Tensor product and related constructions.

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tensor-product.tex

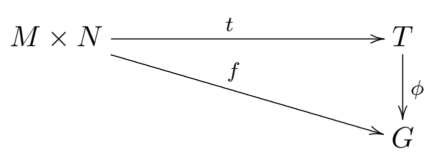
1 Tensor product in the general framework.

1.1 Tensor product in the noncommutative setting.

In all this subsection, R is a ring.

Definition 1.1.1 – *Let M be a right R-module, N be a left R-module and G be an abelian group. A balanced map from $M \times N$ to G is a map $f : M \times N \longrightarrow G$ such that, for all $m, m' \in M$, $n, n' \in N$ and $r \in R$:*
(i) $f(m + m', n) = f(m, n) + f(m', n)$;
(ii) $f(m, n + n') = f(m, n) + f(m, n')$;
(iii) $f(mr, n) = f(m, rn)$.

Definition 1.1.2 – *Let M be a right R-module and N be a left R-module. A tensor product f M and N is a pair (T, t) where T is an abelian group ad $t : M \times N \longrightarrow T$ a balanced map such that, for all abelian group G and all balanced map $f : M \times N \longrightarrow G$, there exists a unique group morphism $\phi : T \longrightarrow G$ such that the diagram*

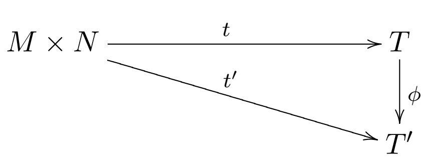


is commutative.

To start with, we first show that, if a tensor product exists, then it must be unique, up to isomorphism.

Proposition 1.1.3 – *Let M be a right R-module and N a left R-module. If (T, t) and (T', t') are tensor products of M and N, then there exists an isomorphism of groups between T and T'.*

Proof: By definition, we have commutative diagrams



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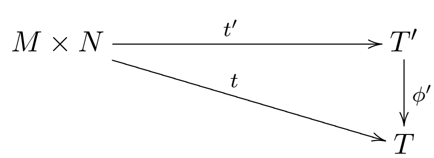
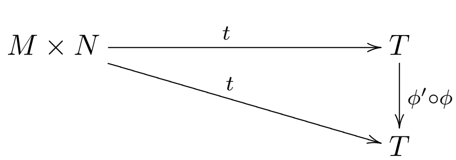


which give rise to two other diagrams



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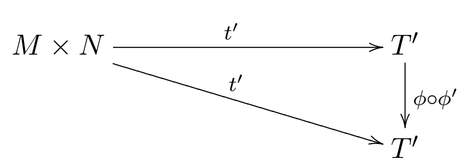


But then, the unicity of the morphism requierred by the definition of the tensor product leads to $\phi \circ \phi' = \mathrm{id}_{T'}$ and $\phi' \circ \phi = \mathrm{id}_T$. ∎

We now show the existence of a tensor product.

Proposition 1.1.4 – *Let M be a right R-module and N be a left R-module. Then, there exists a tensor product of M and N.*

Proof: Denote by F the free abelian group on the set $M \times N$, that is, the set of maps with finite support from $M \times N$ to $\mathbb{Z}$, endowed with the (abelian) group structure inhereted from that of $\mathbb{Z}$. For all $(m, n) \in M \times N$, denote by $\delta_{(m,n)}$ the map wich takes value 1 on (m, n) and 0 on any other element of $M \times N$. Then, the set of elements $\{\delta_{(m,n)}, \, (m, n) \in M \times N\}$ is a $\mathbb{Z}$-basis of F and we have a canonical injection $M \times N \longrightarrow F$, $(m, n) \mapsto \delta_{(m,n)}$. In the sequel, we will abuse notation identifying $(m, n) \in M \times N$ with its image in F.

