain e_1^* or e_2^* or e_3^*. But then, (2.1) shows it contains necessarily e_3^*. That is: V^* is an irred. rep. of highest weight ϖ_2:

$$V^* \cong \Gamma_{(0,1)} .$$

Let now $b \in \mathbb{N}$. By the same method as the one we applied in §1 with the natural rep. it can be shown that $S^b(V^*)$ is an irreducible representation of highest weight $b\varpi_2$ and that $(e_3^*)^b \in S^b(V^*)$ is a highest weight vector of weight $b\varpi_2$.
Therefore:

$$S^b(V^*) \simeq \Gamma_{(a,b)} .$$

§3 How to get $\Gamma_{(a,b)}$ for all $(a,b) \in \mathbb{N}^2$?

Let $(a,b) \in \mathbb{N}^2$. We know from §§1 and 2 that:

1) $S^a(V)$ is a highest weight rep. of weight $a\varpi_1$
2) $S^b(V^*)$ " " " " " $b\varpi_2$.

Hence, by the note "Tensor product and weights", we deduce that $\Gamma_{a,b}$ appears as a subrep. of $S^a(V) \otimes S^b(V^*)$.

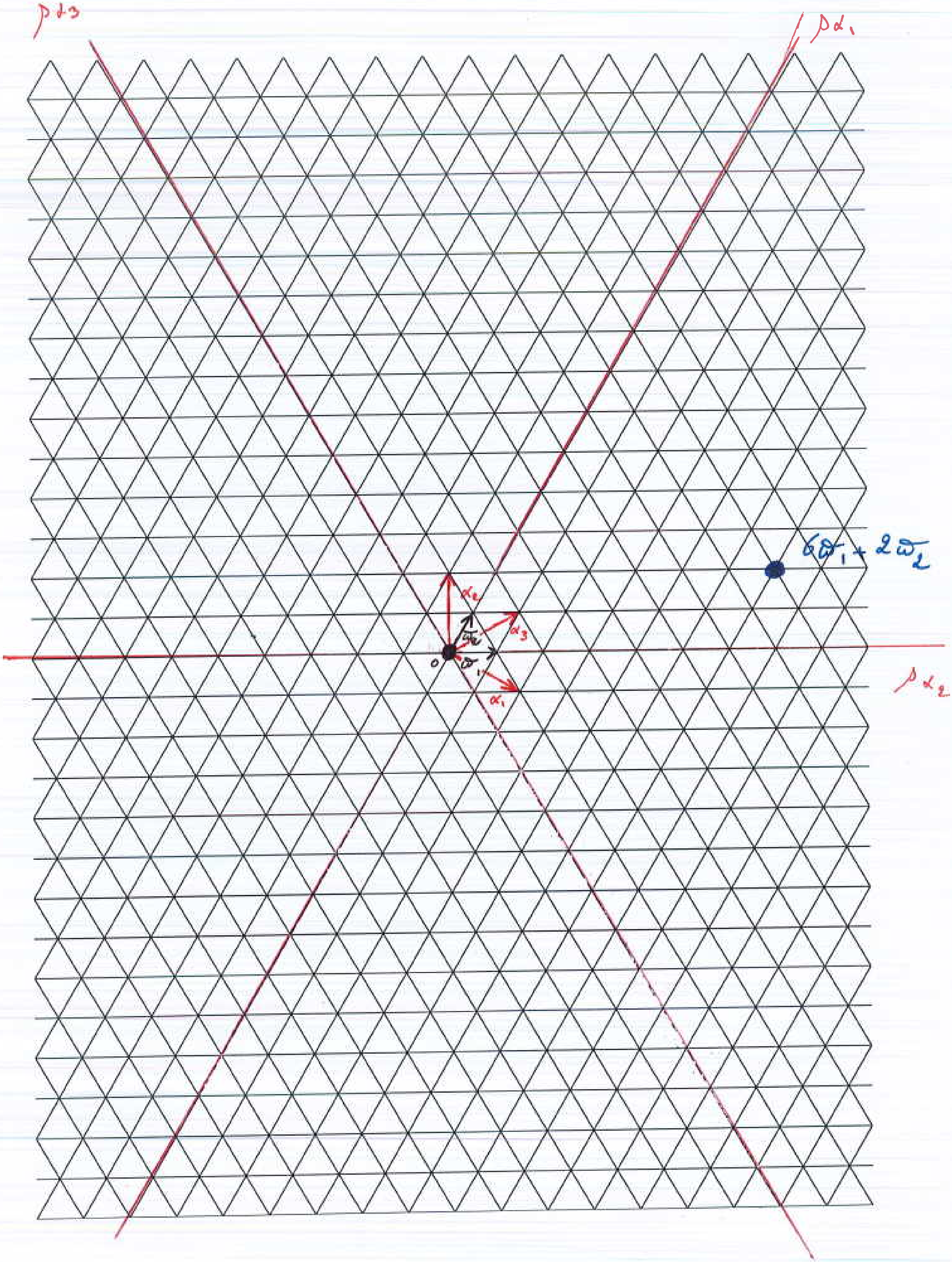
More precisely, since $e_1^a \in S^a(V)$ is a h.w. vector of weight $a\varpi_1$ and $(e_3^*)^b \in S^b(V^*)$ a h.w. vector of weight $b\varpi_2$, then $e_1^a \otimes (e_3^*)^b \in S^a(V) \otimes S^b(V^*)$ is a highest weight vector of weight $a\varpi_1 + b\varpi_2$.
Hence, the subrep. it generates in $S^a(V) \otimes S^b(V^*)$ is iso. to $\Gamma_{a,b}$.

