# REALIZABILITY OF FUSION SYSTEMS BY DISCRETE GROUPS

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Abstract. For a prime $p$, fusion systems over discrete $p$-toral groups are categories that model and generalize the $p$-local structure of Lie groups and certain other infinite groups in the same way that fusion systems over finite $p$-groups model and generalize the $p$-local structure of finite groups. In the finite case, it is natural to say that a fusion system $\mathcal{F}$ is realizable if it is isomorphic to the fusion system of a finite group, but it is less clear what realizability should mean in the discrete $p$-toral case.

In this paper, we look at some of the different types of realizability for fusion systems over discrete $p$-toral groups, including realizability by linear torsion groups and sequential realizability, of which the latter is the most general. After showing that fusion systems of compact Lie groups are always realized by linear torsion groups (hence sequentially realizable), we give some new tools for showing that certain fusion systems are not sequentially realizable, and illustrate it with two large families of examples.

## Introduction

For a fixed prime $p$, let $\mathbb{Z}/p^\infty$ denote the union of an increasing sequence of finite cyclic $p$-groups $\mathbb{Z}/p^n$. A discrete $p$-torus is a group isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty$, and a *discrete $p$-toral group* is a group that contains a discrete $p$-torus with (finite) $p$-power index. For example, if $T$ is a torus in the usual sense (a product of copies of $S^1$), then the group of all elements of $p$-power order in $T$ is a discrete $p$-torus (hence the name).

A *fusion system* over a discrete $p$-toral group $S$ is a category $\mathcal{F}$ whose objects are the subgroups of $S$, whose morphisms are injective homomorphisms between the subgroups including all morphisms induced by conjugation in $S$, and such that $\varphi\colon P \to Q$ in $\mathcal{F}$ implies $\varphi^{-1}\colon \varphi(P) \to P$ is also in $\mathcal{F}$. A fusion system is *saturated* if it satisfies certain additional conditions listed in Definition 1.10. Saturated fusion systems over discrete $p$-toral groups arise in different contexts relevant to algebra and topology (see, e.g., [BLO6] and [KMS1, KMS2]). In [BLO3], we proved their basic properties and showed how they appear naturally in various situations.

When $G$ is a group and $S \leq G$ is a discrete $p$-toral subgroup, we let $\mathcal{F}_S(G)$ (the "fusion system of $G$ with respect to $S$") be the category whose objects are the subgroups of $S$, and whose morphisms are those homomorphisms between subgroups induced by conjugation in $G$. This is always a fusion system, and it is saturated whenever $G$ is finite and $S$ is a Sylow

2020 *Mathematics Subject Classification.* Primary 20D20, 20F50. Secondary 20G15, 55R35, 57T10, 22E15.

*Key words and phrases.* Sylow subgroups, locally finite groups, linear torsion groups, linear algebraic groups, classifying spaces of groups, compact Lie groups, $p$-compact groups.

C. Broto is partially supported by MICINN grant PID2020-116481GB-100 and AGAUR grant 2021-SGR-01015.

B. Oliver is partially supported by UMR 7539 of the CNRS.

All three authors would like to thank the Isaac Newton Institute for Mathematical Sciences and the Gaelic College on Isle of Skye for their support and hospitality during the programme "Topology, representation theory, and higher structures", supported by EPSRC grant no. EP/R014604/1.

$p$-subgroup of $G$. These examples provided Puig with part of his original motivation for defining saturated fusion systems.

When $\mathcal{F}$ is a fusion system over a finite $p$-group $S$, it is natural to say that $\mathcal{F}$ is realizable if it is isomorphic to $\mathcal{F}_S(G)$ for some finite group $G$ with $S \in \mathrm{Syl}_p(G)$. In this situation, the $p$-completed classifying space $BG^\wedge_p$ ($p$-completed in the sense of Bousfield and Kan [BK]) is always homotopy equivalent to the classifying space of the fusion system (see [BLO1, Proposition 1.1] and [BLO2, Definition 1.8]). Many examples have been constructed of fusion systems over finite $p$-groups that do not arise in this way, such as the systems $\mathcal{F}_{\mathrm{Sol}}(q)$ constructed in [LO, LO2] (essentially the only known examples when $p = 2$), and those constructed in [RV] and [COS] for odd primes $p$.

In this paper, we look at the question of what "realizable" should mean for fusion systems over discrete $p$-toral groups. When we first looked at fusion systems over discrete p-toral groups, we were motivated in part by the example of fusion systems of compact Lie groups, but we also showed that such fusion systems arise from classifying spaces of $p$-compact groups, and of torsion subgroups of $GL_n(K)$ when $K$ is a field with $\mathrm{char}(K) \neq p$ — which we call linear torsion groups (see Theorems 9.10, 10.7, and 8.10 in [BLO3]). In view of this, it became clear that it would be much too restrictive to say that $\mathcal{F}$ is realizable only if it is isomorphic to the fusion system of a compact Lie group. The question of what should be the correct concept of realizability in this setting (if there is one) remained open. This paper aims to address this question. As we shall see, rather than a simple answer, there are several forms of realizability, of which the most general that we have found is what we call "sequential realizability".

A fusion system $\mathcal{F}$ over $S$ is *sequentially realizable* if it is the union of an increasing sequence $\mathcal{F}_1 \leq \mathcal{F}_2 \leq \cdots$ of fusion subsystems over finite subgroups of $S$, each of which is realized by a finite group. (Note that we do not assume any relations between the groups realizing the $\mathcal{F}_i$.) We show that fusion systems of compact Lie groups and $p$-fusion systems of linear torsion groups in characteristic different from $p$ are all sequentially realizable. We will show in a later paper that sequentially realizable fusion systems are always saturated (we avoid that question in this paper by assuming saturation when necessary).

Quite surprisingly, it turns out that a fusion system can be sequentially realizable, and at the same time the union of an increasing sequence of fusion subsystems over finite $p$-groups that are not realizable. In Example 2.7, we construct a fusion system $\mathcal{F}$ over a discrete $p$-toral group $S$, together with an increasing sequence of finite saturated fusion subsystems $\mathcal{F}_1 \leq \mathcal{F}_2 \leq \cdots$ whose union is $\mathcal{F}$, where the $\mathcal{F}_i$ alternate between being realizable (by finite groups) and exotic (not realizable). For this reason, while we often use the word "exotic" to mean "not realizable", the phrase "sequentially exotic" does not seem appropriate when we mean "not sequentially realizable".

As one example, we show in Proposition 2.4 that a saturated fusion system $\mathcal{F}$ is sequentially realizable whenever $\mathcal{F} \cong \mathcal{F}_S(G)$ for some locally finite group $G$ and some discrete $p$-toral subgroup $S \leq G$ that is a Sylow $p$-subgroup of $G$. For example, if $G$ is a linear torsion group in characteristic different from $p$, then $G$ is locally finite and has Sylow $p$-subgroups that are discrete $p$-toral, and the fusion system $\mathcal{F}_S(G)$ (for $S \in \mathrm{Syl}_p(G)$) is sequentially realizable.

Our first result shows that fusion systems over finite $p$-groups that are exotic in the earlier sense are still exotic with respect to these new criteria.

**Theorem A** (Theorem 2.3)**.** *Let $\mathcal{F}$ be a saturated fusion system over a finite $p$-group $S$. If $\mathcal{F}$ is sequentially realizable, or if it is realized by some locally finite group $G$ containing*

*$S$ as a maximal $p$-subgroup, then $\mathcal{F}$ is realized by a finite group containing $S$ as a Sylow $p$-subgroup.*

When $G$ is a compact Lie group and $p$ is a prime, we let $\mathrm{Syl}_p(G)$ be the set of all maximal discrete $p$-toral subgroups of $G$. For each $S \in \mathrm{Syl}_p(G)$, there is a maximal torus $T \leq G$ such that $S \cap T$ is the subgroup of elements of $p$-power order in $T$ and $ST/T \in \mathrm{Syl}_p(N_G(T)/T)$ (see, e.g., [BLO3, Proposition 9.3]). We showed in [BLO3, Lemma 9.5] that $\mathcal{F}_S(G)$ is a saturated fusion system for each $S \in \mathrm{Syl}_p(G)$.

**Theorem B** (Theorem 4.3)**.** *Let $G$ be a compact Lie group, and fix a prime $p$ and $S \in \mathrm{Syl}_p(G)$. Then there is a linear torsion group $\Gamma$ in characteristic different from $p$ such that $\mathcal{F}_{S_\Gamma}(\Gamma) \cong \mathcal{F}_S(G)$ for $S_\Gamma \in \mathrm{Syl}_p(\Gamma)$. In particular, $\mathcal{F}_S(G)$ is sequentially realizable.*

In fact, in the situation of Theorem B, for each prime $q \neq p$, there is a group $\Gamma$ that is linear over $\overline{\mathbb{F}}_q$ (hence torsion), such that $B\Gamma^\wedge_p \simeq BG^\wedge_p$, and $\mathcal{F}_S(G) \cong \mathcal{F}_{S_\Gamma}(\Gamma)$ for $S_\Gamma \in \mathrm{Syl}_p(\Gamma)$.

The next theorem shows that sequential realizability imposes strong restrictions on the structure of fusion systems. It is our main tool for proving the nonrealizability of certain fusion systems. Its proof requires the classification of finite simple groups.

**Theorem C** (Theorem 7.4)**.** *Let $\mathcal{F}$ be a saturated fusion system over an infinite discrete $p$-toral group $S$, let $T \trianglelefteq S$ be the identity component, and set $W = \mathrm{Aut}_{\mathcal{F}}(T)$. Assume*

(i) *$S > T$ and $C_S(T) = T$;*

(ii) *no subgroup $T \leq P < S$ is strongly closed in $\mathcal{F}$; and*

(iii) *no infinite proper subgroup of $T$ is invariant under the action of $O^{p'}(W)$.*

*Assume also that $\mathcal{F}$ is sequentially realizable. Then $W$ contains a normal subgroup of index prime to $p$ that is isomorphic to of one of the groups listed in cases (a)–(e) of Proposition 6.10.*

Theorem C is proven as Theorem 7.4. That theorem is stated in slightly greater generality, but Theorem C suffices for our applications here.

As a first application of Theorem C, we determine in Section 8 exactly which fusion systems of simple, connected $p$-compact groups are sequentially realizable. Then, in Section 9, we consider simple fusion systems over nonabelian infinite discrete $p$-toral groups containing a discrete $p$-torus with index $p$ (classified in [OR, § 5]), and determine exactly which of them are sequentially realizable. In all of these cases, either the fusion system in question is realized by an explicitly given linear torsion group, or it fails to be sequentially realizable by Theorem C.

Another result, whose proof is closely related to that of Theorem C and also depends on the classification of finite simple groups, is the following. It is stated more generally in Theorem 8.10 as a result about connected $p$-compact groups.

**Theorem D** (Theorem 8.10(b))**.** *Fix a compact connected Lie group $G$, a prime $p$, and $S \in \mathrm{Syl}_p(G)$, and assume that $p$ divides the order of the Weyl group of $G$. Then there is no linear torsion group in characteristic 0 whose fusion system with respect to a Sylow $p$-subgroup is isomorphic to $\mathcal{F}_S(G)$.*

In particular, Theorem D implies that there is no torsion subgroup $\Gamma \leq G$ with the same fusion system as $G$. Note that by Theorem 4.2, under the hypotheses of Theorem D, $\mathcal{F}_S(G)$ is realized by fusion systems of linear torsion groups in every characteristic except $p$ and 0.

After a brief overview in Section 1 of some general definitions and results about fusion systems, we define sequential realizability and give some of its basic properties in Section 2, and then look at the special case of linear torsion groups in Section 3. We then prove in Section 4 that the fusion system of a compact Lie group is always realized by a linear torsion group (Theorem B). We then show some general results in Sections 5 and 6 which are applied in Section 7 to prove Theorem C. We finish by looking at realizability of fusion systems of $p$-compact groups in Section 8, and in Section 9 that of fusion systems over discrete $p$-toral groups with a discrete $p$-torus of index $p$.

**Notation:** Our notation is fairly standard, with a few exceptions. Composition is always taken from right to left. If $H \le G$ are groups and $x, g \in G$, then ${}^xH = xHx^{-1}$, ${}^xg = xgx^{-1}$, and $c_x^H$ (or $c_x$) denotes the conjugation homomorphism ($g \mapsto {}^xg$) from $H$ to ${}^xH$. If $H_1, H_2, K \le G$, then $\mathrm{Hom}_K(H_1, H_2)$ is the set of all $c_x^{H_1} \in \mathrm{Hom}(H_1, H_2)$ for $x \in K$ such that ${}^xH_1 \le H_2$. Also,

- $\Phi(P)$ denotes the Frattini subgroup of a finite $p$-group $P$;
- $\varOmega_n(P) = \langle x \in P \,|\, x^{p^n} = 1 \rangle$ when $P$ is a $p$-group and $n \ge 1$;
- $H \circ K$ denotes a central product of $H$ and $K$;
- $H.K$ or $H{:}K$ denotes an arbitrary extension or a split extension of $H$ by $K$ (i.e., a group with normal subgroup isomorphic to $H$ and quotient isomorphic to $K$);
- $E_{p^n}$ (or $p^n$ when part of an extension) denotes an elementary abelian $p$-group of rank $n$;
- $p_\pm^{1+2n}$ denotes an extraspecial $p$-group of order $p^{1+2n}$;
- $\mathfrak{Fin}(G)$ denotes the set of finite subgroups of a discrete group $G$;
- $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the rings of $p$-adic integers and $p$-adic rationals, respectively;
- $\mathrm{ord}_p(n)$ denote the order of $n$ in $(\mathbb{Z}/p)^\times$ if $n$ is prime to $p$; and
- $v_p(n)$ denote the $p$-adic valuation of an integer $n$.

Finally, when $X$ is a space, we let $\mathrm{Aut}(X)$ denote the monoid of self homotopy equivalences of $X$, and let $\mathrm{Out}(X)$ be the group of homotopy classes of elements in $\mathrm{Aut}(X)$.

## 1. Saturated fusion systems over discrete $p$-toral groups

We begin by recalling some definitions and notation from [BLO3, Sections 1–2], starting with the definition of a discrete $p$-toral group.

**Definition 1.1.** Let $p$ be a prime.

(a) A *discrete $p$-torus* is a group that is isomorphic to $(\mathbb{Z}/p^\infty)^r$ for some $r \ge 0$, where $\mathbb{Z}/p^\infty \cong \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is the union of the cyclic $p$-groups $\mathbb{Z}/p^n$. The *rank* of a discrete $p$-torus $U \cong (\mathbb{Z}/p^\infty)^r$ is $r = \mathrm{rk}(U)$.

(b) A *discrete $p$-toral group* is a group with a normal subgroup of $p$-power index that is a discrete $p$-torus.

(c) The *identity component* of a discrete $p$-toral group $P$ is the unique discrete $p$-torus of finite index in $P$; equivalently, the intersection of all subgroups of finite index in $P$.

(d) The *order* of a discrete $p$-toral group $P$ with identity component $U$ is the pair $|P| = (\mathrm{rk}(U), |P/U|) \in \mathbb{N}^2$, where $\mathbb{N}^2$ is ordered lexicographically.

Thus if $Q \le P$ is a pair of discrete $p$-toral groups, then $|Q| \le |P|$, and $|Q| = |P|$ if and only if $Q = P$.

We next recall some more terminology.

**Definition 1.2.** (a) A *p-group* (for a prime $p$) is a discrete group all of whose elements have (finite) $p$-power order.

(b) A discrete group $G$ is *locally finite* if every finitely generated subgroup of $G$ is finite.

(c) When $p$ is a prime and $G$ is any group, a *Sylow p-subgroup* of $G$ is a $p$-subgroup of $G$ that contains all other $p$-subgroups up to conjugacy. We let $\mathrm{Syl}_p(G)$ be the (possibly empty) set of Sylow $p$-subgroups of $G$.

(d) A discrete group $G$ is *artinian* if each descending sequence of subgroups of $G$ becomes constant.

Definitions in the literature of "Sylow $p$-subgroups" of infinite discrete groups vary slightly (see, e.g., [KW, p. 85]), but the strict criterion given above is the most appropriate for our purposes.

Discrete $p$-toral groups play an important role when working with compact Lie groups and $p$-compact groups (see, e.g., [DW2, § 6]), and that in turn made it natural for us to consider them when constructing fusion systems for these objects. Since local finiteness and the descending chain condition are used in many of our arguments, the following characterization of discrete $p$-toral groups helps to explain their importance.

**Proposition 1.3** ([BLO3, Proposition 1.2])**.** *A group is discrete p-toral if and only if it is a p-group, artinian, and locally finite.*

Whenever $P$ is a discrete $p$-toral group with identity component $U \trianglelefteq P$, there is a finite subgroup $R \in \mathfrak{Fin}(P)$ such that $P = RU$. This holds since $P$ is locally finite and $P/U$ is finite.

We will need the following variant on the standard result that an inverse limit of a system of finite nonempty sets is nonempty.

**Lemma 1.4.** *Fix a group $\Gamma$, and let $(\dots \xrightarrow{r_4} \Phi_3 \xrightarrow{r_3} \Phi_2 \xrightarrow{r_2} \Phi_1)$ be an inverse system of nonempty $\Gamma$-sets and $\Gamma$-maps such that $\Phi_i/\Gamma$ is finite for each $i \ge 1$. Then $\lim_i(\Phi_i, r_i) \ne \varnothing$.*

*Proof.* Since $\Phi_i/\Gamma$ is finite and nonempty for each $i$, the inverse limit $\lim_i(\Phi_i/\Gamma, r_i/\Gamma)$ is nonempty. Choose an element $(\Psi_i)_{i\ge1}$ in that inverse limit. Thus for each $i$, $\Psi_i \subseteq \Phi_i$ is a $\Gamma$-orbit and $r_i(\Psi_i) = \Psi_{i-1}$. Thus $(\dots \xrightarrow{r_4'} \Psi_3 \xrightarrow{r_3'} \Psi_2 \xrightarrow{r_2'} \Psi_1)$ is an inverse system of $\Gamma$-sets where each map $r_i' = r_i|_{\Psi_i}$ is surjective, and hence $\lim_i(\Phi_i, r_i) \supseteq \lim_i(\Psi_i, r_i') \ne \varnothing$. □

As a first application of Lemma 1.4, we note the following striking property of discrete $p$-toral groups, one which allows us to give a slightly weaker condition for a $p$-subgroup to be Sylow in Proposition 1.6.

**Lemma 1.5.** *Let $P$ and $Q$ be two discrete p-toral groups. If every finite subgroup of $P$ is isomorphic to a subgroup of $Q$, then $P$ is isomorphic to a subgroup of $Q$.*

*Proof.* Choose an increasing sequence $P_1 \le P_2 \le P_3 \le \cdots$ of finite subgroups of $P$ such that $\bigcup_{i=1}^{\infty} P_i = P$. For each $i \ge 1$, let $\mathrm{Inj}(P_i, Q)$ be the set of injective homomorphisms from $P_i$ to $Q$. These sets form an inverse system $(\mathrm{Inj}(P_i, Q), r_i)$ of sets with $\mathrm{Inn}(Q)$-action, where each map $r_i \colon \mathrm{Inj}(P_i, Q) \longrightarrow \mathrm{Inj}(P_{i-1}, Q)$ is defined by restriction. Each orbit set $\mathrm{Inj}(P_i, Q)/\mathrm{Inn}(Q)$ is finite (see [BLO3, Lemma 1.4(a)]), and is nonempty by assumption. So the inverse limit of this system is nonempty by Lemma 1.4. Choose $(\varphi_i)_{i\ge 1} \in \lim_i(\mathrm{Inj}(P_i,Q),r_i)$; then $\bigcup_{i=1}^\infty \varphi_i$ is an injective homomorphism from $P$ to $Q$. □

**Proposition 1.6.** *Fix a prime $p$ and a discrete group $G$. Assume $G$ has Sylow $p$-subgroups that are discrete $p$-toral. Then a $p$-subgroup $P \le G$ is a Sylow $p$-subgroup if every finite $p$-subgroup of $G$ is conjugate to a subgroup of $P$.*

*Proof.* Assume $S \in \mathrm{Syl}_p(G)$ and is discrete $p$-toral. Let $P$ be a $p$-subgroup of $G$ that contains every finite $p$-subgroup of $G$ up to conjugacy. In particular, every finite subgroup of $S$ is isomorphic to a subgroup of $P$, and hence $S$ is isomorphic to a subgroup of $P$ by Lemma 1.5. Then $P$ and $S$ are conjugate in $G$ since $S \in \mathrm{Syl}_p(G)$, and so $P$ is also a Sylow $p$-subgroup. □

The following example helps explain why we needed to assume that $G$ has Sylow $p$-subgroups in Proposition 1.6. It is based on [KW, Example 3.3].