Let S be the subgroup of F generated by the elements $(m+m', n) - (m, n) - (m', n)$, $(m, n+n') - (m, n) - (m, n')$ and $(mr, n) - (m, rn)$, for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. In addition, consider the map

$$t \, : \, M \times N \overset{can.inj.}{\longrightarrow} F \overset{can.proj.}{\longrightarrow} F/S;$$

which, clearly, is balanced.

We intend to show that $(F/S, t)$ is a tensor product of M and N.

Let G be an abelian group and $f \, : \, M \times N \longrightarrow G$ a balanced map. There exists a group morphism $\Phi \, : \, F \longrightarrow G$ such that, for all $(m, n) \in M \times N$, $\Phi((m, n)) = f((m, n))$. It is immediate that $\Phi(S) = 0$, so that Φ induces a group morphism $\phi \, : \, F/S \longrightarrow G$ such that, for all $(m, n) \in M \times N$, $\phi(t(m, n)) = f((m, n))$. Hence, $\phi \circ t = f$.

In addition, any group morphism $\psi \, : \, F/S \longrightarrow G$ such that $\psi \circ t = f$ must coincide since they coincide on the elements $t(m, n)$, $(m, n) \in M \times N$, which form a set of generators of the group F/S. ∎

Remark 1.1.5 – Let M be a right R-module and N a left R-module.
1. Seen the unicity, up to isomorphism, we will speak of *the* tensor product of M and N.
2. The tensor product of M and N constructed in Proposition 1.1.4 will be denotes $M \otimes_R N$ (or sometimes $M \otimes N$ if no confusion can arise). For $(m, n) \in M \times N$, we put $m \otimes n = t((m, n))$. A pure tensor is, by definition, an element of $M \otimes_R N$ of the form $m \otimes n$, where $(m, n) \in M \times N$.
3. If $m, m' \in M$, $n, n' \in N$ and $r \in R$, we have $(m + m') \otimes n = m \otimes n + m' \otimes n$, $m \otimes (n + n') = m \otimes n + m \otimes n'$ and $(mr) \otimes n = m \otimes (rn)$ in $M \otimes_R N$. In particular, for $(m, n) \in M \times N$, $0 \otimes n = m \otimes 0 = 0$ and $-(m \otimes n) = (-m) \otimes n = m \otimes (-n)$.
4. Pure tensors form a set of generators of the $\mathbb{Z}$-module $M \otimes_R N$, but not a basis in general. Therefore, any element of $M \otimes_R N$ may be written as a linear combination of pur tensors, but, in general, not in a unique way.

Proposition 1.1.6 – *Let $f \, : \, M \longrightarrow M'$ be a morphism of right R-modules and $g \, : \, N \longrightarrow N'$ a morphism of left R-modules. There exists a unique morphism of groups $h \, : \, M \otimes_R N \longrightarrow M' \otimes_R N'$ such that, for all $(m, n) \in M \times N$, $h(m \otimes n) = f(m) \otimes g(n)$.*

Proof: The map $M \times N \longrightarrow M' \otimes N'$, $(m, n) \mapsto f(m) \otimes g(n)$ is clearly balanced, hence the existence of h by definition of the tensor product. The unicity of h is obvious since it assigns the image of a generating family of the group $M \otimes_R N$. ∎

Notation 1.1.7 – Let $f : M \longrightarrow M'$ be a morphism of right R-modules and $g : N \longrightarrow N'$ a morphism of left R-modules. The morphism $h : M \otimes_R N \longrightarrow M' \otimes_R N'$ defined in Proposition 1.1.6 will be denoted $f \otimes g$ and called the tensor product of the morphisms f and g.

Remark 1.1.8 – In the notation of Proposition 1.1.6, it is clear that, if $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ are surjective, then so is $f \otimes g$.