Actually, a very satisfactory description of $\Gamma_{a,b}$ can be given. In the sequel we illustrate some of the arguments on examples.

§4. Weight diagram for $\Gamma_{6,2}$.

In this paragraph, we show how to compute the set of weights for an irreducible f. dim. rep. of $\mathfrak{g}$. We deal with the example of $\Gamma_{6,2} = V(6\varpi_1 + 2\varpi_2)$. The generalisation to any $\Gamma_{a,b}$ should be clear.
The set Π of those weights will be represented on a "weight diagram". We start from the lattice of (abstract) weights Λ in $E_{\mathbb{R}}$; it is just the free subgroup $\Lambda = \mathbb{Z}\varpi_1 + \mathbb{Z}\varpi_2$.
We will also make use of the reflections associated to $\alpha_1, \alpha_2, \alpha_3$. Therefore, we start from the empty frame as follows, where the vectors $\varpi_1, \varpi_2, \alpha_1, \alpha_2, \alpha_3$ are indicated, together with the hyperplanes orthogonal to the roots. The blue dot locates (the extremity of) the vector $6\varpi_1 + 2\varpi_2$. Put $\lambda = 6\varpi_1 + 2\varpi_2$.

<u>Obs.</u>: we know from Rmk V.2.17 that $\Pi \subseteq \Lambda$. Therefore, elts of Π will be represented on the diag. by dots which will all be at the intersections of the various black lines of the basic frame. Such points will be called "integral" to simplify.

