Symmetric powers of the nat. rep. of $sl_2(\mathbb{k})$.

$\mathbb{k}$ is a field of charact. zero and alg. closed

$\mathfrak{g} = sl_2(\mathbb{k})$

$V = \mathbb{k}^2$ and $\mathcal{B}$ is the canonical basis of V.

Recall the representation

$$\rho_1 : \mathfrak{g} \longrightarrow g\ell(V)$$

as defined in Section II.4 of the lecture notes.
Notice that in fact, ρ_1 is just the obvious injection of $sl_2(\mathbb{k})$ in $g\ell_2(\mathbb{k}) \simeq g\ell(\mathbb{k}^2)$. That is, ρ_1 is the natural rep. of $\mathfrak{g}$.

Our aim is to study the rep. of $\mathfrak{g}$ in $S(V)$ induced by ρ_1.

1. Consider the basis of $S(V)$ associated to $\mathcal{B}$ by Prop. IV.2.6. Give an explicit formula for the action of x, y and h on this basis.
2. Show that, $\forall n \in \mathbb{N}$, $S^n(V)$ is an irreducible rep. of $sl_2(\mathbb{k})$, of dimension $n+1$.
3. Consider the polynomial ring $\mathbb{k}[X,Y]$ in two indet. X and Y. Let ∂_X, ∂_Y be the usual partial derivatives w.r.t. X and Y resp. and let m_X, m_Y be the endo. of $\mathbb{k}[X,Y]$ of multipl. by X and Y resp. Finaly, consider $E = m_X \circ \partial_Y$, $F = m_Y \circ \partial_X$ and $H = [E, F]$ in $g\ell(\mathbb{k}[X,Y])$

Show that there exists a morphism of Lie algebras as follows:

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathbb{k}[X,Y]) \\ x &\longmapsto E \\ y &\longmapsto F \\ h &\longmapsto H \end{aligned}$$

Show that the corresp. representation of $\mathfrak{g}$ is isomorphic to the representation of $\mathfrak{g}$ on $S(V)$ induced by ℓ_1.

Solution

1. We denote by $\mathcal{B}=\{v, w\}$ the canonical basis of $\mathbb{k}^2$. Therefore, the elts of the canonical basis of $S(V)$ associated to $\mathcal{B}$ are the products $v^i w^j$ where $(i,j) \in \mathbb{N}^2$. Moreover, forall $n \in \mathbb{N}$, the elts $v^i w^j$, $(i,j) \in \mathbb{N}^2$, $i+j=n$ form a basis of the symetric power $S^n(V)$.

Denote by $\tau : \mathfrak{g} \longrightarrow \mathfrak{gl}(S(V))$ the rep. of $\mathfrak{g}$ in $S(V)$ induced by $\mathcal{C}_1$ and, $\forall n \in \mathbb{N}$, denote by $\tau_n : \mathfrak{g} \longrightarrow \mathfrak{gl}(S^n(V))$ the rep. of $\mathfrak{g}$ in $S^n(V)$ induced by τ. By definition:

$$\tau_0 : \mathfrak{g} \longrightarrow \mathfrak{gl}(S^0(V))$$
$$g \longmapsto 0$$

is the trivial rep. of $\mathfrak{g}$ in $S^0(V) \simeq \mathbb{k}$.

$$\tau_1 : \mathfrak{g} \longrightarrow \mathfrak{gl}(S^1(V))$$

is $\mathcal{C}_1$ (up to the isomorphism $S^1(V) \simeq V$).

Put now $n \geq 2$. An immediate calculation gives the action of x, y, h on the basis elts of $S^n(V)$ as follows: $\forall (i,j) \in \mathbb{N}^2$, $i+j=n$:

$$\tau_n(x)\left(v^i w^j\right) = j\, v^{i+1} w^{j-1}$$

$$\tau_n(y)\left(v^i w^j\right) = i\, v^{i-1} w^{j+1}$$

$$\tau_n(h)\left(v^i w^j\right) = (i-j)\, v^i w^j$$

Notice that $\dim_{\mathbb{k}}\left(S^n(V)\right) = n+1$.

2. We see that $\tau_n(h)$ has $n+1$ distinct eigenvalues, equal to: $n, n-2, \dots, -n$, with corresponding eigenvectors: $v^n, v^{n-1}w, \dots, w^n$.

The proof of the irreducibility is not difficult. Suppose $W \subseteq S^n(V)$ is a subrepresentation. Suppose W is non zero. ~~[illegible]~~. By the argument already used many times, W must be the (direct) sum of its intersection with the eigenspaces of $\tau_n(x)$. As $W \neq (0)$ and all the eigenspaces of $\tau_n(x)$ are one dimensional it follows that W contains an elt $v^i w^j$ of the basis of $S^n(V)$ described above. Letting $\tau_n(x)$ and $\tau_n(y)$ act on this elt, we get that all the $v^i w^j$, $(i,j) \in \mathbb{N}$, $i+j=n$ are in W; thas is $W = S^n(V)$. So, $S^n(V)$ is an irreducible rep. of $\mathfrak{g}$, of dimension $n+1$. By § II.4 of the course:

$$\left(S_n(V), \tau_n\right) \simeq \left(\mathbb{k}^{n+1}, \ell_n\right).$$

An alternative argument would consists in establishing an explicit isomorphism between $(S_n(V), \tau_n)$ and $(\mathbb{k}^{n+1}, \ell_n)$. This is very easy using the formulas of the action of $\tau_n(x), \dots$. Indeed, such an iso. must exchange, up to mult. by an adequate scalar, the ~~[illegible]~~ eigenvectors corresp. to the same eigenvalue. Then, the action of x (or y) force the value of the adequate scalar.

3. Since x, y, h is a basis of $sl_2(\mathbb{k})$ there is a linear map:

$$\begin{array}{rccc} \delta: & sl_2(\mathbb{k}) & \longrightarrow & gl(\mathbb{k}[X,Y]) \\ & x & \longmapsto & E \\ & y & \longmapsto & F \\ & h & \longmapsto & H \end{array}$$

Now, applying Ex. I.1.22 of the course, to show that δ is a morphism of Lie algebras, it is enough to check that $\delta([h,x]) = [\delta(h), \delta(x)]$, $\delta([h,y]) = [\delta(h), \delta(y)]$ and $\delta([x,y]) = [\delta(x), \delta(y)]$. All this is straightforward.

It is then easy to check that the isomorphism of algebras given by

$$\begin{array}{rccc} i: & S(V) & \longrightarrow & \mathbb{k}[X,Y] \\ & v & \longmapsto & X \\ & w & \longmapsto & Y \end{array}$$

is an isomorphism of representations.

<u>Note:</u> By the universal property of the polynomial algebra $\mathbb{k}[X,Y]$ in two indeterminates, there exists a (unique) morphism of $\mathbb{k}$-algebras

$$j: \mathbb{k}[X,Y] \longrightarrow S(V)$$

that sends X to v and Y to w. Now, j maps the canonical basis of $\mathbb{k}[X,Y]$ to the basis $\{v^i w^j, (i,j) \in \mathbb{N}\}$ of $S(V)$. So j is an isomorphism. The isom. i above is just its inverse.