Clebsch-Gordan formula

NdT 13/12/20

<u>Set up:</u> $\mathbb{k}$ is a field of char 0 and alg. closed.
We consider two integers $m, n \in \mathbb{N}$ with $m \geq n$.
Let $V(m)$ and $V(n)$ be the corresponding irreducible representations $\mathfrak{sl}_2(\mathbb{k})$ of dim. $m+1$ and $n+1$, resp.
Find integers $p_1, \ldots, p_s \in \mathbb{N}$, $s \in \mathbb{N}^*$ such that

$$(*) \quad V(m) \otimes V(n) \simeq V(p_1) \oplus \cdots \oplus V(p_s).$$

<u>Note:</u> such integers exist by Weyl's Theorem.

1. To start, observe that given a decomposition as in $(*)$, we get s obvious highest weight vectors in $V(m) \otimes V(n)$, each of which is "the" highest weight vector of the subrepresentation corresponding to $V(p_i)$ for some $i \in \{1, \ldots, s\}$.
"The" highest weight vector corresp. to $V(p_i)$ is unique only up to mult. by nonzero scalars and its weight is p_i, $1 \leq i \leq s$.
Hence the idea of looking after highest weight vectors in $V(m) \otimes V(n)$

2. Recall that $V(m) = \mathbb{k}^{m+1}$ and that, if $\{v_0, \ldots, v_m\}$ is the canonical basis of $\mathbb{k}^{m+1}$, we have that

$$(1) \quad h \cdot v_i = (m - 2i) v_i$$

$$(2) \quad x \cdot v_i = (m - i + 1) v_{i-1} \qquad (\text{with } v_{-1} := 0)$$

Similarly, $V(n) = \mathbb{k}^{n+1}$ and we let $\{w_0, \ldots, w_n\}$ denote its canonical basis.

By standard properties of the tensor product, we have a basis of $V(m) \otimes V(n)$ given by:

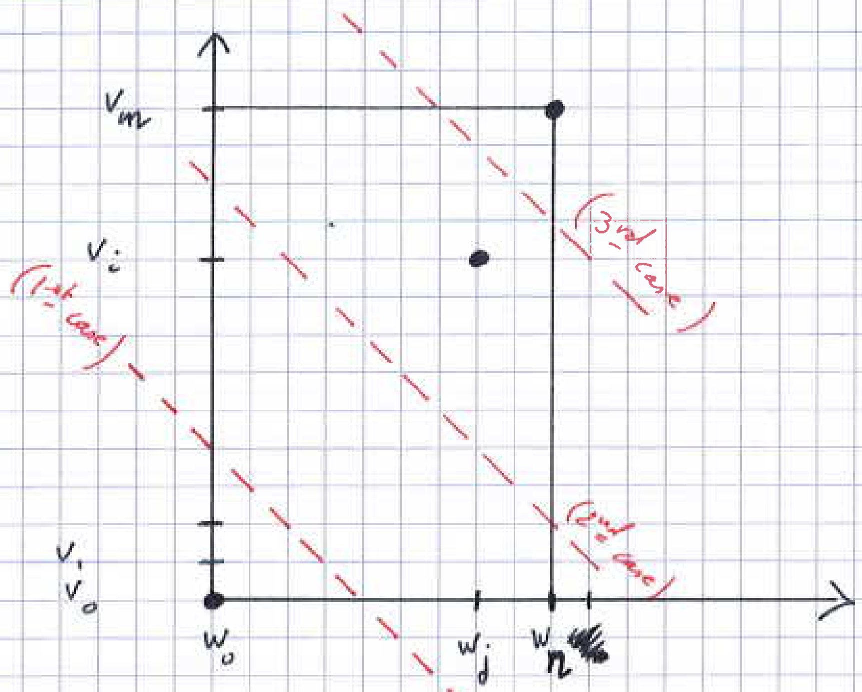
$$\{ v_i \otimes w_j, \quad 0 \leq i \leq m, \ 0 \leq j \leq n \}$$

and this basis consists in weight vectors.

Namely:

$\forall\ 1 \leq i \leq m,\ 1 \leq j \leq n$, $v_i \otimes w_j$ has weight $(m+n) - 2(i+j)$.

It is convenient to represent this basis by a diagram as follows:



The elts of the basis of $V(m) \otimes V(n)$ are represented by dots with integral coeff which belong to the interior and boundary of the above rectangle. Further, two basis elts have the same weight iff they belong to the same antidiagonal (dashed red lines above).

Therefore, weight vectors are linear combinations of basis elts $v_i \otimes w_j$ corresponding to points on a given diagonal.

Clearly, three types of cases will occur when we consider a weight vector of weight $m+n-2k$. Such a weight vector is a linear combination of the type $\sum_{i+j=k} c_{ij}\, v_i \otimes w_j$.

<u>1st case</u>: $0 \leq k \leq n$

The weight elt under consideration may be written

$$t = \sum_{i=0}^{k} c_i \; v_i \otimes w_{k-i}$$

where the c_i are elts of $\Bbbk$.

Then:

$$x \cdot t = \sum_{i=0}^{k} c_i \left[(m-i+1)\, v_{i-1} \otimes w_{k-i} + (n-k+i+1)\, v_i \otimes w_{k-i-1} \right]$$

$$= \sum_{i=1}^{k} c_i (m-i+1)\, v_{i-1} \otimes w_{k-i}$$

$$+ \sum_{i=0}^{k-1} c_i (n-k+i+1)\, v_i \otimes w_{k-i-1}$$

$$= \sum_{i=0}^{k-1} \Big(c_{i+1}(m-i) + c_i (n-k+i+1) \Big)\, v_i \otimes w_{k-i-1}$$

Hence, $x \cdot t = 0$ iff $c_{i+1}(m-i) + c_i(n-k+i+1) = 0$ for all $0 \leq i \leq k-1$. But, clearly, the coeff. $(m-i)$ and $(n-k+i+1)$ are non zero for $0 \leq i \leq k-1$ so that, if we exclude $t=0$, we get that $x \cdot t = 0$ iff

$$t = \sum_{i=0}^{k} (-1)^i \, \frac{\overline{n-k+i}\,!}{\overline{n-k}\,!} \times \frac{\overline{m-i}\,!}{m!} \; v_i \otimes w_{k-i}$$

up to mult. by a non zero scalar.

$\underline{2^{\text{nd}}\text{ Case}}$ $\quad n < k \leq m$

The weight elt now is of the form:

$$t = \sum_{i=k-n}^{k} c_i \, v_i \otimes w_{k-i}$$

A computation as above shows that $x . t = 0$ iff $t = 0$.

$\underline{3^{\text{rd}}\text{ case}}$ $\quad m < k \leq m+n$

The weight elt is of the form

$$t = \sum_{i=k-n}^{m} c_i \, v_i \otimes w_{k-i}$$

Here again, $x . t = 0$ iff $t = 0$.

At this stage, we know that the highest weight vectors of $V(m) \otimes V(n)$ are, up to mult. by non zero scalars, the elts t_k, $0 \leq k \leq n$ given by:

$$t_k = \sum_{i=0}^{k} (-1)^i \frac{\overline{n-k+i}\,!}{\overline{n-k}\,!} \times \frac{\overline{m-i}\,!}{m!} \, v_i \otimes w_{k-i} .$$

It follows that: $V(m) \otimes V(n) \simeq \bigoplus_{k=0}^{n} V(m+n-2k)$.