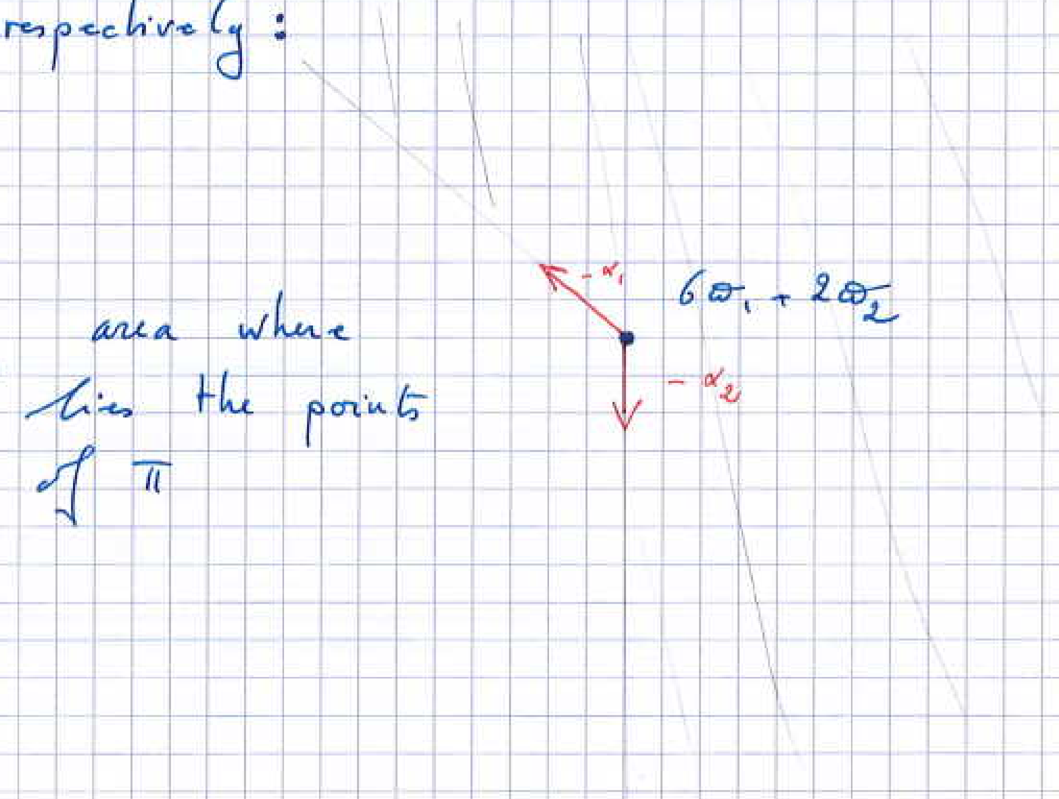


sα3
sα1
sα2
α1
α2
α3
ϖ1
ϖ2
0
6ϖ1 + 2ϖ2

<u>1st step</u>: By theorem V.2.6, we know that the elts of π lie in the set

$6\varpi_1 + 2\varpi_2 - \mathbb{N}\Delta$

which is the set of integral points in the half-cone oriented south west and delimited by the 2 lines containing the extremity of the vector $6\varpi_1 + 2\varpi_2$ and with direction $-\alpha_1$ and $-\alpha_2$, respectively:



Now, we compute the two following sets which are called the α_1-string through λ:

$\pi \cap \{\lambda + i\alpha_1,\ i \in \mathbb{Z}\}$

and the α_2-string through λ:

$\pi \cap \{\lambda + i\alpha_2,\ i \in \mathbb{Z}\}$.

To this aim, we use Lemma V.2.25.

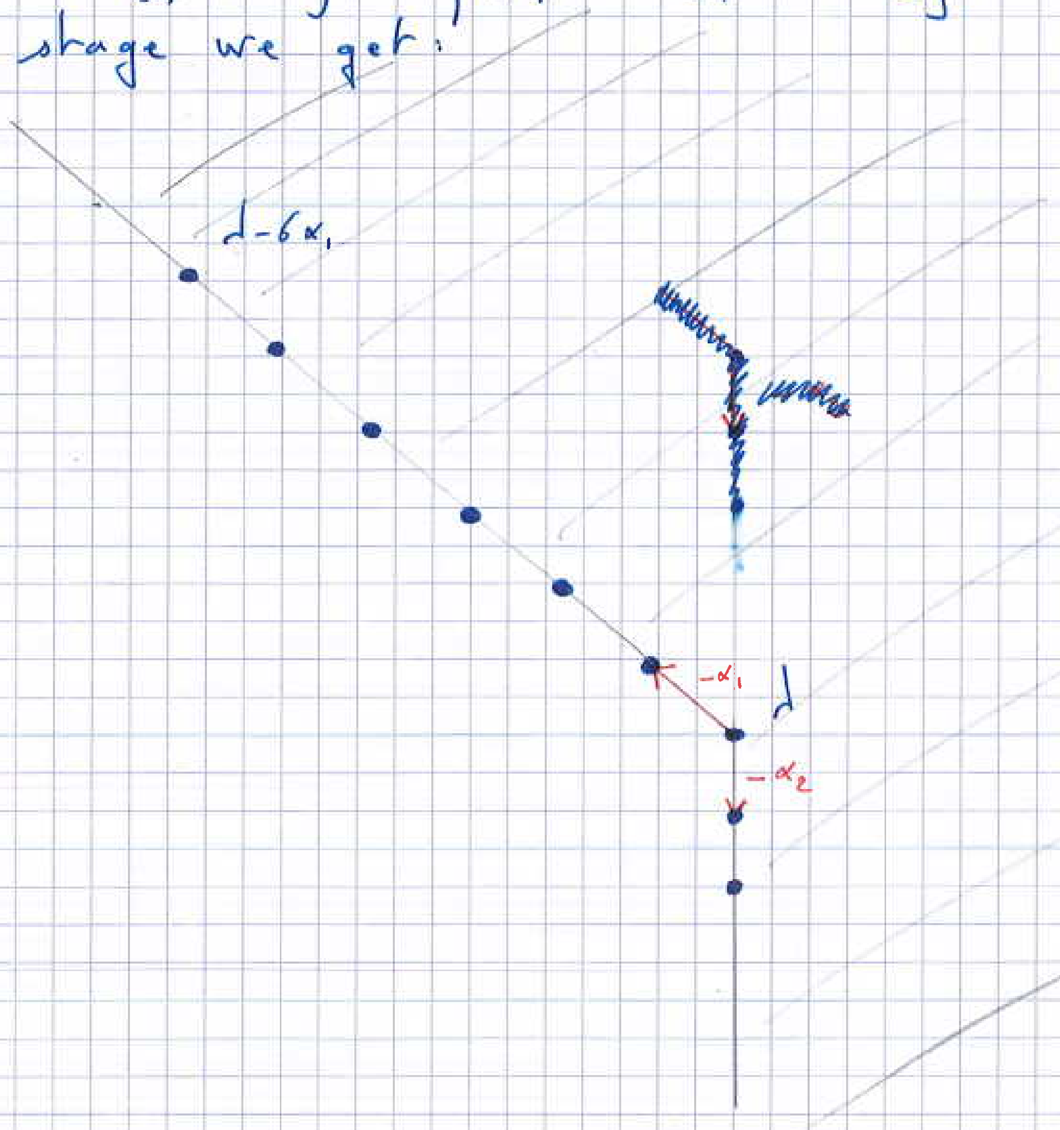
By definition of the fundamental weights, we have that $\langle \lambda, \alpha_1 \rangle = \langle 6\varpi_1 + 2\varpi_2, \alpha_1 \rangle = 6\langle \varpi_1, \alpha_1 \rangle + 2\langle \varpi_2, \alpha_2 \rangle = 6$. Now, applying Lemma V.2.25, and using its notation, we get that $q = 0$ (by the first observation of this step) and therefore $r = 6$. Hence, we have that