**Example 1.7.** Fix two distinct primes $p$ and $q$ and an infinite discrete $p$-toral group $S$. Set $H = \mathbb{F}_q S$, regarded as an abelian $q$-group with action of $S$, and set $G = H \rtimes S$. Then $G$ is locally finite, every $p$-subgroup of $G$ is isomorphic to a subgroup of $S$, and every finite $p$-subgroup of $G$ is conjugate to a subgroup of $S$. In contrast, for each proper infinite subgroup $T < S$, there is a maximal $p$-subgroup $P \le G$ such that $PH = TH$, and $P$ is not conjugate to a subgroup of $S$. In particular, $G$ has no Sylow $p$-subgroups unless $S \cong \mathbb{Z}/p^\infty$.

*Proof.* The group $G$ is locally finite since it is an extension of one locally finite group by another. If $P \le G$ is a $p$-subgroup, then $P \cap H = 1$, so $P \cong PH/H \le G/H \cong S$.

It remains to prove the claims involving conjugacy between $p$-subgroups of $G$. When doing this, it is convenient to consider the groups $\widehat{H} = \mathrm{map}(S,\mathbb{F}_q)$ and $\widehat{G} = \widehat{H} \rtimes S$, where $S$ acts on $\widehat{H}$ by setting

$$g(\xi)(h) = \xi(g^{-1}h) \quad \text{for all } g,h \in S \text{ and } \xi\colon S \longrightarrow \mathbb{F}_q.$$

We consider $H = \mathbb{F}_q S$ as a subgroup of $\widehat{H}$: the subgroup of those $\xi\colon S \longrightarrow \mathbb{F}_p$ in $\widehat{H}$ with finite support. In this way, we have $G \le \widehat{G}$.

Now, $\widehat{H} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}S,\mathbb{F}_q) = \mathrm{Coind}_1^S(\mathbb{F}_q)$ in the notation of [Br, § III.5]. So $H^1(S;\widehat{H}) \cong H^1(1;\mathbb{F}_q) = 0$ by Shapiro's lemma (see [Br, Proposition III.6.2]), and a similar argument shows that $H^1(T;\widehat{H}) = 0$ for all $T \le S$. So every $p$-subgroup $P \le \widehat{G}$ such that $P\widehat{H} = T\widehat{H}$ is conjugate to $T$ by an element of $\widehat{H}$ (see [Br, Proposition IV.2.3]).

Assume $P \le G$ is a finite $p$-subgroup. Then $PH = UH$ for some finite subgroup $U \le S$, and so $P = {}^{\xi}U$ for some $\xi \in \widehat{H}$. Hence for each $g \in U$, we have $[g,\xi] \in H$, and so $g(\xi) - \xi \in \mathrm{map}(S,\mathbb{F}_q)$ has finite support. Since $U$ is finite, this means that $\xi$ is constant on all but finitely many cosets of $U$, and hence that $\xi \in H + C_{\widehat{H}}(U)$. So $P = {}^{\xi}U$ is conjugate to $U$ by an element of $H$. Since every finite $p$-subgroup of $G$ has this form (for some finite $U \le S$), this proves that every finite $p$-subgroup is conjugate to a subgroup of $S$.

Now let $T < S$ be a proper infinite subgroup. We will construct $\xi \in \widehat{H}$ such that ${}^{\xi}T$ is a maximal $p$-subgroup in $G$ and hence not conjugate in $G$ to a subgroup of $S$. To do this, let $1 = T_0 < T_1 < T_2 < T_3 < \cdots$ be a strictly increasing sequence of finite subgroups of $T$ such that $T = \bigcup_{i=1}^\infty T_i$. For each $i \ge 1$, choose $g_i \in T_i \smallsetminus T_{i-1}$ and set $X_i = T_{i-1}g_i$. Define $\xi \in \widehat{H} = \mathrm{map}(S,\mathbb{F}_q)$ by setting $\xi(g) = 1$ if $g \in \bigcup_{i=1}^\infty X_i$ and $\xi(g) = 0$ otherwise. Set $P = {}^{\xi}T$.

We first check that $P \le G$ by showing that $[T,\xi] \le H$. To see this, fix $g \in T$, and let $i$ be such that $g \in T_i$. Then $\xi$ is constant on left $g$-orbits in $T \smallsetminus T_i$, so $g(\xi) - \xi$ has support contained in the finite subgroup $T_i$, and hence lies in $H$. Thus $[g,\xi] \in H$, and since $g \in T$ was arbitrary, we get $[T,\xi] \le H$.

It remains to prove that $P$ is a maximal $p$-subgroup of $G$. Assume otherwise: then there are $U \le S$ and $\eta \in \widehat{H}$ such that $U > T$ and ${}^{\xi}T < {}^{\eta}U \le G$. Thus $[\eta, U] \le H$ and $[\eta - \xi, T] = 1$, and in particular, $\eta - \xi$ is constant on cosets of $T$. Fix an element $g \in U \smallsetminus T$; then by construction, $g(\xi) - \xi$ is nonzero on infinitely many elements of $T$ and zero on infinitely many elements of $T$. Since $g(\eta - \xi) - (\eta - \xi)$ is constant on $T$, the element $g(\eta) - \eta$ has infinite support, contradicting the assumption that $[\eta, U] \le H$. We conclude that $P$ is maximal.

Thus $G$ can have Sylow $p$-subgroups only if $S$ has no proper infinite subgroups; i.e., only if $S \cong \mathbb{Z}/p^\infty$. □

The following well known property of finite $p$-groups will also be needed for discrete $p$-toral groups.

**Lemma 1.8.** *Let $S$ be a discrete $p$-toral group, and let $1 \ne P \trianglelefteq S$ be a nontrivial normal subgroup. Then $P \cap \Omega_1(Z(S)) \ne 1$.*

*Proof.* If $P$ is finite, then $P \cap Z(S)$ is the fixed subgroup of the action of the finite $p$-group $S/C_S(P)$ on $P$, and hence is nontrivial. So $P \cap \Omega_1(Z(S)) = \Omega_1(P \cap Z(S)) \ne 1$.

If $P$ is infinite, let $U \trianglelefteq P$ be its identity component. Then $\Omega_1(U) \trianglelefteq S$, and so $P \cap \Omega_1(Z(S)) \ge \Omega_1(U) \cap \Omega_1(Z(S)) \ne 1$ since $\Omega_1(U)$ is finite. □

We next recall some more definitions.

**Definition 1.9.** A *fusion system* $\mathcal{F}$ over a discrete $p$-toral group $S$ is a category whose objects are the subgroups of $S$, where

$$\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$$

for each $P, Q \le S$, and such that $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ implies $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P), P)$. Here, $\mathrm{Inj}(P,Q)$ is the set of injective homomorphisms from $P$ to $Q$.

When $\mathcal{F}$ is a fusion system over $S$, then for $P \le S$ and $x \in S$ we set

$$P^{\mathcal{F}} = \{\varphi(P) \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)\} \qquad \text{and} \qquad x^{\mathcal{F}} = \{\varphi(x) \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, S)\} :$$

the *$\mathcal{F}$-conjugacy classes* of $P$ and $x$. We also, for each $P \le S$, write

$$\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Hom}_{\mathcal{F}}(P,P) \qquad \text{and} \qquad \mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P),$$

and refer to $\mathrm{Aut}_{\mathcal{F}}(P)$ as the *automizer* of $P$ in $\mathcal{F}$.

**Definition 1.10.** Let $\mathcal{F}$ be a fusion system over a discrete $p$-toral group $S$.

(a) A subgroup $P \le S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \ge |C_S(P^*)|$ for all $P^* \in P^{\mathcal{F}}$.

(b) A subgroup $P \le S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \ge |N_S(P^*)|$ for all $P^* \in P^{\mathcal{F}}$.

(c) A subgroup $P \le S$ is *fully automized in* $\mathcal{F}$ if $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ is finite and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$.

(d) A subgroup $P \le S$ is *receptive in* $\mathcal{F}$ if for each $Q \in P^{\mathcal{F}}$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$, if we set
$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\},$$
then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

(e) The fusion system $\mathcal{F}$ is *saturated* if the following three conditions hold:

- (Sylow axiom) Each subgroup $P \le S$ fully normalized in $\mathcal{F}$ is also fully automized and fully centralized in $\mathcal{F}$.

- (Extension axiom) Each subgroup $P \leq S$ fully centralized in $\mathcal{F}$ is also receptive in $\mathcal{F}$.
- (Continuity axiom) If $P \leq S$, and $\varphi \in \mathrm{Hom}(P,S)$ is an injective homomorphism such that $\varphi|_R \in \mathrm{Hom}_{\mathcal{F}}(R,S)$ for each $R \in \mathfrak{Fin}(P)$, then $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$.

When $\mathcal{F}$ is a fusion system over a finite $p$-group $S$, it follows directly from the definition that every subgroup of $S$ is $\mathcal{F}$-conjugate to a fully normalized and a fully centralized subgroup. When $\mathcal{F}$ is a fusion system over an infinite discrete $p$-toral group $S$, then this is still true, and is a consequence of [BLO3, Lemma 1.6].

**Definition 1.11.** When $G$ is a discrete group and $S \leq G$ is a discrete $p$-toral subgroup, let $\mathcal{F}_S(G)$ be the fusion system over $S$ where for each $P, Q \leq S$,

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \{c_g \mid g \in G,\ {}^gP \leq Q\}.$$

The following are some of the basic definitions for subgroups in a fusion system. Recall that for a finite group $G$, a proper subgroup $H < G$ is *strongly $p$-embedded* if $p \mid |H|$, and for each $x \in G \smallsetminus H$ we have $p \nmid |H \cap {}^xH|$.

**Definition 1.12.** Let $\mathcal{F}$ be a fusion system over a discrete $p$-toral group $S$. For a subgroup $P \leq S$,

- $P$ is *$\mathcal{F}$-centric* if $C_S(Q) \leq Q$ for each $Q \in P^{\mathcal{F}}$;
- $P$ is *$\mathcal{F}$-radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$;
- $P$ is *$\mathcal{F}$-essential* if it is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly $p$-embedded subgroup;
- $P$ is *weakly closed in $\mathcal{F}$* if $P^{\mathcal{F}} = \{P\}$; and
- $P$ is *strongly closed in $\mathcal{F}$* if $x^{\mathcal{F}} \subseteq P$ for each $x \in P$.

We let $\mathcal{F}^{rc} \subseteq \mathcal{F}^c$ denote the sets of subgroups of $S$ that are $\mathcal{F}$-centric and $\mathcal{F}$-radical, or $\mathcal{F}$-centric, respectively.

The next proposition gives some of the basic finiteness properties of fusion systems in this context.

**Proposition 1.13** ([BLO3, Lemma 2.5 and Corollary 3.5]). *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$. Then*

(a) $\mathrm{Hom}_{\mathcal{F}}(P,Q)/\mathrm{Inn}(Q)$ *is finite for each $P, Q \leq S$; and*

(b) $\mathcal{F}^{rc}$ *is the union of finitely many $S$-conjugacy classes of subgroups.*

**Lemma 1.14.** *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$, and assume $P \leq S$ is fully normalized in $\mathcal{F}$. Then for each $Q \in P^{\mathcal{F}}$, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$ such that $\varphi(Q) = P$.*

*Proof.* By definition, every fully normalized subgroup is also fully automized and receptive. So the lemma follows from [BLO6, Lemma 1.7(c)]. □

We will be using the following version of Alperin's fusion theorem for fusion systems over discrete $p$-toral groups.

**Theorem 1.15** ([BLO3, Theorem 3.6]). *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$. Then each morphism in $\mathcal{F}$ is a composite of restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(Q)$ for fully normalized subgroups $Q \in \mathcal{F}^{rc}$.*

We now turn our attention to quotient fusion systems.

**Definition 1.16.** Let $\mathcal{F}$ be a fusion system over a discrete $p$-toral group $S$, and assume $Q \le S$ is weakly closed in $\mathcal{F}$. In particular, $Q \trianglelefteq S$. Let $\mathcal{F}/Q$ be the fusion system over $S/Q$ defined by setting, for each $P, R \le S$ containing $Q$,

$$\mathrm{Hom}_{\mathcal{F}/Q}(P/Q, R/Q) = \{\varphi/Q \mid \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)\};$$

where $\varphi/Q \in \mathrm{Hom}(P/Q, R/Q)$ sends $xQ$ to $\varphi(x)Q$ for all $x \in P$.

**Lemma 1.17.** *Let $\mathcal{F}$ be a fusion system over a discrete $p$-toral group $S$, and assume $Q \le S$ is weakly closed in $\mathcal{F}$. If $\mathcal{F} = \mathcal{F}_S(G)$ for some discrete group $G$ with $S \le G$, then $\mathcal{F}/Q = \mathcal{F}_{S/Q}(N_G(Q)/Q)$.*

*Proof.* The inclusion $\mathcal{F}_{S/Q}(N_G(Q)/Q) \le \mathcal{F}_S(G)/Q = \mathcal{F}/Q$ is clear. Conversely, for each $P, R \le S$ containing $Q$ and each $g \in G$ such that ${}^gP \le R$, since $c_g \in \mathrm{Hom}_{\mathcal{F}}(P, R)$ and $Q$ is weakly closed in $\mathcal{F}$, we have $g \in N_G(Q)$ and hence $c_{gQ} \in \mathrm{Hom}_{\mathcal{F}/Q}(P/Q, R/Q)$. So $\mathcal{F}/Q = \mathcal{F}_{S/Q}(N_G(Q)/Q)$. □

The proof that quotient systems of saturated fusion systems are again saturated is also elementary.

**Lemma 1.18.** *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$, and assume $Q \le S$ is weakly closed in $\mathcal{F}$. Then $\mathcal{F}/Q$ is a saturated fusion system over $S/Q$.*

*Proof.* By [BLO6, Corollary 1.8], it suffices to show that

(i) every subgroup of $S/Q$ is $\mathcal{F}/Q$-conjugate to one that is fully automized and receptive; and

(ii) the continuity axiom holds for $\mathcal{F}/Q$.

**(i)** (The following argument is based on the proof of [AKO, Lemma II.5.4].) Fix a subgroup $P/Q \le S/Q$, and choose $R \in P^{\mathcal{F}}$ such that $R$ is fully normalized in $\mathcal{F}$. Then $R$ is fully automized and receptive in $\mathcal{F}$ since $\mathcal{F}$ is saturated. Also, $R \ge Q$ and $R/Q \in (P/Q)^{\mathcal{F}/Q}$, so it will suffice to prove that $R/Q$ is fully automized and receptive in $\mathcal{F}/Q$.

For each $U, V \le S$ containing $Q$, let

$$\Psi_{U,V} \colon \mathrm{Hom}_{\mathcal{F}}(U, V) \longrightarrow \mathrm{Hom}_{\mathcal{F}/Q}(U/Q, V/Q)$$

be the natural map that sends $\varphi$ to $\varphi/Q$. By definition of $\mathcal{F}/Q$, $\Psi_{U,V}$ is surjective for all $U$ and $V$. Also, $\Psi_{U,V}(\mathrm{Hom}_S(U, V)) = \mathrm{Hom}_{S/Q}(U/Q, V/Q)$ in all cases. Since $\mathrm{Aut}_S(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(R))$ (recall $R$ is assumed fully automized), we have $\mathrm{Aut}_{S/Q}(R/Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}/Q}(R/Q))$, and hence $R/Q$ is fully automized in $\mathcal{F}/Q$.

To see that $R/Q$ is receptive in $\mathcal{F}/Q$, fix an isomorphism $\widehat{\varphi} \in \mathrm{Iso}_{\mathcal{F}/Q}(U/Q, R/Q)$, and choose $\varphi \in \Psi_{U,R}^{-1}(\widehat{\varphi}) \subseteq \mathrm{Iso}_{\mathcal{F}}(U, R)$. Let $N_\varphi \le N_S(U)$ and $N_{\widehat{\varphi}} \le N_{S/Q}(U/Q)$ be as in Definition 1.10(d). For each $g \in N_\varphi$, we have $\varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(R)$, and hence $\widehat{\varphi} c_{gQ} \widehat{\varphi}^{-1} \in \Psi_{R,R}(\mathrm{Aut}_S(R)) = \mathrm{Aut}_{S/Q}(R/Q)$, proving that $gQ \in N_{\widehat{\varphi}}$ and hence that $N_\varphi/Q \le N_{\widehat{\varphi}}$.

Let $N \le N_S(U)$ be such that $N/Q = N_{\widehat{\varphi}}$. Then

$$\Psi_{R,R}(\varphi \mathrm{Aut}_N(U) \varphi^{-1}) = \widehat{\varphi} \mathrm{Aut}_{N_{\widehat{\varphi}}}(U/Q) \widehat{\varphi}^{-1} \le \mathrm{Aut}_{S/Q}(R/Q) = \Psi_{R,R}(\mathrm{Aut}_S(R)).$$

Since $R$ is fully automized, $\mathrm{Aut}_S(R) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(R))$, and hence $\mathrm{Aut}_S(R) \in \mathrm{Syl}_p(\mathrm{Aut}_S(R)\mathrm{Ker}(\Psi_{R,R}))$. So there is $\psi \in \mathrm{Ker}(\Psi_{R,R})$ such that ${}^{\psi\varphi}\mathrm{Aut}_N(U) \le \mathrm{Aut}_S(R)$. Upon replacing $\varphi$ by $\psi\varphi$, we get $N_{\widehat{\varphi}} = N_\varphi/Q$, and still have $\widehat{\varphi} = \varphi/Q$.

Since $R$ is receptive in $\mathcal{F}$, the isomorphism $\varphi$ extends to $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$. So $\overline{\varphi}/Q \in \mathrm{Hom}_{\mathcal{F}/Q}(N_{\widehat{\varphi}}, S/Q)$ is an extension of $\widehat{\varphi}$, and we conclude that $R/Q$ is receptive in $\mathcal{F}/Q$.

**(ii)** Assume $P \geq Q$ and $\widehat{\varphi} \in \mathrm{Hom}(P/Q, S/Q)$ are such that $\widehat{\varphi}|_{R/Q} \in \mathrm{Hom}_{\mathcal{F}/Q}(R/Q, S/Q)$ for each $R/Q \in \mathfrak{Fin}(P/Q)$. Choose $Q \leq P_1 \leq P_2 \leq \cdots \leq P$ such that $|P_i/Q| < \infty$ for each $i$ and $P = \bigcup_{i=1}^{\infty} P_i$. For each $i$, set

$$\Phi_i = \{\psi \in \mathrm{Hom}_{\mathcal{F}}(P_i, S) \,|\, \psi/Q = \widehat{\varphi}|_{P_i/Q}\}.$$

Thus $(\Phi_i)_{i\geq 1}$ is an inverse system of sets via restriction of morphisms, and $\Phi_i \neq \varnothing$ for each $i$ since $\widehat{\varphi}|_{P_i/Q} \in \mathrm{Mor}(\mathcal{F}/Q)$. We claim that $\lim_i(\Phi_i) \neq \varnothing$.