Proposition 1.1.9 – *Let $M \xrightarrow{f} M' \xrightarrow{f'} M''$ be morphisms of right R-modules and $N \xrightarrow{g} N' \xrightarrow{g'} N''$ be morphisms of left R-modules. Then, one has $(f' \circ f) \otimes (g' \circ g) = (f' \otimes f) \circ (f \otimes g)$.*

Proof: This is immediate. ∎

Corollary 1.1.10 – *Let $M \xrightarrow{f} M'$ be an isomorphism of right R-modules and $N \xrightarrow{g} N'$ be an isomorphism of left R-modules. Then, $f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$ is an isomorphism of groupe.*

Proof: It is immediate by Proposition 1.1.9. ∎

Proposition 1.1.11 –
1. Let M be a right R-module and $N \xrightarrow{f} N' \xrightarrow{f'} N'' \longrightarrow 0$ be an exact sequence of left R-modules. Then, $M \otimes_R N \xrightarrow{\mathrm{id}\otimes f} M \otimes_R N' \xrightarrow{\mathrm{id}\otimes f'} M \otimes_R N'' \longrightarrow 0$ is an exact sequence of groups.
2. Let N be a left R-module and $M \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0$ be an exact sequence of right R-modules. Then, $M \otimes_R N \xrightarrow{f\otimes \mathrm{id}} M' \otimes_R N \xrightarrow{f'\otimes \mathrm{id}} M'' \otimes_R N \longrightarrow 0$ is an exact sequence of groups.

Proof: We only prove Point 1, the proof of Point 2 is similar.

The surjectivity of $\mathrm{id} \otimes f'$ follows from that of f' (cf. Remark 1.1.8). As $f' \circ f = 0$, we have $(\mathrm{id} \otimes f') \circ (\mathrm{id} \otimes f) = 0$, which gives the inclusion $\mathrm{im}(\mathrm{id} \otimes f) \subseteq \ker(\mathrm{id} \otimes f')$. It remains to show that $\mathrm{im}(\mathrm{id} \otimes f) \supseteq \ker(\mathrm{id} \otimes f')$.

Put $E = \mathrm{im}(\mathrm{id} \otimes f)$. The morphism $\mathrm{id} \otimes f'$ induces a (surjective) morphism $\overline{\mathrm{id} \otimes f'} : (M \otimes_R N')/E \longrightarrow M \otimes_R N''$. Clearly, to conclude that $\mathrm{im}(\mathrm{id} \otimes f) \supseteq \ker(\mathrm{id} \otimes f')$, it suffices to show that $\overline{\mathrm{id} \otimes f'}$ is injective, whch we do by exhibiting a left inverse to $\overline{\mathrm{id} \otimes f'}$.

Since f' is surjective, it admits a section, that is, a map $s : N'' \longrightarrow N'$ such that $f' \circ s = \mathrm{id}_{N''}$. Consider then the map

$$\begin{array}{rcl} M \times N'' & \longrightarrow & (M \otimes_R N')/E \\ (m, n'') & \mapsto & m \otimes s(n'') + E \end{array}.$$

Using the facts that $f' \circ s = \mathrm{id}_{N''}$ and $\ker(f') = \mathrm{im}(f)$, one easily checks that this map is balanced. Therefore, it induces a group morphism

$$\begin{array}{rcl} M \otimes_R N'' & \longrightarrow & (M \otimes_R N')/E \\ m \otimes n'' & \mapsto & m \otimes s(n'') + E \end{array}.$$

Using again that $f' \circ s = \mathrm{id}_{N''}$ and $\ker(f') = \mathrm{im}(f)$, we get that the latter map is a left inverse to $\overline{\mathrm{id} \otimes f'}$, which therfore is injective. ∎