$$\Pi \cap \{\lambda + i\alpha_1, i \in \mathbb{Z}\} = \{\lambda, \lambda - \alpha_1, \lambda - 2\alpha_1, \dots, \lambda - 6\alpha_1\}$$

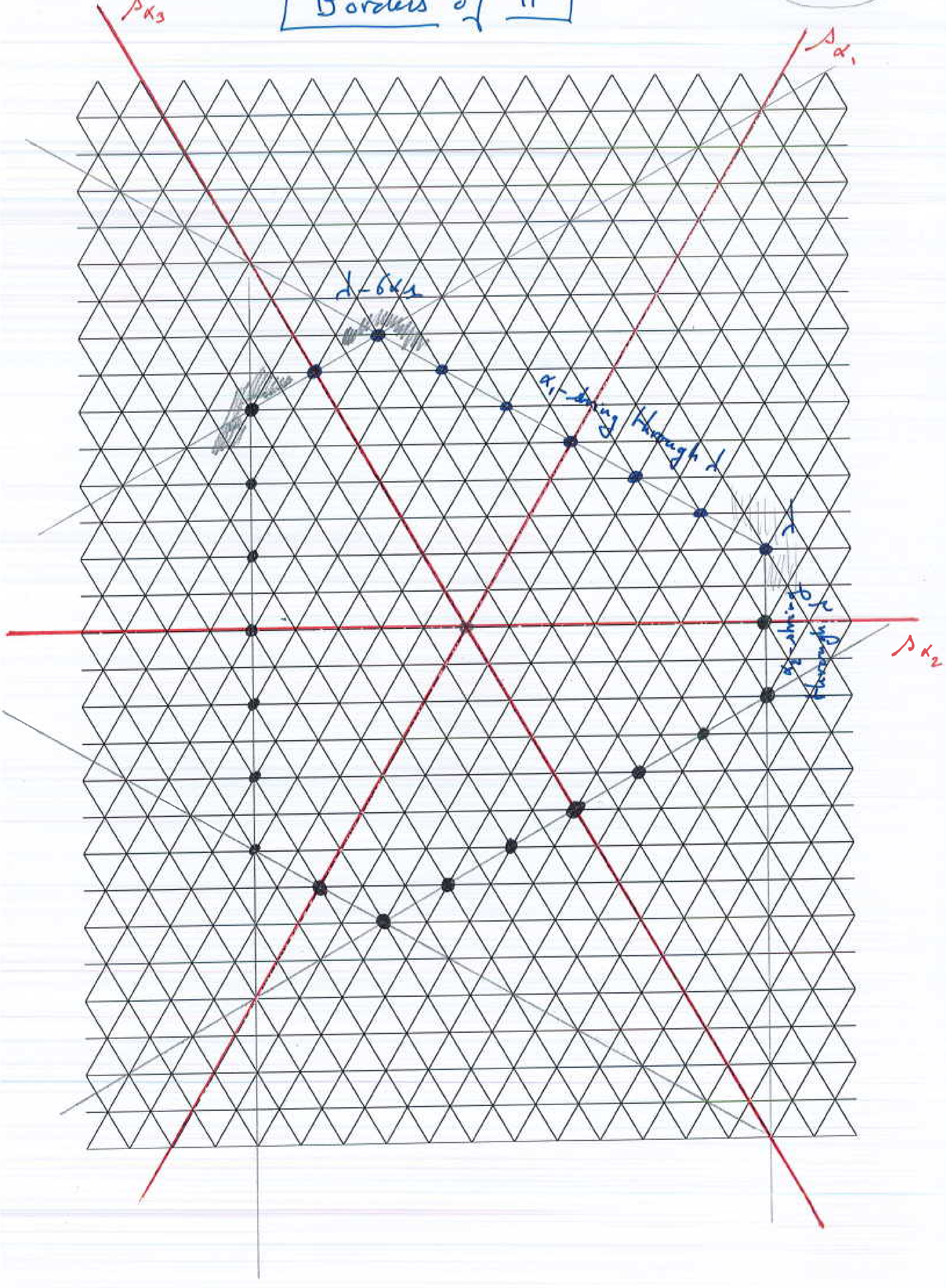
giving us seven weights (see the "border" diagram below). Similarly, we get