Let $\Gamma \leq S$ be such that $Q \trianglelefteq \Gamma$ and $\Gamma/Q = C_{S/Q}(\widehat{\varphi}(P/Q))$. Since $S/Q$ is artinian and the centralizers $C_{S/Q}(\widehat{\varphi}(P_i/Q))$ form a descending sequence intersecting in $\Gamma/Q$, there is $m \geq 1$ such that $\Gamma/Q = C_{S/Q}(\widehat{\varphi}(P_i/Q))$ for each $i \geq m$. Also, for each $i \geq 1$, and each $\psi \in \Phi_i$ and $\gamma \in \Gamma$, we have $c_\gamma \circ \psi \in \Phi_i$ since $[\gamma, \psi(P_i)] \leq Q$ (since $\gamma Q \in C_{S/Q}(\widehat{\varphi}(P_i/Q))$). Thus $\gamma \in \Gamma$ acts on $\Phi_i$ via composition with $c_\gamma$.

Assume $i \geq m$, $\psi \in \Phi_i$, and $x \in S$ are such that $c_x\psi \in \Phi_i$. Then $[x, \psi(P_i)] \leq Q$, so $[xQ, \widehat{\varphi}(P_i/Q)] = 1$, and hence $x \in \Gamma$. Thus two elements of $\Phi_i$ in the same $\mathrm{Inn}(S)$-orbit are in the same $\Gamma$-orbit. So the natural map $\Phi_i/\Gamma \longrightarrow \mathrm{Hom}_{\mathcal{F}}(P_i, S)/\mathrm{Inn}(S)$ is injective, and hence $\Phi_i/\Gamma$ is finite since $\mathrm{Hom}_{\mathcal{F}}(P_i, S)/\mathrm{Inn}(S)$ is finite by Proposition 1.13(a).

We thus have an inverse system $(\Phi_i)_{i\geq m}$ of nonempty $\Gamma$-sets with finite orbit sets, and hence $\lim_i(\Phi_i) \neq \varnothing$ by Lemma 1.4. Choose $(\psi_i)_{i\geq m} \in \lim_i(\Phi_i)$. So $\psi_i|_{P_{i-1}} = \psi_{i-1}$ for each $i > m$, and we can define $\psi = \bigcup_{i=m}^{\infty} \psi_i \in \mathrm{Hom}(P, S)$. Then $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ by the continuity axiom for $\mathcal{F}$, and $\widehat{\varphi} = \psi/Q \in \mathrm{Hom}_{\mathcal{F}/Q}(P/Q, S/Q)$ since $(\psi/Q)_{P_i/Q} = \psi_i/Q = \widehat{\varphi}|_{P_i/Q}$ for each $i \geq m$. □

We also need to work with isomorphisms between fusion systems. Recall, in the following definition, that we write composition from right to left.

**Definition 1.19.** Let $\mathcal{F}_i$ be a fusion system over the discrete $p$-toral group $S_i$ for $i = 1, 2$. An *isomorphism* $(\rho, \widehat{\rho})\colon \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ consists of an isomorphism of groups $\rho\colon S_1 \xrightarrow{\cong} S_2$ and an isomorphism of categories $\widehat{\rho}\colon \mathcal{F}_1 \xrightarrow{\cong} \mathcal{F}_2$ such that $\widehat{\rho}$ sends an object $P$ to $\rho(P)$, and sends a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}_1}(P, Q)$ to the morphism $\widehat{\rho}(\varphi) \in \mathrm{Hom}_{\mathcal{F}_2}(\rho(P), \rho(Q))$ such that

$$\widehat{\rho}(\varphi) \circ (\rho|_P) = (\rho|_Q) \circ \varphi \in \mathrm{Hom}(P, \rho(Q)).$$

Note that in the above definition, the functor $\widehat{\rho}$ is uniquely determined by the isomorphism $\rho\colon S_1 \longrightarrow S_2$. So in practice, we regard an isomorphism of fusion systems as an isomorphism between the Sylow groups that satisfies the extra conditions needed for there to exist an isomorphism of categories.

When $\mathcal{F}$ is a saturated fusion system over a discrete $p$-toral group $S$, a *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of $S$, together with a functor $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}$ that is the inclusion on objects and is surjective on morphism sets and satisfies certain additional conditions. The following definition describes one way to construct centric linking systems under certain conditions on a discrete group $G$ and a discrete $p$-toral subgroup $S \leq G$.

**Definition 1.20.** Let $G$ be a discrete group.

(a) A $p$-subgroup $P \leq G$ is *$p$-centric in $G$* if $Z(P)$ is the unique Sylow $p$-subgroup of $C_G(P)$ (equivalently, $C_G(P)/Z(P)$ has no elements of order $p$).

(b) If $G$ is locally finite and $S \leq G$ is a discrete $p$-toral subgroup, let $\mathcal{L}_S^c(G)$ be the category whose objects are the subgroups of $S$ that are $p$-centric in $G$, and where for each pair of objects $P, Q \leq S$ we set

$$\mathrm{Mor}_{\mathcal{L}_S^c(G)}(P,Q) = \{g \in G \,|\, {}^gP \leq Q\}/O^p(C_G(P)).$$

Here, $O^p(C_G(P))$ means the subgroup generated by all elements of order prime to $p$ in $C_G(P)$.

Since linking systems play a relatively minor role throughout most of this paper (mostly mentioned as "additional information"), we don't give the precise definition here, but instead refer to [BLO3, Definition 4.1]. The exception to this is when we look at fusion systems of compact Lie groups (Section 4) and those of $p$-compact groups (Section 8). Linking systems do play an important role in those two sections, and we list there the precise properties of linking systems that we need to use.

When $\mathcal{L}$ is a centric linking system associated to a fusion system $\mathcal{F}$, we let $|\mathcal{L}|$ denote its geometric realization (see, e.g., [AKO, § III.2.2]). We regard $|\mathcal{L}|_p^\wedge$ as a "classifying space" for $\mathcal{F}$, where $(-)_p^\wedge$ denotes $p$-completion in the sense of Bousfield and Kan (see [AKO, § III.1.4] for a brief summary of some of its elementary properties). By [BLO3, Theorem 7.4], two saturated fusion systems are isomorphic if their classifying spaces are homotopy equivalent, and this is the basis for showing in Section 4 that fusion systems of compact Lie groups can all be realized by certain discrete linear groups.

## 2. Sequential realizability

When working with fusion systems over finite $p$-groups, it is natural to say that a fusion system is realizable if it is isomorphic to the fusion system of a finite group, and is exotic otherwise. When we turn to fusion systems over discrete $p$-toral groups, it is less clear what is the most natural condition for realizability. In this section, we define sequential realizability and look at its basic properties. For example, we show in Corollary 2.5 that the fusion system of a countable locally finite group with Sylow $p$-subgroups is sequentially realizable if it is saturated.

No fusion system over an infinite discrete $p$-toral group $S$ can be the union *as categories* of fusion subsystems over finite subgroups of $S$, since $S$ itself can't be an object in any such union. So before we define sequential realizability, we need to make precise what we mean by an infinite union of fusion systems. The following definition can be thought of as a simplified version of the definition in [Gz, Definition 3.1] of a finite approximation of linking systems.

**Definition 2.1.** Assume $\mathcal{F}_1 \leq \mathcal{F}_2 \leq \mathcal{F}_3 \leq \cdots$ is an increasing sequence of saturated fusion systems over finite subgroups $S_1 \leq S_2 \leq S_3 \leq \cdots$ of a discrete $p$-toral group $S = \bigcup_{i=1}^\infty S_i$. Define $\bigcup_{i=1}^\infty \mathcal{F}_i$ to be the fusion system over $S$ where for each $P, Q \leq S$,

$$\mathrm{Hom}_{\bigcup_{i=1}^\infty \mathcal{F}_i}(P,Q) = \Bigl\{\varphi \in \mathrm{Hom}(P,Q) \Bigm| \forall\, R \in \mathfrak{Fin}(P)\ \exists\, i \geq 1 \\ \text{such that } R \leq S_i \text{ and } \varphi|_R \in \mathrm{Hom}_{\mathcal{F}_i}(R, Q \cap S_i)\Bigr\}.$$

In other words, we define the union of an increasing sequence of fusion systems $\mathcal{F}_i$ over $S_1 \leq S_2 \leq \ldots$ to be the smallest fusion system over $\bigcup_{i=1}^\infty S_i$ that contains the $\mathcal{F}_i$ and satisfies the continuity axiom (see Definition 1.10(e)).

**Definition 2.2.** A fusion system $\mathcal{F}$ over a discrete $p$-toral group $S$ is *sequentially realizable* if there is an increasing sequence $\mathcal{F}_1 \leq \mathcal{F}_2 \leq \ldots$ of saturated fusion subsystems of $\mathcal{F}$ over finite subgroups $S_1 \leq S_2 \leq \ldots$ of $S$, such that $S = \bigcup_{i=1}^\infty S_i$ and $\mathcal{F} = \bigcup_{i=1}^\infty \mathcal{F}_i$, and such that each $\mathcal{F}_i$ is realizable by a finite group.

Note that we do not assume in the definition that $\mathcal{F}$ is saturated. We will show in a later paper that sequentially realizable fusion systems over discrete $p$-toral groups are always saturated. In this paper, when we work with sequentially realizable fusion systems, we always say explicitly whether or not we assume that they are saturated.

In all examples where we can prove that a fusion system is sequentially realizable, we do so by showing that it is realized by a linear torsion group, as defined in the next section.

We next look at fusion systems over finite $p$-groups.

**Theorem 2.3.** *Let $\mathcal{F}$ be a fusion system over a finite $p$-group $S$. Assume that either*

(i) *$\mathcal{F}$ is sequentially realizable; or*

(ii) *$\mathcal{F} \cong \mathcal{F}_S(G)$ for some locally finite group $G$ that contains $S$ as a maximal $p$-subgroup.*

*Then $\mathcal{F} \cong \mathcal{F}_S(G_0)$ for some finite subgroup $G_0 \le G$ that contains $S$ as a Sylow $p$-subgroup.*

*Proof.* **(i)** If $\mathcal{F}$ is sequentially realizable, then $\mathcal{F} = \bigcup_{i=1}^\infty \mathcal{F}_i$ for some increasing sequence $\mathcal{F}_1 \le \mathcal{F}_2 \le \cdots$ of finite subsystems, each of which is realized by a finite group. But since $\mathcal{F}$ is finite, we have $\mathcal{F} = \mathcal{F}_i$ for some $i$, and hence it is realized by a finite group.

**(ii)** Assume $\mathcal{F} = \mathcal{F}_S(G)$ for some locally finite group $G$ which contains $S$ as a maximal $p$-subgroup. Let $\rho\colon \mathrm{Mor}(\mathcal{F}) \longrightarrow G$ be a map of sets that sends a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ to an element $\rho(\varphi)$ such that $\varphi = c_{\rho(\varphi)}|_P$. Set $G_0 = \langle S, \mathrm{Im}(\rho)\rangle$. Then $G_0$ is finitely generated since $\mathrm{Mor}(\mathcal{F})$ is finite, and is finite since $G$ is locally finite. Also, $S \in \mathrm{Syl}_p(G_0)$ since it is a maximal $p$-subgroup of $G$.

By construction, $\mathcal{F} \le \mathcal{F}_S(G_0) \le \mathcal{F}_S(G) = \mathcal{F}$. So $\mathcal{F}$ is realized by $G_0$. □

The next proposition shows that fusion systems of certain locally finite groups are sequentially realizable. By “countable” we always mean “at most countable” (i.e., possibly finite).

**Proposition 2.4.** *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$. Assume $G$ is a locally finite group containing $S$ such that $\mathcal{F} = \mathcal{F}_S(G)$. Then the following hold.*

(a) *There is a countable subgroup $G_* \le G$ such that $S \le G_*$ and $\mathcal{F} = \mathcal{F}_S(G_*)$.*

(b) *If $G$ is countable and $S \in \mathrm{Syl}_p(G)$, then there is an increasing sequence of finite subgroups $\{G_i\}_{i\ge1}$ of $G$ such that $S_i \overset{\text{def}}{=} S \cap G_i \in \mathrm{Syl}_p(G_i)$ for each $i$, and $G = \bigcup_{i=1}^\infty G_i$. For each such sequence,*

  (b.1) *$S = \bigcup_{i=1}^\infty S_i$, and*

  (b.2) *$\mathcal{F} = \bigcup_{i=1}^\infty \mathcal{F}_{S_i}(G_i)$ in the sense of Definition 2.1.*

*Proof.* **(a)** By Proposition 1.13 and since $\mathcal{F}$ is saturated, there are finitely many $S$-conjugacy classes of subgroups in $\mathcal{F}^{rc}$, and $\mathrm{Hom}_{\mathcal{F}}(P,Q)/\mathrm{Inn}(Q)$ is finite for each $P,Q \le S$. Since $S$ is countable, this implies that the object set $\mathcal{F}^{rc}$ is countable, and that $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ is countable for each $P,Q \le S$. Hence the category $\mathcal{F}^{rc}$ has countably many morphisms.

Choose a map of sets $\rho\colon \mathrm{Mor}(\mathcal{F}^{rc}) \longrightarrow G$ that sends a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ (for $P,Q \in \mathcal{F}^{rc}$) to an element $\rho(\varphi)$ such that $\varphi = c_{\rho(\varphi)}|_P$. Set $G_* = \langle S \cup \mathrm{Im}(\rho)\rangle$. Thus $G_*$ is countably generated since $S$ and $\mathrm{Mor}(\mathcal{F}^{rc})$ are countable, and hence is countable. Also, $\mathcal{F} \le \mathcal{F}_S(G_*) \le \mathcal{F}_S(G)$ where the first inclusion holds by Theorem 1.15, and the inclusions are equalities since $\mathcal{F} = \mathcal{F}_S(G)$.

**(b)** Now assume $G$ is countable and $S \in \mathrm{Syl}_p(G)$. Choose a sequence of elements $h_1, h_2, h_3, \ldots$ in $G$ such that $G = \langle h_i \,|\, i \geq 1 \rangle$, and set $H_m = \langle h_1, h_2, \ldots, h_m \rangle$ for each $m \geq 1$. Then each $H_m$ is finite since $G$ is locally finite (Definition 3.1), and we can choose subgroups $T_1 \leq T_2 \leq T_3 \leq \ldots$ such that $T_i \in \mathrm{Syl}_p(H_i)$ for each $i$. Set $T = \bigcup_{i=1}^{\infty} T_i$. Then ${}^xT \leq S$ for some $x \in G$ (since $S \in \mathrm{Syl}_p(G)$), and upon setting $G_i = {}^xH_i$ and $S_i = {}^xT_i$, we get $S_i \leq S \cap G_i$ with equality since $S_i \in \mathrm{Syl}_p(G_i)$. Also, $\bigcup_{i=1}^{\infty} G_i = \bigcup_{i=1}^{\infty} H_i = G$ by construction (and since $x \in G$).

For the rest of the proof, $G_1 \leq G_2 \leq \ldots$ and $S_1 \leq S_2 \leq \ldots$ are arbitrary finite subgroups of $G$ and $S$, respectively, such that $G = \bigcup_{i=1}^{\infty} G_i$ and $S_i \in \mathrm{Syl}_p(G_i)$. Note that $S_i = S \cap G_i$ for each $i$ since $S_i$ is a maximal $p$-subgroup of $G_i$.

**(b.1)** By definition, $\bigcup_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty}(G_i \cap S) = \bigl(\bigcup_{i=1}^{\infty} G_i\bigr) \cap S = S$.

**(b.2)** Set $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$ for each $i$; thus $\mathcal{F}_i \leq \mathcal{F}$ by construction. To prove that $\mathcal{F}$ is the union of the $\mathcal{F}_i$, fix a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ for some $P, Q \leq S$, and let $g \in G$ be such that ${}^gP \leq Q$ and $\varphi = c_g|_P$. Choose $n \geq 1$ such that $g \in G_n$. For each $i \geq n$, if we set $P_i = P \cap G_i$ and $Q_i = Q \cap G_i$, then $c_g^{P_i} \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i)$ is the restriction of $\varphi$. Also, $P = \bigcup_{i=1}^{\infty} P_i$, and so $\mathcal{F}$ is contained in the union of the $\mathcal{F}_i$, with equality since it satisfies the continuity axiom. □

**Corollary 2.5.** *Let $G$ be a locally finite group, and let $S \leq G$ be a discrete $p$-toral subgroup. Assume*

(i) *$S \in \mathrm{Syl}_p(G_*)$ for some countable $G_* \leq G$ such that $\mathcal{F}_S(G_*) = \mathcal{F}_S(G)$; and*

(ii) *$\mathcal{F}_S(G)$ is saturated.*

*Then $\mathcal{F}_S(G)$ is sequentially realizable.*

*Proof.* Set $\mathcal{F} = \mathcal{F}_S(G)$, and let $G_* \leq G$ be as in (i). Thus $\mathcal{F}$ is a saturated fusion system by (ii), $G_*$ is countable, $S \in \mathrm{Syl}_p(G_*)$, and $\mathcal{F}_S(G_*) = \mathcal{F}$. By Proposition 2.4(b) and since $S \in \mathrm{Syl}_p(G_*)$, there is an increasing sequence of finite subgroups $G_1 \leq G_2 \leq \ldots$ of $G_*$ such that $S \cap G_i \in \mathrm{Syl}_p(G_i)$ for each $i$, $S = \bigcup_{i=1}^{\infty}(S \cap G_i)$, and $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_{S\cap G_i}(G_i)$. So $\mathcal{F} = \mathcal{F}_S(G)$ is sequentially realizable. □

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems over discrete $p$-toral groups $S_1$ and $S_2$, respectively, set $S = S_1 \times S_2$, and let $\mathrm{pr}_i \colon S \longrightarrow S_i$ be the projection for $i = 1, 2$. The product $\mathcal{F}_1 \times \mathcal{F}_2$ is defined to be the fusion system over $S$, where for each $P, Q \leq S$ we have

$$\mathrm{Hom}_{\mathcal{F}_1\times\mathcal{F}_2}(P, Q) = \bigl\{(\varphi_1, \varphi_2)|_P \,\big|\, \varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(\mathrm{pr}_i(P), \mathrm{pr}_i(Q)),\ (\varphi_1, \varphi_2)(P) \leq Q\bigr\}.$$

In other words, it is the smallest fusion system over $S$ that contains all morphisms $\varphi_1 \times \varphi_2$ for $\varphi_i \in \mathrm{Mor}(\mathcal{F}_i)$. It is also the largest fusion system $\mathcal{F}$ over $S$ in which $S_1$ and $S_2$ are strongly closed, and such that $\mathcal{F}/S_1 \leq \mathcal{F}_2$ and $\mathcal{F}/S_2 \leq \mathcal{F}_1$. One can show that a product of fusion systems is saturated if each factor is, but we won't need to use that here.

We next check that sequential realizability of fusion systems is preserved by products and quotients.

**Proposition 2.6.** *Let $\mathcal{F}$ be a fusion system over a discrete $p$-toral group $S$.*

(a) *If $\mathcal{F}$ is sequentially realizable, then so is $\mathcal{F}/Q$ for each subgroup $Q \trianglelefteq S$ strongly closed in $\mathcal{F}$.*

(b) *If $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ for some pair of fusion subsystems $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\mathcal{F}$ is sequentially realizable if and only if each of the factors $\mathcal{F}_i$ is sequentially realizable.*

*Proof.* **(a)** Assume $Q \trianglelefteq S$ is strongly closed in $\mathcal{F}$ and $\mathcal{F}$ is sequentially realizable. Thus there are finite subgroups $S_1 \le S_2 \le S_3 \le \dots$ of $S$, fusion subsystems $\mathcal{F}_1 \le \mathcal{F}_2 \le \dots$ of $\mathcal{F}$, and finite groups $G_1, G_2, \dots$ such that $S = \bigcup_{i=1}^{\infty} S_i$ and $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$; and also $S_i \in \mathrm{Syl}_p(G_i)$ and $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$ for each $i$.