Remark 1.1.12 – Denote Ab the category whose objects are abelian groups and whose morphisms are morphisms of groups.
1. We denote $R-\mathsf{Mod}$ the category whose objects are the left R-modules and whose morphisms are morphisms of left R-modules. Let M be a right R-module. The preceding results show that

we may define a functor $F = M \otimes_R -$ from $R - \mathsf{Mod}$ to Ab by putting that, for all object N of $R - \mathsf{Mod}$, $F(N) = M \otimes_R N$ and, for all morphism $f : N \longrightarrow N'$ of $R - \mathsf{Mod}$, $F(f) = \mathrm{id} \otimes f$. Proposition 1.1.11 establishes that the functor $M \otimes_R -$ is right exact.
2. We denote $\mathsf{Mod} - R$ the category whose objects are right R-modules and whose morphisms are morphisms of right R-modules. Let N be a left R-module. The preceding results show that we may define a functor $G = - \otimes_R N$ from $\mathsf{Mod} - R$ to Ab by putting that, for all object M of $\mathsf{Mod} - R$, $G(M) = M \otimes_R N$ and that, for all morphism $f : M \longrightarrow M'$ of $\mathsf{Mod} - R$, $G(f) = f \otimes \mathrm{id}$. Proposition 1.1.11 establishes that the functor $- \otimes_R N$ is right exact.

1.2 Additional structures on the tensor product.

Definition 1.2.1 – *Let R and S be rings. An (R, S)-bimodule is an abelian group M endowed with a left R-module structure and a right S-module structure satisfying the following compatibility condition: for all $r \in R$, $s \in S$ and $m \in M$, $(rm)s = r(ms)$.*

Proposition 1.2.2 – *Let R and S be rings.*
1. If M is an (R, S)-bimodule and N a left S-module, there exists a unique left R-module structure on $M \otimes_S N$ such that, for $r \in R$ and $(m, n) \in M \times N$, $r(m \otimes n) = (rm) \otimes n$.
2. If M is a right S-module and N an (S, R)-bimodule, there exists a unique right R-module structure on $M \otimes_S N$ such that, for all $r \in R$ and $(m, n) \in M \times N$, $(m \otimes n)r = m \otimes (nr)$.

Proof: 1. Unicity is clear.

The datum of a left R-module structure on an abelian group is equivalent to the datum of a ring morphism from R to the ring of endomorphisms of this abelian group. Therefore, we have to build a ring morphism $\phi : R \longrightarrow \mathrm{End}_{\mathbb{Z}}(M \otimes_S N)$.

Let $r \in R$. It is clear that the map $M \times N \longrightarrow M \otimes_S N$, $(m, n) \mapsto (rm) \otimes n$ is balanced. So, it induces an endomorphism $\mu(r) : M \otimes_S N \longrightarrow M \otimes_S N$ of (abelian) groups such that, for all $(m, n) \in M \times N$, $\mu(r)(m \otimes n) = (rm) \otimes n$. To conclude, it remains to show that the map $R \longrightarrow \mathrm{End}_{\mathbb{Z}}(M \otimes_S N)$, $r \mapsto \mu(r)$ is a ring morphism, which is easy.
2. The proof is similar to that of Point 1. ■

Corollary 1.2.3 – *Let R and S be rings.*
1. If M is an (R, S)-bimodule, the functor $M \otimes_S -$ from the category $S - \mathsf{Mod}$ in the category Ab takes values in $R - \mathsf{Mod}$.
2. If N is an (S, R)-bimodule, the functor $- \otimes_S N$ from the category $\mathsf{Mod} - S$ to the category Ab takes values in $\mathsf{Mod} - R$.

Proof: 1. By Proposition 1.2.2, the only think we have to prove is that, if $f : N \longrightarrow N'$ is a morphism of left R-modules, then $\mathrm{id} \otimes f : M \otimes N \longrightarrow M \otimes N'$ is a morphism of left R-modules, which is easy.
2. The proof is similar to that of Point 1. ■

Of course, a ring R is an (R, R)-bimodule. The next statement describe its behavior in a tensor product.

Proposition 1.2.4 – *Let R be a ring.*
1. If M is a left R-module, there exists a unique isomorphism of left R-modules $R \otimes_R M \longrightarrow M$ such that, for all $(r, m) \in R \times M$, $r \otimes m \mapsto rm$.
2. If M is a right R-module, there exists a unique isomorphism of right R-modules $M \otimes_R R \longrightarrow M$ such that, for all $(m, r) \in M \times R$, $m \otimes r \mapsto mr$.