$$\Pi \cap \{\lambda + i\alpha_2, i \in \mathbb{Z}\} = \{\lambda, \lambda - \alpha_2, \lambda - 2\alpha_2\}.$$

At this stage we get:



Borders of Π


s_{α_3}
s_{α_1}
s_{α_2}
$\lambda - 6\alpha_1$
α_1-string through λ
λ
α_2-string through λ

2nd step_ By Theorem V.2.19 (and V.2.16), we know that the set Π is stable under the Weyl group W. Here W is generated by s_{α_1} and s_{α_2} and contains s_{α_3}. From this, it follows that the elements of Π must all lie inside and on the border of the hexagon depicted in the following picture "borders of Π".

3rd step_ We now want to describe Π.

Observation: We know that $\Pi \subseteq \lambda - \mathbb{N}\Delta$. In other terms, an element μ of Π must satisfy $\lambda - \mu \in \mathbb{N}\Delta$. In particular, an elt μ of Π must be congruent to λ modulo the root lattice(*) Λ_r. This, for example exclude the possibility to have $6\varpi_1 \in \Pi$. Indeed $\lambda - 6\varpi_1 = 2\varpi_2 \notin \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$, as one can easily see (cf. $\{\alpha_1, \alpha_2\}$ is a basis of $E_{\mathbb{R}}$) using the first display on page 2.

That is: among integral points lying on the border or inside the hexagon, only those who are congruent to λ modulo Λ_r have a chance to belong to Π.

What actually turns out is that <u>all</u> the points on the border or inside the hexagon <u>and</u> congruent to λ modulo Λ_r are in Π. That is: Π is the intersection of $\lambda + \Lambda_r$ with the area delimited by the hexagon.

(*) Def. III.9.5. Here, $\Lambda_r = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$.