For each $i$, set $Q_i = S_i \cap Q$. Then $Q_i$ is strongly closed in $\mathcal{F}_i$ since $Q$ is strongly closed in $\mathcal{F}$, and $\mathcal{F}_i/Q_i = \mathcal{F}_{S_i/Q_i}(N_{G_i}(Q_i)/Q_i)$ by Lemma 1.17. Also, $\mathcal{F}_i/Q_i$ is isomorphic to the image of $\mathcal{F}_i \le \mathcal{F}$ in $\mathcal{F}/Q$, and so $\mathcal{F}/Q$ is the union of fusion subsystems isomorphic to the $\mathcal{F}_i/Q_i$. Thus $\mathcal{F}/Q$ is sequentially realizable.

**(b)** If $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, where $\mathcal{F}_i$ is a fusion system over $S_i$ and $S = S_1 \times S_2$, then by (a), $\mathcal{F}_1 \cong \mathcal{F}/S_2$ and $\mathcal{F}_2 \cong \mathcal{F}/S_1$ are sequentially realizable if $\mathcal{F}$ is. The converse is clear: if $\mathcal{F}_1$ and $\mathcal{F}_2$ are both sequentially realizable, then so is their product. □

We finish the section with the following example of an increasing sequence of finite saturated fusion systems that alternate between being realizable and exotic. Thus the union of the fusion systems is sequentially realizable (in fact, realized by a linear torsion group), but it is also the union of an increasing sequence of finite exotic subsystems.

**Example 2.7.** Fix an odd prime $p$ and a prime $q \equiv 1 \pmod{p^2}$. Set $K = \bigcup_{i=0}^{\infty} \mathbb{F}_{q^{p^i}} \subseteq \overline{\mathbb{F}}_q$. Set $G = PSL_p(K)$, and $G_i = PSL_p(q^{p^i})$ for all $i \ge 0$. Then there are Sylow $p$-subgroups $S_i \in \mathrm{Syl}_p(G_i)$ and $S \in \mathrm{Syl}_p(G)$ such that $S_i = G_i \cap S$ for each $i \ge 0$, and such that if we set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{F}_{2i} = \mathcal{F}_{S_i}(G_i)$ for all $i \ge 0$, then there are exotic saturated fusion systems $\mathcal{F}_{2i+1}$ over $S_{i+1}$ for all $i$ such that $\mathcal{F}_{2i} \le \mathcal{F}_{2i+1} \le \mathcal{F}_{2i+2}$. Thus $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$, where the $\mathcal{F}_i$ are realizable for even $i$ and exotic for odd $i$.

*Proof.* Set $e = v_p(q-1)$; thus $e \ge 2$ by assumption. Then $v_p(q^{p^i} - 1) = e + i$ for each $i \ge 0$ (see, e.g., [BMO2, Lemma 1.13]). Let $\widehat{A} \le SL_p(K)$ be the group of all diagonal matrices of $p$-power order, and set $Z = Z(SL_p(K)) = \langle \zeta \cdot \mathrm{Id} \rangle$ where $\zeta \in \mathbb{F}_q^\times$ has order $p$. Set $\widehat{A}_i = \widehat{A} \cap SL_p(q^{p^i})$ for each $i \ge 0$, and also $A = \widehat{A}/Z \le G$ and $A_i = \widehat{A}_i/Z \le G_i$. Thus $\widehat{A}_i$ is homocyclic of rank $p-1$ and exponent $p^{e+i}$, and hence $A_i$ has rank $p-1$, exponent $p^{e+i}$, and order $p^{(e+i)(p-1)-1}$.

Let $x \in G_0 = SL_p(q)$ be a permutation matrix of order $p$ (i.e., a matrix with a unique entry 1 in each row an column and otherwise 0). Set $S = A\langle x \rangle$ (the subgroup of $G$ generated by $A$ and $x$), and set $S_i = A_i\langle x \rangle$ for each $i$. Then $S_i \in \mathrm{Syl}_p(G_i)$ (see [Wr, § 2]), and hence $S = \bigcup_{i=0}^{\infty} S_i \in \mathrm{Syl}_p(G)$ by Proposition 1.6 (every finite $p$-subgroup of $G$ is contained in $G_i$ for some $i$, and hence is conjugate to a subgroup of $S$). By definition, $S_i = S \cap G_i$ for each $i$.

For each $i \ge 0$, let $\mathcal{H}^{(i)}$ be the set of subgroups $E \le S_i$ such that $E \cong C_p \times C_p$ and $E \cap A_i = Z(S_i) \cong C_p$. By [COS, Lemma 2.2(f)], there are exactly $p$ $S_i$-conjugacy classes of subgroups in $\mathcal{H}^{(i)}$, and for $x \in S_i \smallsetminus A_i$ and $a \in A_i$, $Z(S_i)\langle x \rangle$ is $S_i$-conjugate to $Z(S_i)\langle ax \rangle$ if and only if $a \in [S_i, S_i] = [A_i, x]$. (Note that $[S_i, S_i]$ has index $p$ in $A_i$.)

Now assume $i \ge 1$. Since $A_{i-1} \le [S_i, S_i]$, the members of $\mathcal{H}^{(i-1)}$ are all contained in one of the $S_i$-conjugacy classes $\mathcal{H}_0^{(i)} \subseteq \mathcal{H}^{(i)}$. By case (iii) of [COS, Theorem 2.8], or case (a.i) of [O1, Theorem 2.8] if $p = 3$, there is a unique saturated fusion system $\mathcal{F}_{2i-1} \le \mathcal{F}_{2i}$ over $S_i$ whose essential subgroups are the members of $\mathcal{H}_0^{(i)}$, together with $A_i$ if $p \ge 5$. Furthermore, this fusion system is exotic by [COS, Table 2.2] or the last statement in [O1, Theorem 2.8].

It remains only to check that $\mathcal{F}_{2i-1} \ge \mathcal{F}_{2i-2}$. In both fusion systems, the members of $\mathcal{H}^{(i-1)}$ are essential, and we claim that they all have automizer $SL_2(p)$. This is shown in case (a.i) of [O1, Theorem 2.8] when $p = 3$, and follows by a similar argument when $p \ge 5$.

Since $\mathcal{F}_{2i-1}$ and $\mathcal{F}_{2i-2}$ are generated by $N_{\mathcal{F}_{2i}}(T_i) \geq N_{\mathcal{F}_{2i-2}}(T_{i-1})$ and these automizers, this shows that $\mathcal{F}_{2i-1} \geq \mathcal{F}_{2i-2}$. □

## 3. Linear torsion groups and LT-realizability

In this section, we recall the definition of linear torsion groups and prove that every fusion system that is realized by such a group is also sequentially realizable. In fact, this is our only tool so far for showing this: whenever we prove that a fusion system over an infinite discrete $p$-toral group is sequentially realizable, we do so by showing that it's realized by a linear torsion group.

**Definition 3.1.** (a) A *linear torsion group in characteristic* $q$ is a (discrete) group $G$, each of whose elements has finite order, that is isomorphic to a subgroup of $GL_n(K)$ for some $n \geq 1$ and some field $K$ of characteristic $q$.

(b) A fusion system $\mathcal{F}$ over a discrete $p$-toral group $S$ is LT-*realizable* (in characteristic $q$) if $\mathcal{F} \cong \mathcal{F}_{S^*}(G)$ for some linear torsion group $G$ in characteristic $q \neq p$ with $S^* \in \mathrm{Syl}_p(G)$.

Our interest in this class of groups is due mostly to the following proposition, shown in [BLO3]. Recall the definition of $\mathcal{L}_S^c(G)$ in Definition 1.20.

**Proposition 3.2.** *Fix a prime $p$, a field $K$ with* $\mathrm{char}(K) \neq p$*, and a linear torsion group $G \leq GL_n(K)$ (some $n \geq 1$). Then $G$ is locally finite, all $p$-subgroups of $G$ are discrete $p$-toral,* $\mathrm{Syl}_p(G) \neq \varnothing$*, and every maximal $p$-subgroup of $G$ is a Sylow $p$-subgroup. For each $S \in \mathrm{Syl}_p(G)$, the fusion system $\mathcal{F}_S(G)$ is saturated, $\mathcal{L}_S^c(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$, and $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$.*

*Proof.* By [BLO3, Proposition 8.8], $G$ is locally finite and all $p$-subgroups of $G$ are discrete $p$-toral. By Proposition 8.9 in the same paper, for each increasing sequence $A_1 \leq A_2 \leq \cdots$ of finite abelian $p$-subgroups of $G$, there is $r \geq 1$ such that $C_G(A_i) = C_G(A_r)$ for all $i \geq 1$. So all maximal $p$-subgroups of $G$ are conjugate to each other by [KW, Theorem 3.4], and hence they are Sylow $p$-subgroups. (Since the union of an increasing sequence of $p$-subgroups of $G$ is again a $p$-subgroup, every $p$-subgroup is contained in a maximal $p$-subgroup.)

For each $S \in \mathrm{Syl}_p(G)$, the fusion system $\mathcal{F}_S(G)$ is saturated, $\mathcal{L}_S^c(G)$ is a centric linking system, and $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$ by Theorem 8.7 or 8.10 in [BLO3]. □

The results in the last section imply that LT-realizable fusion systems are sequentially realizable.

**Proposition 3.3.** *Every LT-realizable fusion system is sequentially realizable.*

*Proof.* If $\mathcal{F} = \mathcal{F}_S(G)$ for a linear torsion group $G$ in characteristic different from $p$ and for $S \in \mathrm{Syl}_p(G)$, then $\mathcal{F}$ is saturated by Proposition 3.2. By Proposition 2.4(a), there is a countable subgroup $G_* \leq G$ such that $S \leq G_*$ and $\mathcal{F} = \mathcal{F}_S(G_*)$. Then $S$ is a maximal $p$-subgroup of $G_*$ and hence a Sylow $p$-subgroup by Proposition 3.2 again. So $\mathcal{F}$ is sequentially realizable by Corollary 2.5. □

It seems unlikely that the converse of Proposition 3.3 holds, but we know of no example of a sequentially realizable fusion system that is not LT-realizable, and it seems very difficult to construct one.

We now focus attention on the question of in which characteristics a given fusion system is LT-realizable.

When $G$ is a group and $p$ is a prime, the *$p$-rank* of $G$ is defined by setting

$$\mathrm{rk}_p(G) = \sup\{\mathrm{rk}(P) \mid P \le G \text{ a finite abelian } p\text{-subgroup}\}.$$

We note the following:

**Lemma 3.4.** *Let $G$ be a (discrete) group, and let $N \trianglelefteq G$ be a finite normal subgroup. Then $\mathrm{rk}_q(G/N) = \mathrm{rk}_q(G)$ for each prime $q$ with $q \nmid |N|$.*

*Proof.* If $H/N$ is a finite abelian $q$-subgroup of $G/N$, then $H$ is finite, and each $S \in \mathrm{Syl}_q(H)$ is a finite abelian $q$-subgroup of $G$ isomorphic to $H/N$. Thus $\mathrm{rk}_q(G) \ge \mathrm{rk}_q(G/N)$, and the opposite inequality is clear. □

It is not hard to see that Lemma 3.4 also holds if one assumes that $N$ be a normal discrete $p$-toral subgroup for some prime $p \ne q$.

**Definition 3.5.** For every group $G$ and every prime $p$, the *sectional $p$-rank* of $G$ is defined by setting

$$\begin{aligned}\mathrm{srk}_p(G) &= \sup\bigl\{\mathrm{rk}(P/\Phi(P)) \bigm| P \le G \text{ a finite } p\text{-subgroup of } G\bigr\}\\ &= \sup\bigl\{r \bigm| \exists\, N \trianglelefteq H \le G \text{ such that } H \text{ finite, } H/N \cong E_{p^r}\bigr\}.\end{aligned}$$

**Lemma 3.6.** *Fix a locally finite group $G$, a field $K$, and $n \ge 1$ such that $G \le GL_n(K)$. Then for each prime $p$,*

(a) *if $p \ne \mathrm{char}(K)$, then $\mathrm{rk}_p(G) \le n$ and $\mathrm{srk}_p(G) \le pn/(p-1)$; and*

(b) *if $\mathrm{char}(K) = p$, then $G$ has no elements of order $p^k$ if $n \le p^{k-1}$.*

*Proof.* We can assume without loss of generality that $K$ is algebraically closed.

**(a)** Let $D \le GL_n(K)$ be the subgroup of diagonal matrices. If $p \ne \mathrm{char}(K)$, then by elementary representation theory, all irreducible $K$-representations of a finite abelian group of order not divisible by $\mathrm{char}(K)$ are 1-dimensional. Thus every finite abelian $p$-subgroup of $GL_n(K)$ is conjugate to a subgroup of $D$, and hence $\mathrm{rk}_p(G) \le \mathrm{rk}_p(D) = n$.

Set $M = N_{GL_n(K)}(D)$; thus $M/D \cong \Sigma_n$. By [Se, §8.5, Theorem 16], every finite $p$-subgroup of $GL_n(K)$ is conjugate to a subgroup of $M$. So $\mathrm{srk}_p(G) \le n + m$, where $m$ is the sectional $p$-rank of $M/D \cong \Sigma_n$. Since $\Sigma_n$ has Sylow $p$-subgroups of order $p^e$ where $e = [n/p] + [n/p^2] + [n/p^3] + \cdots$, we have

$$\mathrm{srk}_p(G) \le n + m \le n + e \le n\bigl(1 + \tfrac{1}{p} + \tfrac{1}{p^2} + \tfrac{1}{p^3} + \cdots\bigr) = pn/(p-1).$$

**(b)** If $\mathrm{char}(K) = p > 0$, and $g \in G \le GL_n(K)$ has order $p^k$, then the minimal polynomial for $g$ divides $(X^{p^k} - 1) = (X-1)^{p^k}$, hence has the form $(X-1)^m$ for some $m \le n$, where $m > p^{k-1}$ since otherwise $|g| \le p^{k-1}$. □

**Remark 3.7.** *In fact, one can show that for $n \ge 1$ and $K$ an algebraically closed field with $\mathrm{char}(K) \ne p$, we have $\mathrm{srk}_p(GL_n(K)) = n$ if $p$ is odd, and $\mathrm{srk}_p(GL_n(K)) = [3n/2]$ if $p = 2$. But the above bound is good enough for our purposes here. When $p = 2$, the bound $[3n/2]$ is realized by taking a direct product of $[n/2]$ copies of $C_4 \circ D_8$, with an additional factor $C_2$ if $n$ is odd.*

Thus if $G$ is a linear torsion group in characteristic $q$, then $\mathrm{rk}_p(G) < \infty$ and $\mathrm{srk}_p(G) < \infty$ for every prime $p \ne q$, while if $q > 0$, then there is a bound on the orders of elements of $q$-power order in $G$.

**Example 3.8.** For each prime $q$, the group $\bigoplus_{i=1}^{\infty} \mathbb{Z}/q$ is a linear torsion group in characteristic $q$ (a subgroup of $SL_2(\overline{\mathbb{F}}_q)$), but not in any characteristic different from $q$. The group $\mu_\infty \le \mathbb{C}^\times$ of all complex roots of unity is a linear torsion group in characteristic 0 (a subgroup of $GL_1(\mathbb{C})$), but not in any other characteristic.

The following example shows how one particular fusion system can be LT-realizable in certain characteristics but not in others. (The 2-fusion system of $SO(3)$ is LT-realizable in all odd characteristics by Theorem 4.2.) More examples of fusion systems that are not LT-realizable in characteristic 0 are given in Proposition 7.5 and Theorem 8.10(b).

**Example 3.9.** Set $p = 2$, choose $S \in \mathrm{Syl}_2(SO(3))$, and set $\mathcal{F} = \mathcal{F}_S(SO(3))$. Then $\mathcal{F}$ is not isomorphic to the fusion system of any linear torsion group in characteristic 0.

*Proof.* Assume otherwise: assume $\mathcal{F} = \mathcal{F}_S(G)$ for some torsion subgroup $G \le GL_n(K)$ for $n \ge 1$ and $K$ a field of characteristic 0. Upon replacing $G$ by $G/O_{p'}(G)$, we can assume that $O_{p'}(G) = 1$. By Proposition 2.4, there is an increasing sequence of finite subgroups $\{G_i\}_{i \ge 1}$ such that upon setting $S_i = S \cap G_i$ and $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$, we have $S_i \in \mathrm{Syl}_p(G_i)$, $S = \bigcup_{i=1}^{\infty} S_i$, and $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$.

Since $S$ is a Sylow 2-subgroup of $SO(3)$, we have $S \cong (\mathbb{Z}/2^\infty) \rtimes C_2$. So there is $m \ge 1$ such that $S_i$ is dihedral of order at least 16 for all $i \ge m$. Choose $T \le S_m$ with $T \cong E_4$; then $T$ is $\mathcal{F}$-radical (all subgroups of $S$ isomorphic to $E_4$ are $\mathcal{F}$-radical), so $\mathrm{Aut}_G(T) \cong \Sigma_3$, and there is $n \ge m$ such that $\mathrm{Aut}_{\mathcal{F}_i}(T) = \mathrm{Aut}_{G_i}(T) \cong \Sigma_3$ for all $i \ge n$. So $T$ is $\mathcal{F}_i$-radical for all $i \ge n$.

Thus for $i \ge n$, $S_i \cong D_{2^k}$ for $k \ge 4$, and at least one of the two conjugacy classes of subgroups of $S_i$ isomorphic to $E_4$ is $\mathcal{F}_i$-radical. So by the Gorenstein-Walter theorem [G, §16.3], $G_i/O_{2'}(G_i)$ is isomorphic to $PSL_2(q_i)$ or $PGL_2(q_i)$ for some odd prime power $q_i$, or is an extension of one of these by a group of field automorphisms of odd order.

For each $i \ge n$, the group $G_i/(G_i \cap O_{2'}(G_{i+1}))$ is isomorphic to a subgroup of $G_{i+1}/O_{2'}(G_{i+1})$, and is nonsolvable of order a multiple of 16. The subgroups of $PSL_2(q_{i+1})$ were described by Dickson (see [GLS3, Theorem 6.5.1]), and from his list we see that each nonsolvable subgroup of $PGL_2(q_{i+1})$ of order a multiple of 16 is isomorphic to $PSL_2(q')$ or $PGL_2(q')$ for some $q'$ of which $q_{i+1}$ is a power. It follows that $q_{i+1}$ is a power of $q_i$.

Thus there is a prime $q$ of which each $q_i$ is a power. The $q_i$ must include arbitrarily large powers of $q$ since the Sylow subgroups $S_i$ get arbitrarily large. Hence the sectional $q$-rank of $G_i$ becomes arbitrarily large for large $i$. But since $G \le GL_n(K)$ where $\mathrm{char}(K) = 0$, we have $\mathrm{srk}_q(G) \le pn/(p-1)$ by Lemma 3.6(a), which is impossible. □

More generally, we will show in Theorem 8.10(b) that a similar result holds whenever $\mathcal{F}$ is the fusion system of a compact connected Lie group $G$ over a Sylow $p$-subgroup and $p$ divides the order of the Weyl group of $G$: such an $\mathcal{F}$ cannot be realized by a linear torsion group in characteristic 0. However, in contrast to the above result, the proof of Theorem 8.10(b) depends on the classification of finite simple groups.

Using results and arguments similar to those in Section 6 (and the classification of finite simple groups), we can also prove that

- if $p = 3$ and $\mathcal{F}$ is the fusion system of the 3-compact group $X_{12}$, then $\mathcal{F}$ is realized by a linear torsion group only in characteristic 2; and
- if $p = 5$ and $\mathcal{F}$ is the fusion system of the 5-compact group $X_{31}$, then $\mathcal{F}$ is realized by a linear torsion group only in characteristic $q$ for $q \equiv \pm 2 \pmod 5$.