Proof: 1. Unicity is clear. The map $R \times M \longrightarrow M$, $(r, m) \mapsto rm$ is clearly balanced. Therefore, it induces a group morphism $\mu : R \otimes_R M \longrightarrow M$ such that, for all $(r, m) \in R \times M$, $\mu(r \otimes m) = rm$. Further, μ is a morphism of left R-modules, as is easily verified. On the other hand, it is obvious that the map $M \longrightarrow R \otimes_R M$, $m \mapsto 1 \otimes m$ is a morphism of left R-modules, which is a right and left inverse of μ.
2. The proof is similar to that of Point 1. ∎

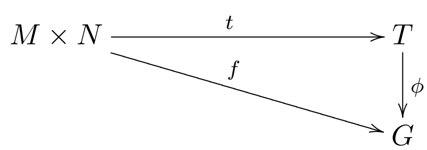
1.3 Case where the base ring is commutative.

In this subsection, R is a commutative ring.

Notice that, R being commutative, any (left) R-module may be seen as a right R-module and as a (R, R)-bimodule.

Definition 1.3.1 – *Let M, N and G be R-modules. A bilinear map from $M \times N$ to G is a map $f : M \times N \longrightarrow G$ such that, for all $m, m' \in M$, $n, n' \in N$ and $r \in R$:*
(i) $f(m + m', n) = f(m, n) + f(m', n)$;
(ii) $f(m, n + n') = f(m, n) + f(m, n')$;
(iii) $f(rm, n) = f(m, rn) = rf(m, n)$.

We can consider the following universal problem. Let M and N be R-modules. Does there exist a pair (T, t) where T is an R-module and $t : M \times N \longrightarrow T$ a bilinear map such that, for all R-module G and all bilinear map $f : M \times N \longrightarrow G$, there exists a unique morphism of R-modules $\phi : T \longrightarrow G$ such that the diagram



is commutative?

The proof of Proposition 1.1.3 adapts easily to show that, if such a module T exists, then it is unique, up to isomorphism.

Next, given R-modules M and N, we may very well consider M as a right R-module, N as a left R-module and consider their tensor product $(M \otimes_R N, t)$, as definied in subsection 1.1. Further, using Proposition 1.2.2 and considering M as an (R, R)-bimodule, we get an R-module structure on $M \otimes_R N$. It is easy to see that, actually, the balanced map $t : M \times N \longrightarrow M \otimes_R N$ is bilinear.

Proposition 1.3.2 – *Let M and N be R-modules. The pair $(M \otimes_R N, t)$ as definied in subsection 1.1 is a solution to the above universal problem.*

Proof: Consider a R-module G and a bilinear (and therfore obvioulsy balanced) map $M \times N \longrightarrow G$. By definition of $(M \otimes_R N, t)$, there exists a unique morphism of groups $\phi : M \otimes_R N \longrightarrow G$ such that $\phi \circ t = f$. It only remains to show that ϕ is a morphisme of R-modules. Let $r \in R$ and $(m, n) \in M \times N$. We have $\phi(r(m \otimes n)) = \phi(rm \otimes n) = \phi \circ t(rm, n) = f(rm, n) = rf(m, n) = r\phi(m \otimes n)$. It follows that ϕ is a morphism of R-modules. ∎

Remark 1.3.3 – In a course on commutative algebra, the tensor product of two modules is defined as a solution to the latter universal problem. Proposition 1.3.2 shows that this is consistant with the noncommutative point of view.

1.4 Some useful isomorphisms.

We collect, in the present subsection, a result on the associativity of the tensor product and a result on its distributivity with respect to direct sums.

Proposition 1.4.1 – *Let R, S be rings. Consider a right R-module L, an (R,S)-bimodule M and a left S-module N. Then, there exists a unique isomorphism of groups*

$$\phi \ : \ L \otimes_R (M \otimes_S N) \ \longrightarrow \ (L \otimes_R M) \otimes_S N \ ,$$

such that, for all $(\ell, m, n) \in L \times M \times N$, $\phi(\ell \otimes (m \otimes n)) = (\ell \otimes m) \otimes n$.

