NdT 03/12/2020

Symmetric algebra, symmetric powers and representations

We use notation and results of the Course Complement "Exterior algebra, ext. powers and representations" and of the § IV.2 of the Lecture Notes.

Let $\mathbb{k}$ be a field.

Let $\mathfrak{g}$ be a Lie algebra and (V, ρ) a rep. of $\mathfrak{g}$.

1. Recall the induced representation of $\mathfrak{g}$ in $T(V)$:

$$\tau : \mathfrak{g} \longrightarrow gl(T(V))$$

as defined in the Course Complement above.

2. Recall the ideal I of $T(V)$ generated by the elts $v \otimes w - w \otimes v \in T^2(V)$, $v, w \in V$. It is easy to check, using the definition of τ that, $\forall v, w \in V$, $\forall x \in \mathfrak{g}$, $\tau(x)(v \otimes w - w \otimes v)$ again belongs to I. As, $\forall x \in \mathfrak{g}$, $\tau(x)$ is a derivation, it follows that I is a subrepresentation of $(T(V), \tau)$. Therefore, τ induces a representation as follows:

$$\begin{aligned} T : \mathfrak{g} &\longrightarrow gl(S(V)) \\ x &\longmapsto T(x) \end{aligned}$$

where, $\forall x \in \mathfrak{g}$, $\tau(x)$ is the endomorph. of $S(V)$ induced by $\sigma(x)$. Clearly, $\tau(x)$ is a deriv. of $S(V)$ (since $\sigma(x)$ is a deriv. of $T(V)$).

3. Recall that $S(V)$ is a graded (comm.) algebra:

$$S(V) = \bigoplus_{n \in \mathbb{N}} S^n(V).$$

It is easy to check that, $\forall n \in \mathbb{N}$, $S^n(V)$ is a subrepresentation of $(S(V), \tau)$.

<u>Def.</u>: $\forall n \in \mathbb{N}$ the subrepr. $S^n(V)$ of $S(V)$ induced by σ is called the n-th symmetric power of (V, ρ).

By the above, if we denote the morphism of Lie algebras that defines it by τ_n, we have that:

$$\begin{aligned} \tau_n : \mathfrak{g} &\longrightarrow \mathfrak{gl}(S^n(V)) \\ x &\longmapsto \tau_n(x) \end{aligned}$$

where, $\forall v_1, \dots, v_n \in V$, $\forall x \in \mathfrak{g}$

$$\tau_n(x)(v_1 \cdots v_n) = \sum_{1 \le i \le n} v_1 \cdots v_{i-1}\, \rho(v_i)\, v_{i+1} \cdots v_n.$$