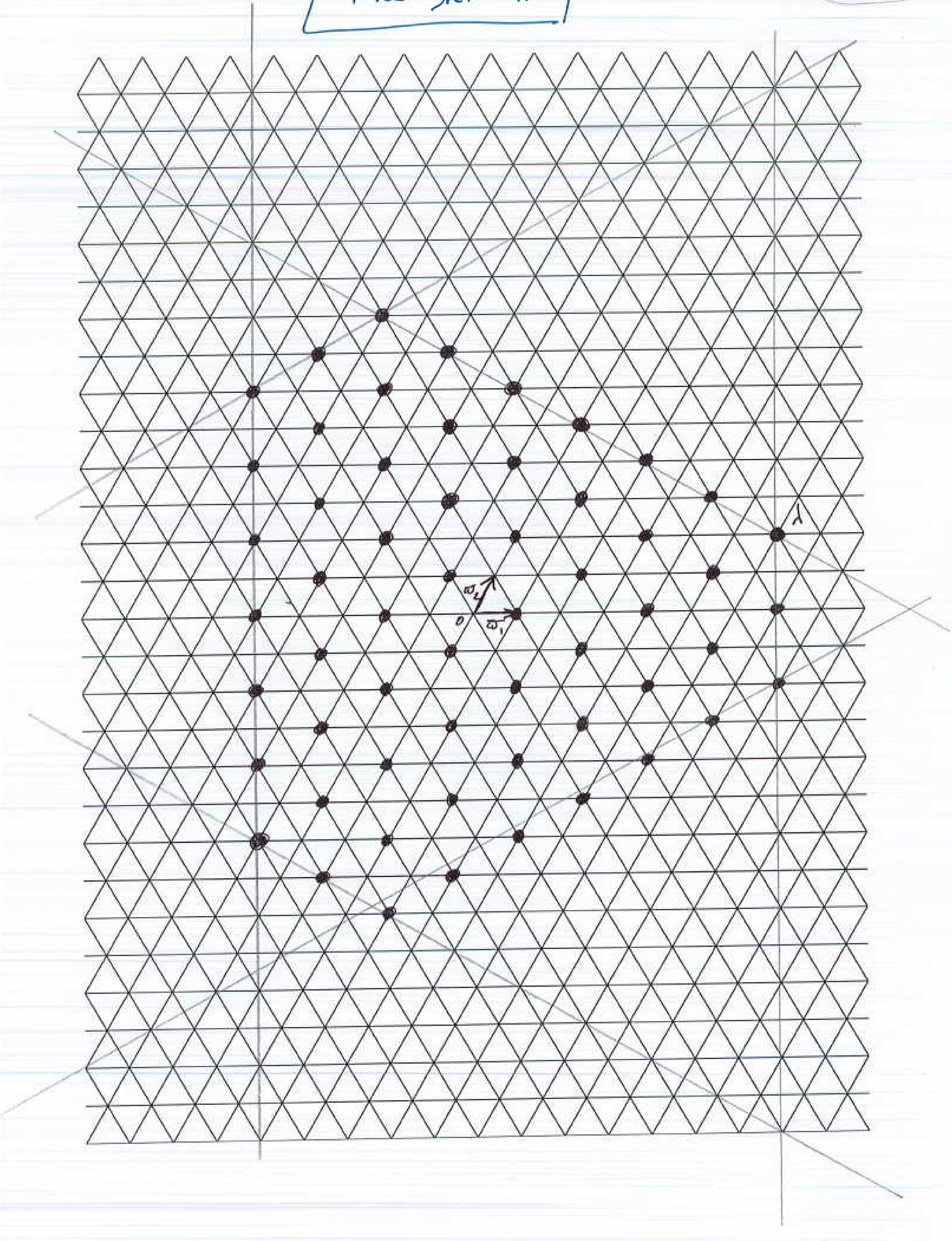
Here is an illustration on an example of how to prove it.

Consider $\mu = \lambda - \alpha_1$. This is an element of Π since it is in the α_1-string through in λ. Since Π is left invariant by s_{α_2}, $s_{\alpha_2}(\lambda - \alpha_1) = (\lambda - \alpha_1) - 3\alpha_2 \in \Pi$. So μ and $\mu - 3\alpha_2$ are elts of the α_2-string through μ. By Lemma V.2.25, we deduce that $\mu - \alpha_2$ and $\mu - 2\alpha_2$ are also in Π.

Playing this game with well chosen elts of the border of the hexagon gives the result.

<u>Conclusion:</u> Π is the set of elts which are congruent to λ modulo Λ_r and in the hexagon.

The set Π



<u>Bibliographie</u>: Very interesting complements can be found in

* W. Fulton, J. Harris. Representation Theory, a first course. GTM 129, New-York, Springer-Verlag 1991.
  See chap 12, 13.
* B. C. Hall. Lie groups, Lie algebras and Representations. An elementary introduction. 2nd ed. GTM 222, Springer, 2015.
  See chap. 6.