Proof: Unicity is clear. Let $\ell \in L$. It is easy to check that the map $M \times N \longrightarrow (L \otimes_R M) \otimes_S N$, $(m,n) \mapsto (\ell \otimes m) \otimes n$, is balanced. Hence the existence of a morphism of groups

$$f_\ell \, : \, M \otimes_S N \longrightarrow (L \otimes_R M) \otimes_S N$$

such that, for all $(m,n) \in M \times N$, $f_\ell(m \otimes n) = (\ell \otimes m) \otimes n$. Now, we are in position to define a map

$$\begin{array}{ccc} L \times (M \otimes_S N) & \longrightarrow & (L \otimes_R M) \otimes_S N \\ (\ell, p) & \mapsto & f_\ell(p) \end{array} .$$

Let $r \in R$ and $\ell, \ell' \in L$, we have $f_{\ell+\ell'} = f_\ell + f_{\ell'}$ and, for all $p \in M \otimes_S N$, $f_{\ell r}(p) = f_\ell(rp)$. It follows that the above map is balanced so that there exists a morphism of groups

$$\phi \ : \ L \otimes_R (M \otimes_S N) \ \longrightarrow \ (L \otimes_R M) \otimes_S N$$

which, for all $(\ell, m, n) \in L \times M \times N$, maps $\ell \otimes (m \otimes n)$ to $(\ell \otimes m) \otimes n$. In the same way, we can define a morphism of groups

$$(L \otimes_R M) \otimes_S N \ \longrightarrow \ L \otimes_R (M \otimes_S N)$$

which, for $(\ell, m, n) \in L \times M \times N$, maps $(\ell \otimes m) \otimes n$ to $\ell \otimes (m \otimes n)$.

The result follows. ∎

Remark 1.4.2 – Consider the context and notation of Proposition 1.4.1 and its proof. Let in addition Q, T be rings.
1. If L is an (Q,R)-bimodule, $L \otimes_R (M \otimes_S N)$ and $(L \otimes_R M) \otimes_S N$ are left Q-modules and ϕ is a morphism of Q-modules.
2. If N is an (S,T)-bimodule, $L \otimes_R (M \otimes_S N)$ and $(L \otimes_R M) \otimes_S N$ are right T-modules and ϕ is a morphism of T-modules.

Proposition 1.4.3 – *Let R be a ring, M a right R-module and $(N_i)_{i \in I}$ a family, indexed by the nonempty set I, of left R-modules. There exists a unique isomorphism of groups*

$$\Theta \ : \ M \otimes_R (\textstyle\bigoplus_{i \in I} N_i) \ \longrightarrow \ \textstyle\bigoplus_{i \in I} (M \otimes_R N_i)$$

such that, for $m \in M$ and $(n_i)_{i \in I} \in \bigoplus_{i \in I} N_i$, $\Theta(m \otimes (n_i)_{i \in I}) = ((m \otimes n_i)_{i \in I})$.

Proof: Exercise.

Remark 1.4.4 –
1. Retain the notation of Proposition 1.4.3. Lett Q, S des anneaux. If M is a (Q,R)-bimodule, $M \otimes_R (\bigoplus_{i \in I} N_i)$ and $\bigoplus_{i \in I} (M \otimes_R N_i)$ are endowed with left Q-module structures and the map Θ is a morphism of Q-modules. In addition, if, for all $i \in I$, N_i is an (R,S)-bimodule, $M \otimes_R (\bigoplus_{i \in I} N_i)$ and $\bigoplus_{i \in I} (M \otimes_R N_i)$ are right S-modules and the map Θ is a morphism of S-modules.
2. Of course, the results of Proposition 1.4.3 and of Point 1 above remain correct whenever the direct sum appears on the left of the tensor product rather than on the right.

References.

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