Here, $X_n$ denotes a connected $p$-compact group whose Weyl group is the $n$-th group in the Shephard-Todd list [ST, Table VII]. The classifying spaces $BX_{12}$ and $BX_{31}$ were first constructed by Zabrodsky [Za, § 4.3], and a different construction of these spaces as well as of $BX_{29}$ and $BX_{34}$ was given by Aguadé [Ag].

The following is an obvious question, but one that seems quite difficult.

**Question 3.10.** *Is there a fusion system over a discrete $p$-toral group that is realized by a linear torsion group in characteristic $0$ but not by one in any positive characteristic?*

## 4. Fusion systems of compact Lie groups

The main result in this section is Theorem 4.3 (Theorem B): for every prime $p$ and every compact Lie group $G$, the $p$-fusion system of $G$ is realized by a linear torsion group $\Gamma$ in characteristic different from $p$ (hence is also sequentially realizable). We begin by reducing this to a question of showing that the $p$-completed classifying spaces of $G$ and $\Gamma$ are homotopy equivalent. Here, "$p$-completed" means in the sense of Bousfield and Kan [BK].

**Lemma 4.1.** *Fix a prime $p$, a compact Lie group $G$, and a linear torsion group $\Gamma$ in characteristic different from $p$ such that $BG^\wedge_p \simeq B\Gamma^\wedge_p$. Then for $S \in \mathrm{Syl}_p(G)$ and $S_\Gamma \in \mathrm{Syl}_p(\Gamma)$, we have $\mathcal{F}_S(G) \cong \mathcal{F}_{S_\Gamma}(\Gamma)$.*

*Proof.* By [BLO3, Theorems 8.10 and 9.10], there are centric linking systems $\mathcal{L}^c_S(G)$ and $\mathcal{L}^c_{S_\Gamma}(\Gamma)$, associated to $\mathcal{F}_S(G)$ and $\mathcal{F}_{S_\Gamma}(\Gamma)$, respectively, such that

$$|\mathcal{L}^c_S(G)|^\wedge_p \simeq BG^\wedge_p \simeq B\Gamma^\wedge_p \simeq |\mathcal{L}^c_{S_\Gamma}(\Gamma)|^\wedge_p.$$

So $\mathcal{F}_S(G) \cong \mathcal{F}_{S^*}(\Gamma)$ by [BLO3, Theorem 7.4]. □

We next look at the special case of Theorem B where we assume that the Lie groups are connected. This follows from a theorem of Friedlander and Mislin.

**Theorem 4.2.** *Let $p$ be a prime, let $G$ be a compact connected Lie group, and fix $S \in \mathrm{Syl}_p(G)$. Then $\mathcal{F}_S(G)$ is* LT*-realizable. More precisely, if $G$ is a maximal compact subgroup of $\mathbb{G}(\mathbb{C})$ where $\mathbb{G}$ is a connected, reductive algebraic group scheme over $\mathbb{Z}$, then $\mathcal{F}_S(G)$ is realized by the linear torsion group $\mathbb{G}(\overline{\mathbb{F}}_q)$ for each prime $q \neq p$.*

*Proof.* Let $G$ be a compact connected Lie group. By [BD, Propositions III.8.2–4], there is a complex connected algebraic group $G_\mathbb{C}$ containing $G$ such that $L(G_\mathbb{C}) \cong \mathbb{C} \otimes_\mathbb{R} L(G)$ (the Lie algebras). Hence $G$ is a maximal compact subgroup of $G_\mathbb{C}$, and $BG_\mathbb{C} \simeq BG$ since $G_\mathbb{C}/G$ is diffeomorphic to a Euclidean space by [Ho, Theorem XV.3.1]. Also, $G_\mathbb{C} \cong \mathbb{G}(\mathbb{C})$ for some connected reductive algebraic group scheme over $\mathbb{Z}$, and $B\mathbb{G}(\mathbb{C})^\wedge_p \simeq B\mathbb{G}(\overline{\mathbb{F}}_q)^\wedge_p$ for each prime $q \neq p$ by a theorem of Friedlander and Mislin [FM, Theorem 1.4]. So $\mathcal{F}_S(G)$ is realized by $\mathbb{G}(\overline{\mathbb{F}}_q)$ for each such $q$ by Lemma 4.1. □

We are now ready to prove Theorem B, in the following slightly more precise form.

**Theorem 4.3.** *Let $p$ be a prime, let $G$ be a compact Lie group, and fix $S \in \mathrm{Syl}_p(G)$. Then $\mathcal{F}_S(G)$ is* LT*-realizable. More precisely, if $G_e \trianglelefteq G$ denotes the identity connected component of $G$, then for each prime $q \neq p$, there are linear torsion groups $\Gamma_e \trianglelefteq \Gamma \leq GL_n(\overline{\mathbb{F}}_q)$ (for some $n$) such that $\Gamma/\Gamma_e \cong G/G_e$, and for $S_\Gamma \in \mathrm{Syl}_p(\Gamma)$, we have $\mathcal{F}_S(G) \cong \mathcal{F}_{S_\Gamma}(\Gamma)$ and $\mathcal{F}_{S\cap G_e}(G_e) \cong \mathcal{F}_{S_\Gamma \cap \Gamma_e}(\Gamma_e)$.*

*Proof.* Set $\pi = G/G_e = \pi_0(G)$, and let $\delta_G\colon G \longrightarrow \pi$ be the natural surjective homomorphism. Set $G_s = [G_e, G_e]$: the "semisimple part" of $G_e$. Since $Z(G_s)$ is finite, we can replace $G$ by $G/O_{p'}(Z(G_s))$, and arrange that $Z(G_s)$ be a finite abelian $p$-group (without changing the fusion system of $G$ or of $G_e$).

In general, in what follows, when $X \longrightarrow B\pi$ is a fibration, we let $\overline{X}$ denote the fiberwise $p$-completion of $X$ as defined in [BK, § I.8]. Thus if $F$ is the fiber of $X \longrightarrow B\pi$, then $F^\wedge_p$ is the fiber of $\overline{X} \longrightarrow B\pi$. Also, when $F$ is $p$-good, the natural map $X \longrightarrow \overline{X}$ is a mod $p$ homology equivalence by the Serre spectral sequences for the fibrations and since $H^*(F;\mathbb{F}_p) \cong H^*(F^\wedge_p;\mathbb{F}_p)$, and so $\overline{X}^\wedge_p \simeq X^\wedge_p$ by [BK, Lemma I.5.5].

In each of the cases considered below, we construct a linear torsion group $\Gamma$, together with a surjective homomorphism $\delta_\Gamma\colon \Gamma \longrightarrow \pi$ and a fiber homotopy equivalence

$$(\overline{B\Gamma} \longrightarrow B\pi) \simeq (\overline{BG} \longrightarrow B\pi).$$

In particular, this implies that $B\Gamma^\wedge_p \simeq BG^\wedge_p$, and hence by Lemma 4.1 that $\mathcal{F}_S(G) \cong \mathcal{F}_{S_\Gamma}(\Gamma)$ (for $S_\Gamma \in \mathrm{Syl}_p(\Gamma)$). So $\mathcal{F}_S(G)$ is LT-realizable. Also, $(B\Gamma_e)^\wedge_p \simeq (BG_e)^\wedge_p$, so $\mathcal{F}_{S\cap G_e}(G_e) \cong \mathcal{F}_{S_\Gamma\cap\Gamma_e}(\Gamma_e)$.

**Case 1: $\boldsymbol{Z(G_e) = 1}$.** In this case, $G_e$ is a product of simple groups with trivial center. As in the proof of Theorem 4.2, let $\mathbb{G}$ be a connected, semisimple group scheme over $\mathbb{Z}$ such that $G_e$ is a maximal compact subgroup of $\mathbb{G}(\mathbb{C})$, and set $\Gamma_e = \mathbb{G}(\overline{\mathbb{F}}_q)$. Thus $\Gamma_e$ is a product of simple groups with $Z(\Gamma_e) = 1$. By the Friedlander-Mislin theorem [FM, Theorem 1.4], there is a homotopy equivalence

$$\psi\colon (B\Gamma_e)^\wedge_p \xrightarrow{\ \simeq\ } B\mathbb{G}(\mathbb{C})^\wedge_p \simeq (BG_e)^\wedge_p.$$

Let $R$ denote the root system of $G_e$ and of $\Gamma_e$. We regard the roots as elements in the dual $V^*$ of a real vector space $V$ that can be identified with the universal cover of a maximal torus in $G_e$. We fix a Weyl chamber $C \subseteq V$ and let $R^+$ denote the corresponding set of positive roots. Thus $R^+$ is the set of all $r \in R$ such that $r(x) > 0$ for $x \in C$, while $C$ is the set of $x \in V$ such that $r(x) > 0$ for all $r \in R^+$. Also, $R = \{\pm r \,|\, r \in R^+\}$. Let $\mathrm{Isom}(R)$ be the group of all isometries of $R$, and let $\mathrm{Isom}^+(R)$ be the subgroup of those isometries that permute the positive roots; equivalently, those that send the Weyl chamber $C$ to itself.

Let $\mathrm{Aut}(\Gamma_e)$ be the group of all automorphisms of $\Gamma_e$ as an algebraic group, and set $\mathrm{Out}(\Gamma_e) = \mathrm{Aut}(\Gamma_e)/\mathrm{Inn}(\Gamma_e)$. By [GLS3, Theorem 1.15.2], the group $\mathrm{Out}(\Gamma_e)$ is isomorphic to $\mathrm{Isom}^+(R)$. By [Bb9, § 4.10, Proposition 17], $\mathrm{Out}(G_e) \cong \mathrm{Isom}(R)/W$, where $W \trianglelefteq \mathrm{Isom}(R)$ is the Weyl group. Since the Weyl group permutes the Weyl chambers simply and transitively [Bb4-6, § VI.1.5, Théorème 2], this shows that there is a natural isomorphism

$$\theta\colon \mathrm{Out}(G_e) \cong \mathrm{Isom}(R)/W \xrightarrow{\ \cong\ } \mathrm{Isom}^+(R) \cong \mathrm{Out}(\Gamma_e).$$

Let $\rho_G\colon \mathrm{Out}(G_e) \longrightarrow \mathrm{Out}((BG_e)^\wedge_p)$ and $\rho_\Gamma\colon \mathrm{Out}(\Gamma_e) \longrightarrow \mathrm{Out}((B\Gamma_e)^\wedge_p)$ be the homomorphisms induced by the functor from topological groups to their $p$-completed classifying spaces. Let $\eta_G\colon \pi \longrightarrow \mathrm{Out}(G_e)$ be induced by conjugation, set $\eta_\Gamma = \theta\circ\eta_G$, and consider the following diagram

$$\begin{array}{ccccc}
\pi & \xrightarrow{\eta_G} & \mathrm{Out}(G_e) & \xrightarrow{\rho_G} & \mathrm{Out}((BG_e)^\wedge_p) \\
\Big\| & & \cong\Big\downarrow\theta & & \cong\Big\downarrow c_\psi^{-1} \\
\pi & \xrightarrow{\eta_\Gamma} & \mathrm{Out}(\Gamma_e) & \xrightarrow{\rho_\Gamma} & \mathrm{Out}((B\Gamma_e)^\wedge_p).
\end{array} \tag{4-1}$$

The left hand square commutes by definition.

To see that the right hand square in (4-1) commutes, fix $\alpha \in \mathrm{Aut}(G_e)$. We just saw that its class $[\alpha] \in \mathrm{Out}(G_e)$ is induced by an isometry of the root system of $G$, and hence $[\alpha] = [\beta(\mathbb{C})|_G]$ for some $\beta \in \mathrm{Aut}(\mathbb{G})$. Also, $\theta([\alpha]) = \theta([\beta(\mathbb{C})|_G]) = [\beta(\overline{\mathbb{F}}_q)]$ by the above definition of $\theta$. The equivalence $\psi\colon B\mathbb{G}(\overline{\mathbb{F}}_q)^\wedge_p \longrightarrow B\mathbb{G}(\mathbb{C})^\wedge_p$ is natural as a map of functors from reductive group schemes over $\mathbb{Z}$ to the homotopy category, since it is induced by the projection of the ring of Witt vectors $\mathbb{W}(\overline{\mathbb{F}}_q)$ onto $\overline{\mathbb{F}}_q$ together with a choice of embedding of $\mathbb{W}(\overline{F}_q)$ into $\mathbb{C}$ (see [FP, Corollary 2]). So $(B\beta(\mathbb{C})|_{BG})^\wedge_p \circ \psi = \psi \circ (B\beta(\overline{\mathbb{F}}_q))^\wedge_p$, and hence

$$c_\psi^{-1}(\rho_G([\alpha])) = c_\psi^{-1}([(B\beta(\mathbb{C})|_{BG})^\wedge_p]) = [(B\beta(\overline{\mathbb{F}}_q))^\wedge_p] = \rho_\Gamma([\beta(\overline{\mathbb{F}}_q)]) = \rho_\Gamma\theta([\alpha])\,.$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_e & \xrightarrow{\text{incl}} & G & \xrightarrow{\delta_G} & \pi & \longrightarrow & 1 \\ & & {\scriptstyle c_1}\downarrow\cong & & {\scriptstyle c_2}\downarrow & & {\scriptstyle \eta_G}\downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(G_e) & \xrightarrow{\text{incl}} & \mathrm{Aut}(G_e) & \longrightarrow & \mathrm{Out}(G_e) & \longrightarrow & 1 \end{array} \tag{4-2}$$

where the rows are short exact sequences and $c_1$ and $c_2$ are induced by conjugation. Thus $c_1$ is an isomorphism since $Z(G_e) = 1$. So $G$ is isomorphic to a pullback of $\pi$ and $\mathrm{Aut}(G_e)$ over $\mathrm{Out}(G_e)$. Define $\Gamma$ to be the analogous pullback of $\pi$ and $\mathrm{Aut}(\Gamma_e)$ over $\mathrm{Out}(\Gamma_e)$; then a similar diagram (but with $G$ replaced by $\Gamma$) shows that we can identify $\Gamma_e$ with a subgroup of $\Gamma$ such that $\Gamma/\Gamma_e \cong \pi$.

By [BLO3, Theorem 7.1] and since $Z(G_e) = 1 = Z(\Gamma_e)$, the space $B\mathrm{Aut}((BG_e)^\wedge_p)$ is an Eilenberg-MacLane space with fundamental group $\mathrm{Out}((BG_e)^\wedge_p)$. Hence the fibration sequence $(BG_e)^\wedge_p \longrightarrow \overline{BG} \longrightarrow B\pi$ is classified by the map

$$B\pi \xrightarrow{B(\rho_G\eta_G)} B\mathrm{Out}((BG_e)^\wedge_p) \simeq B\mathrm{Aut}((BG_e)^\wedge_p).$$

(see [BGM, Theorem IV.5.6]). By the commutativity of (4-1), $c_\psi^{-1}\rho_G\eta_G = \rho_\Gamma\eta_\Gamma$, where $c_\psi$ is an isomorphism. Hence $((BG_e)^\wedge_p \longrightarrow \overline{BG} \longrightarrow B\pi)$ is fiberwise homotopy equivalent to the fibration sequence classified by $B(\rho_\Gamma\eta_\Gamma)$, and that sequence is the fiberwise completion of the fibration sequence $B\Gamma_e \longrightarrow B\Gamma \longrightarrow B\pi$ defined above. So $BG^\wedge_p \cong B\Gamma^\wedge_p$, and the fusion systems of $G$ and $\Gamma$ are isomorphic by Lemma 4.1.

Finally, $\Gamma$ has a finite dimensional representation over $\overline{\mathbb{F}}_q$ since $\Gamma_e$ does, and since $|\Gamma/\Gamma_e| = |\pi| < \infty$.

**Case 2: $G_e$ is a torus.** Set $T = G_e$ to simplify notation, and let $T_f \le T$ be the subgroup of elements of finite order. Then $T/T_f$ is uniquely divisible, so $H^2(\pi; T/T_f) = 0$, and the group $G/T_f$ is a semidirect product of $T/T_f$ with $\pi = G/T$.

Let $s\colon \pi \longrightarrow G/T_f$ be a splitting of the projection $G/T_f \longrightarrow \pi$, and set $G_f/T_f = s(\pi)$. Thus $G_f \le G$, where $T_f = T \cap G_f \trianglelefteq G_f$ and $G_f/T_f \cong \pi$. Also, $TG_f = G$. Set $\Gamma = G_f/O_{p'}(T_f)$ and $\Gamma_e = T_f/O_{p'}(T_f)$, where $O_{p'}(T_f)$ denotes the subgroup of all elements of order prime to $p$. Thus $\Gamma_e$ is a discrete $p$-torus, and $\Gamma/\Gamma_e \cong \pi$.

The natural homomorphisms $T \hookleftarrow T_f \longrightarrow \Gamma_e$ induce mod $p$ homology equivalences $BT \longleftarrow BT_f \longrightarrow B\Gamma_e$, and they induce weak equivalences

$$(BT)^\wedge_p \simeq (BT_f)^\wedge_p \simeq (B\Gamma_e)^\wedge_p.$$

So after fiberwise completion of $BG$, $BG_f$, and $B\Gamma$ over $B\pi$, we also get weak equivalences

$$\overline{BG} \simeq \overline{BG_f} \simeq \overline{B\Gamma}.$$

Hence $BG^\wedge_p \simeq B\Gamma^\wedge_p$.

**Case 3: $Z(G_s) = 1$.** Set $G_1 = G/Z(G_e)$ and $G_2 = G/G_s$, and let $G_{1e} = G_e/Z(G_e)$ and $G_{2e} = G_e/G_s$ be their identity components. Then $G_e = Z(G_e)G_s$: every compact connected Lie group is a central product of a semisimple group with a torus (see [MTo, Corollary 5.5.31]), and the torus factor is clearly contained in the center. Also, $Z(G_e)\cap G_s = Z(G_s) = 1$ by assumption. So $G$ is isomorphic to the pullback of $G_1$ and $G_2$ over $\pi = G/G_e$, and this in turn restricts to an isomorphism $G_e \cong G_{1e} \times G_{2e}$.

Since $Z(G_{1e}) = 1$ and $G_{2e}$ is a torus, by Cases 1 and 2, there are pairs of groups $\Gamma_{1e} \trianglelefteq \Gamma_1$ and $\Gamma_{2e} \trianglelefteq \Gamma_2$ such that for $i = 1, 2$, the fibration sequences

$$(B\Gamma_{ie})^\wedge_p \longrightarrow \overline{B\Gamma_i} \longrightarrow B\pi \qquad \text{and} \qquad (BG_{ie})^\wedge_p \longrightarrow \overline{BG_i} \longrightarrow B\pi$$

are fiberwise homotopy equivalent and $\Gamma_i/\Gamma_{ie} \cong \pi$. So if we let $\Gamma$ be the pullback of $\Gamma_1$ and $\Gamma_2$ over $\pi$, and set $\Gamma_e = \Gamma_{1e} \times \Gamma_{2e}$, then

$$(B\Gamma_e)^\wedge_p \longrightarrow \overline{B\Gamma} \longrightarrow B\pi \qquad \text{and} \qquad (BG_e)^\wedge_p \longrightarrow \overline{BG} \longrightarrow B\pi$$

are also fiberwise homotopy equivalent and $\Gamma/\Gamma_e \cong \pi$. Here, $\overline{(-)}$ again denotes fiberwise $p$-completion over $B\pi$. So $B\Gamma^\wedge_p \simeq BG^\wedge_p$, and the fusion systems are isomorphic by Lemma 4.1.

By construction, $\Gamma_e$ is the product of a semisimple algebraic group over $\overline{\mathbb{F}}_q$ (some prime $q \neq p$) with a discrete $p$-torus. So $\Gamma_e$ and $\Gamma$ are linear torsion groups over $\overline{\mathbb{F}}_q$, and $\mathcal{F} = \mathcal{F}_S(G)$ is LT-realizable.

**General case:** Now assume $G$ is arbitrary. Set $Z = Z(G_s)$: a finite abelian $p$-group by the assumption at the beginning of the proof. Set $G^* = G/Z$ and $G^*_e = G_e/Z$. By Case 3 and since $Z((G^*)_s) = Z(G_s/Z) = 1$ (see [Bb2-3, § III.9.8, Proposition 29]), there is a pair of linear torsion groups $\Gamma^*_e \trianglelefteq \Gamma^*$ such that $\Gamma^*/\Gamma^*_e \cong \pi$, and such that the fibration sequences

$$(B\Gamma^*_e)^\wedge_p \longrightarrow \overline{B\Gamma^*} \longrightarrow B\pi \qquad \text{and} \qquad (BG^*_e)^\wedge_p \longrightarrow \overline{BG^*} \longrightarrow B\pi$$

are fiberwise homotopy equivalent. By the construction in Case 3, we can also assume that $\Gamma^*_e$ is a product of a semisimple algebraic group over $\overline{\mathbb{F}}_q$ (some $q \neq p$) with a discrete $p$-torus.

The extensions $1 \longrightarrow Z \longrightarrow G_e \longrightarrow G^*_e \longrightarrow 1$ and $1 \longrightarrow Z \longrightarrow G \longrightarrow G^* \longrightarrow 1$ induce a commutative diagram of spaces

$$\begin{array}{ccccc} BZ & \longrightarrow & (BG_e)^\wedge_p & \longrightarrow & (BG^*_e)^\wedge_p \\ \Vert & & \downarrow & & \downarrow \\ BZ & \longrightarrow & \overline{BG} & \longrightarrow & \overline{BG^*}. \end{array} \tag{4-3}$$

Here $BZ$ is $p$-complete since $Z$ is a finite $p$-group. So the top row in (4-3) is a fibration sequence by [BK, Lemma II.5.1] and since $Z \leq Z(G_e)$. Hence the bottom row is also a fibration sequence after fiberwise $p$-completion over $B\pi$. Similarly, if $\Gamma$ is any discrete group together with a surjection $\chi\colon \Gamma \longrightarrow \Gamma^*$ such that $\mathrm{Ker}(\chi) = Z \leq Z(\chi^{-1}(\Gamma^*_e))$, and we define $\Gamma_e = \chi^{-1}(\Gamma^*_e)$, then we get the following commutative diagram whose rows are fibration sequences:

$$\begin{array}{ccccc} BZ & \longrightarrow & (B\Gamma_e)^\wedge_p & \longrightarrow & (B\Gamma^*_e)^\wedge_p \\ \Vert & & \downarrow & & \downarrow \\ BZ & \longrightarrow & \overline{B\Gamma} & \xrightarrow{B\chi} & \overline{B\Gamma^*}. \end{array} \tag{4-4}$$

It remains to choose the pair $(\Gamma, \chi)$ so that the bottom rows in (4-3) and (4-4) are fiber homotopy equivalent. By [BGM, Theorem IV.5.6], fibrations over a space $B$ with fiber $BZ$ are classified by homotopy classes of maps $B \longrightarrow B\mathrm{Aut}(BZ)$. Let $\mathrm{Aut}_*(BZ) \subseteq \mathrm{map}_*(BZ, BZ)$

be the spaces of pointed self equivalences and pointed self maps, respectively. Since $BZ$ is an Eilenberg-MacLane space, we have

$$\pi_i(\mathrm{map}_*(BZ,BZ)) \cong [BZ, \Omega^i(BZ)]_* = \begin{cases} H^1(Z;Z) \cong \mathrm{Hom}(Z,Z) & \text{if } i=0 \\ 0 & \text{if } i>0. \end{cases}$$

From this, together with the fibration sequence $\mathrm{Aut}_*(BZ) \longrightarrow \mathrm{Aut}(BZ) \longrightarrow BZ$, we see that $\pi_1(B\mathrm{Aut}(BZ)) \cong \mathrm{Aut}(Z)$, $\pi_2(B\mathrm{Aut}(BZ)) \cong Z$, and $\pi_i(B\mathrm{Aut}(BZ)) = 0$ for all $i \geq 3$.

Let $[B\Gamma^*, B\mathrm{Aut}(BZ)]_0$ and $[\overline{B\Gamma^*}, B\mathrm{Aut}(BZ)]_0$ denote the sets of homotopy classes of maps for which the induced homomorphism in $\pi_1$ factors through $\Gamma^*/\Gamma^*_e \cong \pi$. Since $H^*(\overline{B\Gamma^*};A) \cong H^*(B\Gamma^*;A)$ for every finite abelian $p$-group $A$ with action of $\pi$ (since the map $B\Gamma^*_e \to (B\Gamma^*_e)^\wedge_p$ between the $\pi$-covers is a mod $p$ homology equivalence), the natural map

$$[\overline{B\Gamma^*}, B\mathrm{Aut}(BZ)]_0 \longrightarrow [B\Gamma^*, B\mathrm{Aut}(BZ)]_0 \tag{4-5}$$

is a bijection.

Since $\overline{B\Gamma^*} \simeq \overline{BG^*}$, there is a fibration $X \xrightarrow{\ \nu\ } \overline{B\Gamma^*}$ with fiber $BZ$ that is fiberwise homotopy equivalent to the fibration $\overline{BG} \longrightarrow \overline{BG^*}$. Since $Z \leq Z(G_e)$, the fibration $\nu$ is classified by a map in $[\overline{B\Gamma^*}, B\mathrm{Aut}(BZ)]_0$. So by (4-5), there is also a fibration $Y \longrightarrow B\Gamma^*$ with fiber $BZ$ whose fiberwise $p$-completion over $B\pi$ is $X \longrightarrow \overline{B\Gamma^*}$. Upon putting these together, we get the following commutative diagram, each of whose rows is a fibration sequence:

$$\begin{array}{ccccc} BZ & \longrightarrow & \overline{BG} & \longrightarrow & \overline{BG^*} \\ \| & & \downarrow{\simeq} & & \downarrow{\simeq} \\ BZ & \longrightarrow & X & \xrightarrow{\ \nu\ } & \overline{B\Gamma^*} \\ \| & & \uparrow & & \uparrow \\ BZ & \longrightarrow & Y & \xrightarrow{\ \nu_0\ } & B\Gamma^* \end{array} \tag{4-6}$$

Set $\Gamma = \pi_1(Y)$; then there is a surjection $\chi = \pi_1(\nu_0)\colon \Gamma \longrightarrow \Gamma^*$ with kernel $Z$. Set $\Gamma_e = \chi^{-1}(\Gamma^*_e)$; then after fiberwise completion we have $\overline{B\Gamma} \simeq \overline{Y} \simeq X \simeq \overline{BG}$. It now follows that $B\Gamma^\wedge_p \simeq BG^\wedge_p$, and hence by Lemma 4.1 that $\mathcal{F}_S(G) \cong \mathcal{F}_{S_\Gamma}(\Gamma)$.

By construction, $Z \leq Z(\Gamma_e)$, and

$$\Gamma^*_e = \Gamma_e/Z = \Gamma^*_1 \times \cdots \times \Gamma^*_k \times T^*,$$

where $T^*$ is a discrete $p$-torus and each $\Gamma^*_i$ is a simple algebraic group over $\overline{\mathbb{F}}_q$. By Theorems 3.1–3.3 and 4.1 in [St2], for each $i$, the universal central extension of $\Gamma^*_i$ is itself an algebraic group over $\overline{\mathbb{F}}_q$. (It's important here that we are working over an algebraic extension of a finite field.) So $\Gamma_e$ is a central product of simple algebraic groups over $\overline{\mathbb{F}}_q$ and a discrete $p$-torus, where a discrete $p$-torus of rank $r$ is contained in the algebraic group $(\overline{\mathbb{F}}_q^\times)^r$. Thus $\Gamma_e$ is contained in a central product of algebraic groups over $\overline{\mathbb{F}}_q$, and this in turn is an algebraic group over $\overline{\mathbb{F}}_q$ by [Sp, Proposition 5.5.10] (the quotient of a linear algebraic group by a closed normal subgroup is again a linear algebraic group). So $\Gamma_e$ is a linear torsion group over $\overline{\mathbb{F}}_q$. Since $\Gamma_e$ has finite index in $\Gamma$, the group $\Gamma$ is also linear over $\overline{\mathbb{F}}_q$, finishing the proof that $\mathcal{F} = \mathcal{F}_S(G)$ is LT-realizable. □

## 5. Increasing sequences of finite fusion subsystems

In this section and the next, we show some preliminary results that will be needed in Section 7 to prove that certain saturated fusion systems are not sequentially realizable. In this section, we mostly look at the question of how a saturated fusion system $\mathcal{F}$ over an infinite discrete $p$-toral group $S$ is approximated by sufficiently large finite fusion subsystems. The following lemma is a first step towards doing that.

**Lemma 5.1.** *Let $\mathcal{F}$ be a saturated fusion system over an infinite discrete $p$-toral group $S$, and let $T \trianglelefteq S$ be the identity component. Assume $C_S(T) = T$. Then there is $n \geq 1$ such that $C_S(\Omega_n(T)) = T$, and such that for each $A \leq T$ that contains $\Omega_n(T)$:*

(a) *$A$ is fully centralized in $\mathcal{F}$, and $\varphi(A) \leq T$ for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(A, S)$; and*

(b) *if $A$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant, then it is weakly closed in $\mathcal{F}$, and restriction induces an isomorphism $\rho_A \colon \mathrm{Aut}_{\mathcal{F}}(T) \xrightarrow{\cong} \mathrm{Aut}_{\mathcal{F}}(A)$.*

*Proof.* Since $\{C_S(\Omega_i(T))\}_{i\geq 1}$ is a descending sequence of subgroups with intersection $C_S(T) = T$ (and since $S$ is artinian), there is $n_0 \geq 1$ such that $C_S(\Omega_{n_0}(T)) = T$. Let $k$ be such that $S/T$ has exponent $p^k$; i.e., such that $s^{p^k} \in T$ for all $s \in S$. Set $n_1 = n_0 + k$.

Let $A$ be such that $\Omega_{n_1}(T) \leq A \leq T$. For each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(A, S)$,

$$\varphi(\Omega_{n_0}(T)) \leq \varphi(\{a^{p^k} \mid a \in A\}) \leq T,$$

so $\varphi(\Omega_{n_0}(T)) = \Omega_{n_0}(T)$, and $\varphi(A) \leq C_S(\Omega_{n_0}(T)) = T$. Furthermore, $C_S(\varphi(A)) = T = C_S(A)$, and since $\varphi \in \mathrm{Hom}_{\mathcal{F}}(A, S)$ is arbitrary, $A$ is fully centralized in $\mathcal{F}$. This proves (a).

Assume in addition that $A$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant. By [BLO3, Lemma 2.4(b)], $\varphi$ is the restriction of some element of $\mathrm{Aut}_{\mathcal{F}}(T)$, and so $\varphi(A) = A$ by the assumption. Thus $A$ is weakly closed in $\mathcal{F}$, and restriction defines a surjective homomorphism $\rho_A$ from $\mathrm{Aut}_{\mathcal{F}}(T)$ to $\mathrm{Aut}_{\mathcal{F}}(A)$.

Consider the descending sequence $\{\mathrm{Ker}(\rho_{\Omega_i(T)})\}_{i\geq n_1}$. This sequence is constant for $i$ large since $\mathrm{Aut}_{\mathcal{F}}(T)$ is finite, and the intersection of its terms is the trivial subgroup. So there is $n \geq n_1$ such that $\rho_{\Omega_n(T)}$ is injective. Hence $\rho_A$ is injective for every $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant subgroup $A \leq T$ containing $\Omega_n(T)$, and this finishes the proof of (b). □

We now look more closely at unions of increasing sequences of finite fusion subsystems, and make more precise what we mean by a subsystem being "large enough".

**Lemma 5.2.** *Let $\mathcal{F}$ be a saturated fusion system over an infinite discrete $p$-toral group $S$, let $T \trianglelefteq S$ be the identity component, and assume $C_S(T) = T$. Let $\mathcal{F}_1 \leq \mathcal{F}_2 \leq \mathcal{F}_3 \leq \cdots$ be fusion subsystems of $\mathcal{F}$ over finite subgroups $S_1 \leq S_2 \leq S_3 \leq \cdots$ in $S$ such that $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$, and set $T_i = T \cap S_i$. Then there is $n \geq 1$ such that for each $i \geq n$ and each $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant subgroup $A$ such that $T_n \leq A \leq T_i$,*

(a) *$A$ is weakly closed in $\mathcal{F}$ and hence in $\mathcal{F}_i$,*

(b) *restriction to $A$ induces an isomorphism $\rho_A \colon \mathrm{Aut}_{\mathcal{F}}(T) \xrightarrow{\cong} \mathrm{Aut}_{\mathcal{F}}(A)$, and*

(c) *$\mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_{\mathcal{F}_i}(A)$.*

*Furthermore, $n$ can be chosen so that $T_i$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant for all $i \geq n$.*

*Proof.* Let $\rho_A \colon \mathrm{Aut}_{\mathcal{F}}(T) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(A)$ be the homomorphism induced by restriction for each $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant subgroup $A \leq T$. By Lemma 5.1(b), there is $n_0 \geq 1$ such that for each $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant subgroup $\Omega_{n_0}(T) \leq A \leq T$, $A$ is weakly closed in $\mathcal{F}$ and $\rho_A$ is an isomorphism. Choose $n_1 \geq 1$ such that $T_{n_1} \geq \Omega_{n_0}(T)$; then the same conclusion holds whenever $T_{n_1} \leq A \leq T$.

Since $\mathcal{F}$ is the union of the $\mathcal{F}_i$, we have $\mathrm{Aut}_{\mathcal{F}}(T_{n_1}) = \bigcup_{i=n_1}^{\infty} \mathrm{Aut}_{\mathcal{F}_i}(T_{n_1})$. This is an increasing union, and $\mathrm{Aut}_{\mathcal{F}}(T_{n_1})$ is finite since $T_{n_1}$ is finite. Hence there is $n \geq n_1$ such that $\mathrm{Aut}_{\mathcal{F}}(T_{n_1}) = \mathrm{Aut}_{\mathcal{F}_i}(T_{n_1})$ for all $i \geq n$.

Now assume $i \geq n$ and $T_n \leq A \leq T_i$, where $A$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant. We already showed that (a) and (b) hold in this situation ($A$ is weakly closed in $\mathcal{F}_i$ since it is weakly closed in $\mathcal{F}$). Since $A$ and $T_{n_1}$ are both weakly closed in $\mathcal{F}$ and in $\mathcal{F}_i$, they are both fully centralized in $\mathcal{F}_i$. For each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(A)$, we have $\alpha|_{T_{n_1}} \in \mathrm{Aut}_{\mathcal{F}}(T_{n_1}) = \mathrm{Aut}_{\mathcal{F}_i}(T_{n_1})$, and by the extension axiom for $\mathcal{F}_i$ (and since $A \leq C_S(T_{n_1})$), this extends to $\alpha' \in \mathrm{Aut}_{\mathcal{F}_i}(A) \leq \mathrm{Aut}_{\mathcal{F}}(A)$. Since $\rho_A$ and $\rho_{T_{n_1}}$ are isomorphisms, restriction induces an isomorphism from $\mathrm{Aut}_{\mathcal{F}}(A)$ to $\mathrm{Aut}_{\mathcal{F}}(T_{n_1})$, and hence $\alpha' = \alpha$. Thus $\mathrm{Aut}_{\mathcal{F}_i}(A) = \mathrm{Aut}_{\mathcal{F}}(A)$, and (c) holds.

It remains to prove the last statement. Choose $m, n' \geq 1$ such that $T_n \leq \Omega_m(T) \leq T_{n'}$. By Lemma 5.1, we can do this so that $C_S(\Omega_m(T)) = T$, and so that each subgroup of $T$ containing $\Omega_m(T)$ is fully centralized in $\mathcal{F}$. Set $A = \Omega_m(T)$, and note that $A$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant and hence (a), (b), and (c) hold for $A$. Fix $i \geq n'$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$. Then $\alpha|_A \in \mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_{\mathcal{F}_i}(A)$ by (c), and since $A$ is weakly closed (hence fully centralized) in $\mathcal{F}_i$ by (a), this extends to some $\beta \in \mathrm{Hom}_{\mathcal{F}_i}(T_i, S_i)$ by the extension axiom. Also, $\beta(T_i) \leq C_{S_i}(A) = C_{S_i}(\Omega_m(T)) = T_i$. Since $T_i$ is fully centralized in $\mathcal{F}$, $\beta$ extends to $\gamma \in \mathrm{Aut}_{\mathcal{F}}(T)$ by the extension axiom again. Then $\gamma|_A = \alpha|_A$, so $\alpha = \gamma$ by (b), and $\alpha(T_i) = T_i$. This proves that $T_i$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant for all $i \geq n'$, and so the last statement holds upon replacing $n$ by $n'$. □

We next focus on strongly closed subgroups in fusion systems over discrete $p$-toral groups.

**Definition 5.3.** For each fusion system $\mathcal{F}$ over a discrete $p$-toral group $S$, let $\mathcal{F}^{\mathfrak{sc}}$ be the set of all *nontrivial* subgroups $1 \neq R \leq S$ strongly closed in $\mathcal{F}$. Set $\mathfrak{minsc}(\mathcal{F}) = \bigcap_{R \in \mathcal{F}^{\mathfrak{sc}}} R$.

Clearly, $\mathfrak{minsc}(\mathcal{F})$ is always strongly closed in $\mathcal{F}$. But it can be trivial.

**Lemma 5.4.** *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$, and let $\{\mathcal{F}_i\}_{i \geq 1}$ be an increasing sequence of saturated fusion subsystems of $\mathcal{F}$ over $S_1 \leq S_2 \leq \cdots$ such that $S = \bigcup_{i=1}^{\infty} S_i$ and $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$. Then there is $n \geq 1$ such that $S_n \geq \Omega_1(Z(S))$ and $C_S(S_n) = Z(S)$. For each such $n$, we have $\mathfrak{minsc}(\mathcal{F}_i) \leq \mathfrak{minsc}(\mathcal{F}_{i+1})$ for all $i \geq n$, and $\mathfrak{minsc}(\mathcal{F}) = \bigcup_{i=n}^{\infty} \mathfrak{minsc}(\mathcal{F}_i)$.*

*Proof.* Since $\Omega_1(Z(S))$ is finite, we have $S_n \geq \Omega_1(Z(S))$ for $n$ large enough. Since $Z(S) = \bigcap_{i=1}^{\infty} C_S(S_i)$, and $S$ is artinian, we have $C_S(S_n) = Z(S)$ for $n$ large enough.

Now fix $n \geq 1$ such that $S_n \geq \Omega_1(Z(S))$ and $C_S(S_n) = Z(S)$. It will be convenient to set $\mathcal{F}_\infty = \mathcal{F}$ and $S_\infty = S$, and then refer to indices $n \leq i \leq \infty$. We first claim that for each $n \leq j < i \leq \infty$,

$$S_j \geq \Omega_1(Z(S_i)), \quad \mathcal{F}_j^{\mathfrak{sc}} \supseteq \{R \cap S_j \mid R \in \mathcal{F}_i^{\mathfrak{sc}}\}, \quad \text{and} \quad \mathfrak{minsc}(\mathcal{F}_j) \leq \mathfrak{minsc}(\mathcal{F}_i). \tag{5-1}$$

The first statement holds since $S_j \geq S_n \geq \Omega_1(Z(S)) = \Omega_1(C_S(S_i)) \geq \Omega_1(Z(S_i))$. For each $R \in \mathcal{F}_i^{\mathfrak{sc}}$, the subgroup $R \cap S_j$ is strongly closed in $\mathcal{F}_j$ since $R$ is strongly closed in $\mathcal{F}_i$, and $R \cap S_j \geq R \cap \Omega_1(Z(S_i)) \neq 1$ where the last inequality holds by Lemma 1.8 and since $R \trianglelefteq S_i$. This proves the second statement, and the third holds since

$$\mathfrak{minsc}(\mathcal{F}_i) = \bigcap_{R \in \mathcal{F}_i^{\mathfrak{sc}}} R \geq \bigcap_{R \in \mathcal{F}_i^{\mathfrak{sc}}} (R \cap S_j) \geq \bigcap_{R \in \mathcal{F}_j^{\mathfrak{sc}}} R = \mathfrak{minsc}(\mathcal{F}_j).$$

By (5-1) when $i=\infty$, we have $\bigcup_{i=n}^{\infty}\mathfrak{minsc}(\mathcal{F}_i)\le\mathfrak{minsc}(\mathcal{F})$, and it remains only to show that this is an equality. For each $x\in\bigcup_{i=n}^{\infty}\mathfrak{minsc}(\mathcal{F}_i)$ and each $y\in x^{\mathcal{F}}$, there is $i\ge n$ such that $x\in\mathfrak{minsc}(\mathcal{F}_i)$, $x,y\in S_i$, and $y\in x^{\mathcal{F}_i}$. Then $y\in\mathfrak{minsc}(\mathcal{F}_i)$ since that subgroup is strongly closed. Thus $\bigcup_{i=n}^{\infty}\mathfrak{minsc}(\mathcal{F}_i)$ is also strongly closed in $\mathcal{F}$. So either $\mathfrak{minsc}(\mathcal{F})$ is the union of the $\mathfrak{minsc}(\mathcal{F}_i)$, or $\mathfrak{minsc}(\mathcal{F}_i)=1$ for all $n\le i<\infty$ while $\mathfrak{minsc}(\mathcal{F})\ne 1$. It remains to show that this last situation cannot occur.

For the rest of the proof, we assume that $\mathfrak{minsc}(\mathcal{F})\ne 1$, while $\mathfrak{minsc}(\mathcal{F}_i)=1$ for all $n\le i<\infty$. Set $U=\Omega_1(Z(S))$ for short. Thus $\mathfrak{minsc}(\mathcal{F})\cap U\ne 1$ since $\mathfrak{minsc}(\mathcal{F})\trianglelefteq S$ (see Lemma 1.8). For each $i\ge 1$ and $V\le U$, let $\mathscr{C}_i^{(V)}$ be the set of all $R\in\mathcal{F}_i^{\mathfrak{sc}}$ such that $R\cap U=V$. Thus $\mathcal{F}_i^{\mathfrak{sc}}$ is the (finite) union of the $\mathscr{C}_i^{(V)}$ for $V\subseteq U$. Also,

$$R\in\mathscr{C}_i^{(V)} \implies R\cap S_j\in\mathscr{C}_j^{(V)} \quad \forall\, n\le j<i \qquad\text{and}\qquad \mathscr{C}_i^{(1)}=\varnothing \quad \forall\, i\ge n\ : \tag{5-2}$$

the implication holds by (5-1) and the second statement since for each $1\ne P\trianglelefteq S_i$,

$$P\cap U=P\cap\Omega_1(Z(S))=P\cap\Omega_1(C_S(S_i))\ge P\cap\Omega_1(Z(S_i))\ne 1.$$

Set $W=\mathfrak{minsc}(\mathcal{F})\cap U\ne 1$. If, for some finite $i\ge n$, we have $R\ge W$ for each $R\in\mathcal{F}_i^{\mathfrak{sc}}$, then $\mathfrak{minsc}(\mathcal{F}_i)\ge W\ne 1$, contradicting our assumption. Hence for each $n\le i<\infty$, there is $V_i\le U$ such that $V_i\ngeq W$ and $\mathscr{C}_i^{(V_i)}\ne\varnothing$. Since $U$ has only finitely many subgroups, there is $V\le U$ such that $V\ngeq W$ and $\mathscr{C}_i^{(V)}\ne\varnothing$ for infinitely many $i$, and hence by (5-2) for all $m\le i<\infty$ (some $m\ge n$). By (5-2) again, $V\ne 1$.

For this fixed subgroup $V$, the $\mathscr{C}_i^{(V)}$ form an inverse system of sets, nonempty for all $i\ge m$, where $\mathscr{C}_i^{(V)}\longrightarrow\mathscr{C}_{i-1}^{(V)}$ sends $R$ to $R\cap S_{i-1}$. Also, $\mathscr{C}_i^{(V)}$ is finite for each $i$ since its members are subgroups of the finite $p$-group $S_i$. So the inverse limit is nonempty: there is a sequence of subgroups $R_m\le R_{m+1}\le\cdots$ with $R_i\in\mathscr{C}_i^{(V)}$ for each $i$. Set $R=\bigcup_{i=m}^{\infty}R_i$. Then $R\cap U=V$, and $R$ is strongly closed in $\mathcal{F}_i$ for each $m\le i<\infty$ and hence in $\mathcal{F}$. In particular, $R\in\mathcal{F}^{\mathfrak{sc}}$ and $R\ngeq\mathfrak{minsc}(\mathcal{F})$, a contradiction. □

The first part of the following lemma implies that if $Q$ is a finite strongly closed subgroup in a saturated fusion system $\mathcal{F}$ over a discrete $p$-toral group $S$, then there is a morphism of fusion systems from $\mathcal{F}$ onto $\mathcal{F}/Q$. For fusion systems over finite $p$-groups, this is originally due to Puig [Pg, Proposition 6.3], and the proof below is based on that of [Cr, Theorem II.5.14].

**Lemma 5.5.** *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$, and let $Q\trianglelefteq S$ be a finite subgroup strongly closed in $\mathcal{F}$.*

(a) *For each $P,R\le S$ and each $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,R)$, there is $\widehat{\varphi}\in\mathrm{Hom}_{\mathcal{F}}(PQ,RQ)$ such that $\varphi(g)\equiv\widehat{\varphi}(g)$ (mod $Q$) for all $g\in P$.*

(b) *If $R\le S$ is strongly closed in $\mathcal{F}$, then $RQ/Q$ is strongly closed in $\mathcal{F}/Q$. Conversely, if $R\ge Q$ and $R/Q$ is strongly closed in $\mathcal{F}/Q$, then $R$ is strongly closed in $\mathcal{F}$. In particular, if $R\le S$ is strongly closed in $\mathcal{F}$, then so is $RQ$.*

*Proof.* **(a)** By Alperin's fusion theorem (Theorem 1.15), it suffices to prove this when $\varphi\in\mathrm{Aut}_{\mathcal{F}}(P)$ and $P$ is fully normalized in $\mathcal{F}$. We will do so by induction on $|P\cap Q|$. If $P\ge Q$, then there is nothing to prove, so we assume $P\ngeq Q$ and hence $|P\cap Q|<|Q|$.

Set

$$K=\mathrm{Ker}\big[\mathrm{Aut}_{\mathcal{F}}(P)\longrightarrow\mathrm{Aut}(PQ/Q)\big] \qquad\text{and}\qquad N_S^K(P)=\{g\in N_S(P)\,|\,c_g^P\in K\}.$$

Set $P_1 = PN_S^K(P)$. We will show that $|P_1 \cap Q| > |P \cap Q|$, and also that there is $\varphi_1 \in \mathrm{Aut}_{\mathcal{F}}(P_1)$ such that $\varphi_1(g) \equiv \varphi(g) \pmod{Q}$ for each $g \in P$. Since $Q$ is finite, we can continue this procedure, and after finitely many steps construct a morphism $\widehat{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(PQ)$ whose restriction to $P$ is congruent to $\varphi$ modulo $Q$.

Now, $PQ > P$ since $P \not\geq Q$, so $PN_Q(P) = N_{PQ}(P) > P$ (see [BLO3, Lemma 1.8]), and hence $N_Q(P) \not\leq P$. Choose $x \in N_Q(P) \smallsetminus P$; then $c_x^P \in K$, and so $x \in (P_1 \cap Q) \smallsetminus P$. This proves that $P_1 \cap Q > P \cap Q$.

Since $K \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$, we have $K \cap \mathrm{Aut}_S(P) \in \mathrm{Syl}_p(K)$. So by the Frattini argument (see [G, Theorem 1.3.7]),

$$\mathrm{Aut}_{\mathcal{F}}(P) = K \cdot N_{\mathrm{Aut}_{\mathcal{F}}(P)}(K \cap \mathrm{Aut}_S(P)).$$

Thus $\varphi = \chi\psi$, where $\chi \in K$ and $\psi \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(K \cap \mathrm{Aut}_S(P))$. In particular, $\psi(g) \equiv \varphi(g) \pmod{Q}$ for each $g \in P$. By the extension axiom, $\psi$ extends to some $\varphi_1 \in \mathrm{Aut}_{\mathcal{F}}(P_1)$ (recall $P_1 = PN_S^K(P)$), which is what we needed to show.

**(b)** If $R \leq S$ and $RQ/Q$ is not strongly closed in $\mathcal{F}/Q$, then there are elements $x \in R$ and $y \in S \smallsetminus RQ$, and a map $\varphi \in \mathrm{Hom}_{\mathcal{F}/Q}(\langle xQ\rangle, \langle yQ\rangle)$ that sends $xQ$ to $yQ$. Hence there is $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(Q\langle x\rangle, Q\langle y\rangle)$ such that $\widehat{\varphi}(x) \in yQ$, and in particular, $\widehat{\varphi}(x) \notin R$. So $R$ is not strongly closed in $\mathcal{F}$.

Conversely, assume $R \geq Q$ is not strongly closed in $\mathcal{F}$. Thus there are elements $x \in R$ and $y \in x^{\mathcal{F}} \smallsetminus R$, and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, \langle y\rangle)$ with $\varphi(x) = y$. By (a), there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q\langle x\rangle, Q\langle y\rangle)$ with $\psi(x) \in yQ$, and in particular, $\psi(x) \notin R$. Then $(\psi/Q)(xQ) = yQ$ in $\mathcal{F}/Q$, so $R/Q$ is not strongly closed in $\mathcal{F}/Q$.

The last statement ($R$ strongly closed implies $RQ$ strongly closed) follows immediately from the first two. □

In the following lemma, we show among other things that under certain conditions on a fusion system over $S$, the strongly closed subgroups properly contained in $S$ are all finite of bounded order.

**Lemma 5.6.** *Let $\mathcal{F}$ be a saturated fusion system over an infinite discrete p-toral group $S$, and let $T \trianglelefteq S$ be the identity component. Assume that $S > T$ and $C_S(T) = T$.*

(a) *Assume there are elements $s_1, \ldots, s_k \in S \smallsetminus T$ such that $s_i^{\mathcal{F}} \cap T \neq \varnothing$ for each $i$ and $S = T\langle s_1, \ldots, s_k\rangle$. Then there is no proper subgroup $R < S$ containing $T$ that is strongly closed in $\mathcal{F}$.*

(b) *Assume that*
   (i) *there is no proper subgroup $T \leq R < S$ that is strongly closed in $\mathcal{F}$; and*
   (ii) *no infinite proper subgroup of $T$ is invariant under the action of $\mathrm{Aut}_{\mathcal{F}}(T)$.*

   *Let $Q \leq S$ be the subgroup generated by all proper subgroups of $S$ that are strongly closed in $\mathcal{F}$. Then $Q \leq T$, $Q$ is finite and strongly closed in $\mathcal{F}$, and $\mathfrak{minsc}(\mathcal{F}/Q) = S/Q$.*

*Proof.* **(a)** Let $s_1, \ldots, s_k \in S \smallsetminus T$ be as assumed. If $R \geq T$ and is strongly closed in $\mathcal{F}$, then for each $1 \leq i \leq k$, we have $s_i \in R$ since it is $\mathcal{F}$-conjugate to an element in $T \leq R$, and hence $R \geq T\langle s_1, \ldots, s_k\rangle = S$.

**(b)** Assume (i) and (ii). If $R < S$ is strongly closed in $\mathcal{F}$, then $[R, T] \leq R \cap T$ since $R, T \trianglelefteq S$, and $R \cap T$ is invariant under the action of $\mathrm{Aut}_{\mathcal{F}}(T)$ since $R$ is strongly closed. Also, $R \cap T < T$ since $R \not\geq T$ by (i), so $R \cap T$ is finite by (ii).

If $R < S$ is strongly closed in $\mathcal{F}$ and $R \not\leq T$, then for $x \in R \smallsetminus T$, we have $[x, T] \neq 1$ since $C_S(T) = T$, and $[x, T]$ is infinitely divisible since $T$ is abelian and infinitely divisible and normalized by $x$. Thus $R \cap T \geq [x, T]$ is infinite, which we just saw is impossible. So $R \leq T$, and $R = R \cap T$ is finite.

Let $\mathscr{C}$ be the set of all subgroups of $T$ strongly closed in $\mathcal{F}$. For each $n \geq 1$, let $\mathscr{C}_n$ be the set of members of $\mathscr{C}$ contained in $\Omega_n(T)$. Thus each set $\mathscr{C}_n$ is finite, $\mathscr{C} = \bigcup_{i=1}^{\infty} \mathscr{C}_i$ since all members of $\mathscr{C}$ are finite, and there are maps $\omega_n \colon \mathscr{C} \longrightarrow \mathscr{C}_n$ that send $R \in \mathscr{C}$ to $R \cap \Omega_n(T) \in \mathscr{C}_n$. If $\mathscr{C}$ is infinite, then we can inductively choose indices $1 \leq n_0 < n_1 < n_2 < \cdots$ together with subgroups $R_i \in \mathscr{C}_{n_i}$ such that $\omega_{n_i}^{-1}(R_i)$ is infinite, and such that $\omega_{n_i}(R_{i+1}) = R_i$ and $R_{i+1} > R_i$ for each $i$. But then $\bigcup_{i=1}^{\infty} R_i \leq T < S$ is infinite and strongly closed in $\mathcal{F}$, which we just showed is impossible.

Thus the set $\mathscr{C}$ is finite, and hence the subgroup $Q = \langle \mathscr{C} \rangle$ is finite. By Lemma 5.5(b), $Q$ is strongly closed in $\mathcal{F}$ and no proper nontrivial subgroup of $S/Q$ is strongly closed in $\mathcal{F}/Q$. So $\mathfrak{minsc}(\mathcal{F}/Q) = S/Q$. □

## 6. Large abelian subgroups of finite simple groups

The main results in this section are Propositions 6.3, 6.4, and 6.10, which together with the classification of finite simple groups (CFSG) are our main tools for proving that certain fusion systems are exotic. In Propositions 6.3 and 6.4, we show that if a known finite simple group $G$ has a "large abelian $p$-subgroup" (in a sense made precise in Definition 6.5), then $G$ must be of Lie type in characteristic different from $p$. In Proposition 6.10, we show that if $G$ is of Lie type in characteristic different from $p$ and has a large abelian $p$-subgroup $A$, then $\mathrm{Aut}_G(A)$ must be one of the groups appearing in a very short list. All of these results are summarized by Corollary 6.11.

All results in this section are independent of CFSG. We begin with the following definition.

**Definition 6.1.** Fix a prime $p$. For each finite $p$-group $S$, set $\underline{\mathbf{ex}}(S) = p^n$ where $n \geq 1$ is the smallest integer for which there is a sequence of subgroups $1 = S_0 < S_1 < S_2 < \cdots < S_k = S$, all of them normal in $S$, such that $|S_i| = p^i$ for each $i$, and such that for each $1 \leq i \leq k$ there is an element $g_i \in S_i \smallsetminus S_{i-1}$ of order at most $p^n$. For each finite group $G$, set $\underline{\mathbf{ex}}_p(G) = \underline{\mathbf{ex}}(S)$ if $S \in \mathrm{Syl}_p(G)$. We call $\underline{\mathbf{ex}}(S)$ the *subexponent* of the $p$-group $S$, and call $\underline{\mathbf{ex}}_p(G)$ the $p$-subexponent of $G$.

Note that the conditions on the $S_i$ in Definition 6.1 are equivalent to requiring that they form a chief series for $S$. Note also that $\underline{\mathbf{ex}}(S) = 1$ if and only if $S = 1$, and hence $\underline{\mathbf{ex}}_p(G) = 1$ if and only if $p \nmid |G|$.

We first list some of the basic properties of $\underline{\mathbf{ex}}(-)$. All of them follow easily from the definition. Let $\mathrm{expt}(S)$ denote the *exponent* of a finite $p$-group $S$.

**Lemma 6.2.** *Fix a prime $p$, and let $S$ be a finite $p$-group.*

(a) *In all cases, $\underline{\mathbf{ex}}(S) \leq \mathrm{expt}(S)$, with equality if $S$ is abelian.*

(b) *If $S = T \times U$, then $\underline{\mathbf{ex}}(S) = \max\{\underline{\mathbf{ex}}(T), \underline{\mathbf{ex}}(U)\}$.*

(c) *If $T \trianglelefteq S$, then $\underline{\mathbf{ex}}(S/T) \leq \underline{\mathbf{ex}}(S) \leq \mathrm{expt}(T) \cdot \underline{\mathbf{ex}}(S/T)$.*

(d) *If $S = TU$ where $T \trianglelefteq S$, then $\underline{\mathbf{ex}}(S) \leq \max\{\mathrm{expt}(T), \underline{\mathbf{ex}}(U)\}$.*

(e) *If $S \cong D_{2^k}$, $Q_{2^k}$, or $SD_{2^k}$ for $k \geq 4$, then $\underline{\mathbf{ex}}(S) = 2^{k-2}$.*

*Proof.* Point (a) follows directly from Definition 6.1, and point (b) is clear. In point (e), $\underline{\mathbf{ex}}(S) \geq 2^{k-2}$ since $S$ contains a unique normal subgroup of index 4 and it is cyclic, and the opposite inequality is easily checked.

The first inequality in (c) is clear. To see the second, set $p^n = \underline{\mathbf{ex}}(S/T)$ and $p^m = \mathrm{expt}(T)$, and let $1 = S_0 < S_1 < S_2 < \cdots < S_\ell = T$ be an arbitrary sequence of subgroups normal in $S$ such that $|S_i| = p^i$. Let $T = S_\ell < S_{\ell+1} < \cdots < S_{\ell+k} = S$ be normal subgroups such that for each $i$, $|S_{\ell+i}/T| = p^i$ and $(S_{\ell+i}/T) \smallsetminus (S_{\ell+i-1}/T)$ contains an element of order at most $p^n$. Then for each $1 \le i \le \ell+k$, the set $S_i \smallsetminus S_{i-1}$ contains an element of order at most $p^{m+n}$, and hence $\underline{\mathbf{ex}}(S) \le p^{m+n} = \mathrm{expt}(T) \cdot \underline{\mathbf{ex}}(S/T)$.

Now assume $S = TU$ where $T \trianglelefteq S$. Let $1 = S_0 < S_1 < \cdots < S_\ell = T$ be subgroups normal in $S$ such that $|S_i| = p^i$. Set $\underline{\mathbf{ex}}(U) = p^n$, and let $1 = U_0 < U_1 < \cdots < U_k = U$ be subgroups normal in $U$ such that $U_i \smallsetminus U_{i-1}$ contains an element of order at most $p^n$ for each $1 \le i \le k$. Set $S_{\ell+i} = TU_i$ for each $1 \le i \le k$. For each such $i$, the subgroup $S_{\ell+i}$ contains $T$ and is normalized by $U$, and hence is normal in $S = TU$. Also, $U_i = U_{i-1}\langle x_i\rangle$ for some $x_i \in U_i$ of order at most $p^n$, so $S_{\ell+i} = TU_i = TU_{i-1}\langle x_i\rangle = S_{\ell+i-1}\langle x_i\rangle$. Finally, $|S_i| \le p^i$ for each $i$, so upon removing duplicated terms, we get a sequence of the form given in Definition 6.1, proving that $\underline{\mathbf{ex}}(S) \le \max\{\mathrm{expt}(T), \underline{\mathbf{ex}}(U)\}$. □

The main property of these functions $\underline{\mathbf{ex}}(-)$ that we need here is the following:

**Proposition 6.3.** *Fix a prime $p$, let $S$ be a finite nonabelian $p$-group, and assume $A \trianglelefteq S$ is a normal abelian subgroup.*

(a) *If $\underline{\mathbf{ex}}(S) = p^n$, then there is $x \in S \smallsetminus A$ of order at most $p^n$ such that $[x, S] \le \Omega_n(A)$.*

(b) *If $1 \le k \le m$ are such that $C_S(\Omega_k(A)) = A$ and $\Omega_m(A)$ is homocyclic of exponent $p^m$ (i.e., every element of $\Omega_1(A)$ is a $p^{m-1}$-st power in $A$), then $\underline{\mathbf{ex}}(S) \ge p^{m-k+1}$.*

*Proof.* **(a)** By assumption, there is a sequence $1 = S_0 < S_1 < \cdots < S_k = S$ of subgroups normal in $S$, where $|S_i| = p^i$ for each $i = 1, \ldots, k$, and where there is an element of order at most $p^n$ in $S_i \smallsetminus S_{i-1}$ for each $i = 1, \ldots, k$. Let $\ell \le k-1$ be such that $S_\ell \le A$ but $S_{\ell+1} \nleq A$ (recall that $S$ is nonabelian, so $S \ne A$). Then $\mathrm{expt}(S_\ell) = \underline{\mathbf{ex}}(S_\ell) \le p^n$ by Lemma 6.2(a) and since $S_\ell$ is abelian, and hence $S_\ell \le \Omega_n(A)$.

Now, $S_{\ell+1}/S_\ell$ is normal of order $p$ in $S/S_\ell$, hence lies in $Z(S/S_\ell)$, and so $[S, S_{\ell+1}] \le S_\ell$. By assumption, there is $x \in S_{\ell+1} \smallsetminus S_\ell$ of order at most $p^n$. Then $x \notin A$ since $S_{\ell+1} \cap A = S_\ell$, and $[x, S] \le [S_{\ell+1}, S] \le S_\ell \le \Omega_n(A)$.

**(b)** For each $x \in S \smallsetminus A$, since $C_S(\Omega_k(A)) = A$, there is $a_0 \in \Omega_k(A)$ such that $[a_0, x] \ne 1$. Choose $a \in A$ such that $a^{p^{m-k}} = a_0$; then $[a, x]$ has order at least $p^{m-k+1}$ and lies in $[A, x]$. Hence $[A, x] \nleq \Omega_{m-k}(A)$, so $\underline{\mathbf{ex}}(S) \ge p^{m-k+1}$ by (a). □

We will use $\underline{\mathbf{ex}}_p(-)$ to characterize certain simple groups of Lie type in characteristic different from $p$. The next proposition is the key to doing this.

**Proposition 6.4.** *Fix a prime $p$. Let $G$ be an alternating group, a simple group of Lie type in defining characteristic $p$, the Tits group ${}^2F_4(2)'$ (if $p = 2$), or a sporadic simple group. Then $\underline{\mathbf{ex}}_p(G) \le p^3$, and $\underline{\mathbf{ex}}_p(G) \le p^2$ if $p$ is odd.*

*Proof.* We consider the three cases separately. Fix $S \in \mathrm{Syl}_p(G)$.

**Case 1:** Assume $G = A_n$ and $S \in \mathrm{Syl}_p(G)$. If $p$ is odd, then $S$ is a product of iterated wreath products $C_p \wr \cdots \wr C_p$, so $\underline{\mathbf{ex}}_p(A_n) = \underline{\mathbf{ex}}(S) = p$ by Lemma 6.2(b,d). If $p = 2$, then $S \cong E \rtimes T$ where $E$ is elementary abelian of rank $[n/2] - 1$ and $T$ is a product of iterated wreath products $C_2 \wr \cdots \wr C_2$, so $\underline{\mathbf{ex}}_2(S) = 2$ by Lemma 6.2(b,d) again.

**Case 2:** If $S \in \mathrm{Syl}_p(G)$ where $G$ is a finite group of Lie type in defining characteristic $p$, then $S$ has a normal series where each term is the semidirect product of the previous term with a root group. This follows from [GLS3, Theorem 3.3.1] when $G$ is a Chevalley group or a Steinberg group; from [R, (3.4,3.8)] when $G \cong {}^2F_4(q)$ (and $p = 2$); and holds when $G \cong \mathrm{Sz}(q)$ $(p = 2)$ or $G \cong {}^2G_2(q)$ $(p = 3)$ since $S$ is a root group in those cases. By [GLS3, Table 2.4], these root groups are all elementary abelian, except when $G$ is an odd dimensional unitary group or a Suzuki or Ree group in which case they can have exponent $p^2$. So $\underline{\mathbf{ex}}_p(G) \le p^2$ in all cases (and $\underline{\mathbf{ex}}_p(G) = p$ if $G$ is a Chevalley group, or a Steinberg group other than an odd dimensional unitary group).

By Lemma 1 and Section 3 in [Pa], the Tits group ${}^2F_4(2)'$ contains a Sylow 2-subgroup of the form $T = J\langle x\rangle$, where $J$ is an extension of $E = [J, J] \cong E_{2^5}$ by $E_{2^4}$ and $|x| = 4$. So $\underline{\mathbf{ex}}_2({}^2F_4(2)') \le 4$.

**Case 3:** In Table 6.1, for each sporadic group $G$, and each prime $p$ such that $S \in \mathrm{Syl}_p(G)$ is neither elementary abelian nor extraspecial of exponent $p$, we give an upper bound for $\underline{\mathbf{ex}}_p(G)$, based on a chosen subgroup $H_p \le G$ containing $S$. These subgroups are all listed in the Atlas [Atlas], and more precise references in many cases are given in Tables 2.1–2.2 and 3.2–3.3 in [O2]. In most cases, the bound follows immediately from the description of $H_p$ together with Lemma 6.2.

By [Jn, § 5] or [A1, § 13], if $G = J_3$ and $p = 3$, then $\Omega_1(S) \cong E_{3^3}$ and $S/\Omega_1(S) \cong E_9$, so $\underline{\mathbf{ex}}_3(J_3) = 9$. If $G = F_3$ and $p = 3$, then by [A1, 14.2(1,5)], $S$ is an extension of $E_{3^5}$ by $C_3 \times (C_3 \wr C_3)$, so $\underline{\mathbf{ex}}_3(F_3) \le 9$ by Lemma 6.2(c). If $G = Fi_{23}$, then $H_3 \cong O_8^+(3){:}3$ where a Sylow 3-subgroup of $O_8^+(3)$ has exponent at most 9 by Lemma 3.6(b), so $\underline{\mathbf{ex}}_3(Fi_{23}) \le 9$ by Lemma 6.2(d).

If $G = F_5$ and $p = 2$, then $H_2$ is an extension of the group $2_+^{1+8}$ of exponent 4 by $A_5 \wr C_2$. Since $\underline{\mathbf{ex}}_2(A_5 \wr C_2) = 2$, we have $\underline{\mathbf{ex}}_2(F_5) \le 8$. □

Propositions 6.3(b) and 6.4 motivate the following definition.

**Definition 6.5.** A *large abelian subgroup* of a finite $p$-group $S$ is a normal abelian subgroup $A \trianglelefteq S$ such that $C_S(\Omega_2(A)) = A$ and $\Omega_5(A)$ is homocyclic of exponent $p^5$. A large abelian $p$-subgroup of an arbitrary finite group $G$ is a large abelian subgroup of a Sylow $p$-subgroup of $G$.

Our next main result, Proposition 6.10, describes the possible automizers of a large abelian subgroup of a simple group $G$. Some technical lemmas are first needed.

**Lemma 6.6.** *Let $B \le A$ be finite abelian groups, and set*

$$G = \{\alpha \in \mathrm{Aut}(A) \,|\, \alpha|_B = \mathrm{Id},\ [\alpha, A] \le B\}.$$

*Then $G$ is abelian, and* $\mathrm{expt}(G) = \gcd(\mathrm{expt}(B), \mathrm{expt}(A/B))$.

*Proof.* We write the groups $A$ and $B$ additively. Set

$$D = \{\varphi \in \mathrm{End}(A) \,|\, \mathrm{Im}(\varphi) \le B \le \mathrm{Ker}(\varphi)\} \cong \mathrm{Hom}(A/B, B),$$

regarded as an additive group. For all $\rho, \sigma \in D$, we have $\rho \circ \sigma = 0$, so $(\rho + \mathrm{Id}_A) \circ (\sigma + \mathrm{Id}_A) = (\rho + \sigma + \mathrm{Id}_A)$. In particular, this shows that $\rho + \mathrm{Id}_A \in G$ for all $\rho \in D$, so there is an injective homomorphism $\chi\colon D \longrightarrow G$ defined by $\chi(\rho) = \rho + \mathrm{Id}_A$. Also, $\alpha - \mathrm{Id}_A \in D$ for each $\alpha \in G$ by definition of $G$, so $\chi$ is an isomorphism, and hence

$$\mathrm{expt}(G) = \mathrm{expt}(D) = \mathrm{expt}(\mathrm{Hom}(A/B, B)) = \gcd(\mathrm{expt}(A/B), \mathrm{expt}(B)).$$ □

**Lemma 6.7.** *Fix a prime $p$ and an integer $m \ge 1$. Let $S$ be a finite, nonabelian $p$-group, and let $A \trianglelefteq S$ be a normal abelian subgroup such that*

(i) $A = C_S(\Omega_m(A))$, *and*

| $G$ | $M_{11}$ | $M_{12}$ | $M_{22}$ | $M_{23}$ | $M_{24}$ | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $Co_3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\underline{\mathbf{ex}}_2(G)$ | 4 | $\le 4$ | 2 | 2 | 2 | 2 | $\le 4$ | $\le 4$ | 2 | $\le 4$ |
| $H_2$ | $SD_{16}$ | $4^2{:}D_{12}$ | $2^4{:}A_6$ | $2^4{:}A_7$ | $2^4{:}A_8$ | $2^3$ | $2^{2+4}{:}S_3$ | $2^{2+4}{:}S_3$ | $2^{11}{:}M_{24}$ | $2^4{\cdot}A_8$ |
| $\underline{\mathbf{ex}}_3(G)$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 9 | 3 | 3 |
| $H_3$ | $3^2$ | $3^{1+2}_+$ | $3^2$ | $3^2$ | $3^{1+2}_+$ | 3 | $3^{1+2}_+$ | $3^3.3^2$ | $3^{1+2}_+$ | $3^5{:}M_{11}$ |

| $G$ | $Co_2$ | $Co_1$ | $HS$ | $McL$ | $Suz$ | $He$ | $Ly$ | $Ru$ | $O'N$ |
|---|---|---|---|---|---|---|---|---|---|
| $\underline{\mathbf{ex}}_2(G)$ | 2 | 2 | 4 | 2 | $\le 4$ | 2 | $\le 4$ | $\le 4$ | $\le 8$ |
| $H_2$ | $2^{10}{:}M_{22}{:}2$ | $2^{11}{:}M_{24}$ | $4^3{:}L_3(2)$ | $M_{22}$ | $2^{4+6}{:}3A_6$ | $2^6{:}3S_6$ | $2A_{11}$ | $2^{3+8}{:}L_3(2)$ | $4^3{\cdot}L_3(2)$ |
| $\underline{\mathbf{ex}}_3(G)$ | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $H_3$ | $McL$ | $3^6{:}2M_{12}$ | $S_8$ | $3^4{:}A_6$ | $3^5{:}M_{11}$ | $3^{1+2}_+$ | $3^5{:}M_{11}$ | $3^{1+2}_+$ | $3^4$ |
| $\underline{\mathbf{ex}}_5(G)$ | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| $H_5$ | $5^{1+2}_+$ | $5^3{:}A_5$ | $5^{1+2}_+$ | $5^{1+2}_+$ | $5^2$ | $5^2$ | $G_2(5)$ | $5^{1+2}_+$ | 5 |

| $G$ | $Fi_{22}$ | $Fi_{23}$ | $Fi'_{24}$ | $F_5$ | $F_3$ | $F_2$ | $F_1$ |
|---|---|---|---|---|---|---|---|
| $\underline{\mathbf{ex}}_2(G)$ | 2 | $\le 4$ | $\le 4$ | $\le 8$ | $\le 4$ | $\le 8$ | $\le 8$ |
| $H_2$ | $2^{10}{:}M_{22}$ | $2^{11}{\cdot}M_{23}$ | $2^{11}{\cdot}M_{24}$ | $2^{1+8}_+.(A_5 \wr 2)$ | $2^5{\cdot}L_5(2)$ | $2^{1+22}_+{\cdot}Co_2$ | $2^{1+24}_+{\cdot}Co_1$ |
| $\underline{\mathbf{ex}}_3(G)$ | 3 | $\le 9$ | $\le 9$ | 3 | $\le 9$ | $\le 9$ | $\le 9$ |
| $H_3$ | $O_7(3)$ | $O_8^+(3){:}3$ | $3^7{\cdot}O_7(3)$ | $3^4{:}2(A_4\times A_4)$ | $3^5.3^4.GL_2(3)$ | $Fi_{23}$ | $3^8{\cdot}O_8^-(3)$ |
| $\underline{\mathbf{ex}}_5(G)$ | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| $H_5$ | $5^2$ | $5^2$ | $5^2$ | $5^{1+4}_+{:}(2^{1+4}_-.5)$ | $5^{1+2}_+$ | $F_5$ | $5^{1+6}_+{:}4J_2$ |
| $\underline{\mathbf{ex}}_7(G)$ | 7 | 7 | 7 | 7 | 7 | 7 | 7 |
| $H_7$ | 7 | 7 | $7^{1+2}_+$ | 7 | $7^2$ | $7^2$ | $7^{1+4}_+{:}2S_7$ |

TABLE 6.1. Upper bounds for $\underline{\mathbf{ex}}_p(G)$ for sporadic groups $G$. In each case, $H_p$ is a subgroup of $G$ of index prime to $p$, except when in brackets in which case it is some group whose Sylow $p$-subgroups are isomorphic to those of $G$. The groups are described using [Atlas] notation; in particular, $H{:}K$ and $H{\cdot}K$ denote split and nonsplit extensions, respectively.

(ii) *$\Omega_{2m}(A)$ is homocyclic of exponent $p^{2m}$.*

*Then $A$ is the only normal abelian subgroup of $S$ that satisfies* (i).

*Proof.* Let $A^*$ be an arbitrary normal abelian subgroup of $S$ such that $A^* = C_S(\Omega_m(A^*))$. Set $B = A \cap A^*$. Then $[A, \Omega_m(A^*)] \le A \cap \Omega_m(A^*) = \Omega_m(B)$ since $A$ and $\Omega_m(A^*)$ are both normal. So $\mathrm{Aut}_A(\Omega_m(A^*))$ has exponent at most $p^m$ by Lemma 6.6, applied with $\Omega_m(B) \le \Omega_m(A^*)$ in the role of $B \le A$. Also, $\mathrm{Aut}_A(\Omega_m(A^*)) \cong A/C_A(\Omega_m(A^*)) = A/(A \cap A^*) = A/B$ by assumption, so $A/B$ has exponent at most $p^m$.

Since $\Omega_{2m}(A)$ is homocyclic of exponent $p^{2m}$ by (ii), every element of $\Omega_m(A)$ is a $p^m$-power in $A$, and hence lies in $B$. So $\Omega_m(A) = \Omega_m(B) \le \Omega_m(A^*)$, and hence $A = C_S(\Omega_m(A)) \ge C_S(\Omega_m(A^*)) = A^*$ by (i). So $\Omega_m(A) = \Omega_m(A^*)$, and hence $A = A^*$. □

In particular, Lemma 6.7 shows that if $A \trianglelefteq S$ is a large abelian subgroup, and $B \trianglelefteq S$ is such that $C_S(\Omega_2(B)) = B$, then $B = A$.

**Lemma 6.8.** *Let $T$ be a discrete $p$-torus, and let $G \le \mathrm{Aut}(T)$ be a finite group of automorphisms. Then $G$ acts faithfully on $\Omega_1(T)$ if $p$ is odd, and on $\Omega_2(T)$ if $p = 2$.*

*Proof.* Set $q = p^k$, where $k = 1$ if $p$ is odd and $k = 2$ if $p = 2$. Let $\theta\colon \mathrm{Aut}(T) \longrightarrow \mathrm{Aut}(\Omega_k(T))$ be the homomorphism induced by restriction; we must show that $\theta|_G$ is injective. Upon identifying $\mathrm{Aut}(T) = GL_n(\mathbb{Z}_p)$ and $\mathrm{Aut}(\Omega_k(T)) = GL_n(\mathbb{Z}/q)$, we see that it suffices to show that the multiplicative group $\mathrm{Ker}(\theta) = I + qM_n(\mathbb{Z}_p)$ contains no nonidentity elements of finite order. Note that this is not true when $q = 2$, since $-I \in I + 2M_n(\mathbb{Z}_p)$.

Assume otherwise: let $0 \ne X \in M_n(\mathbb{Z}_p)$ and $n > 1$ be such that $(I + qX)^n = I$. Let $i \ge k$ and $Y \in M_n(\mathbb{Z}_p)$ be such that $qX = p^iY$ and $p \nmid Y$. Write $n = p^jm$ where $p \nmid m$. Then

$$\begin{aligned} I = (I + qX)^n = (I + p^iY)^{p^jm} &= I + p^{i+j}mY + p^{2i+j}m\big(\tfrac{n-1}{2}\big)Y^2 + \dots \\ &\equiv I + p^{i+j}mY \pmod{p^{i+j+1}M_n(\mathbb{Z}_p)}, \end{aligned}$$

which is impossible. (This argument does not work when $p = 2$ and $i = 1$, since $p^{2i+j} = p^{i+j+1}$ and the factor $\frac{n-1}{2}$ need not be in $\mathbb{Z}_p$.) □

**Notation 6.9.** *For all $4 \le i \le 37$, let $\boldsymbol{ST_i}$ denote the $i$-th group in the Shephard-Todd list* [ST, Table VII] *of irreducible unitary reflection groups. For each prime $p$, each $k \mid m \mid (p-1)$, and each $n \ge 2$, set*

$$G(m,k,n) = \big\langle \mathrm{diag}(u_1,\dots,u_n) \,\big|\, u_1^m = \dots = u_n^m = 1,\ (u_1u_2\dots u_n)^{m/k} = 1 \big\rangle\, \mathrm{Perm}_n \le GL_n(\mathbb{F}_p);$$

*i.e., the group of all monomial matrices in $GL_n(\mathbb{F}_p)$ whose nonzero entries are $m$-th roots of unity with product an $(m/k)$-th root of unity.*

Thus for each $m$, $k$, and $n$ as above, $G(m,k,n)$ is normal of index $k$ in $G(m,1,n) \cong C_m \wr \Sigma_n$.

We are now ready to apply Lemma 6.7 to show that if a finite simple group $G$ of Lie type contains a large abelian $p$-subgroup $A$, then its automizer is one of the groups on a very short list.

**Proposition 6.10.** *Fix a prime $p$, and let $G$ be a finite simple group of Lie type in characteristic different from $p$. Assume $G$ has a large abelian $p$-subgroup $A \le G$. Set $k = 2$ if $p = 2$, or $k = 1$ if $p$ is odd. Then $(\mathrm{Aut}_G(A), \Omega_k(A)) \cong (W, M)$ where either*

(a) *$W \cong \mathrm{Aut}_{\mathbf{G}}(\mathbf{T})$ for some simple compact connected Lie group $\mathbf{G}$ with maximal torus $\mathbf{T}$, and $M$ is the group of elements of order dividing $p^k$ in $\mathbf{T}$; or*

(b) *$W \cong G(m,1,n) \cong C_m \wr \Sigma_n$ where $3 \le m \mid (p-1)$, $n \ge p$, and $M \cong (\mathbb{F}_p)^n$ has the natural action of $W \le GL_n(p)$; or*

(c) *$W \cong G(2m,2,n)$ where $2 \le m \mid (p-1)/2$, $n \ge p$, and $M \cong (\mathbb{F}_p)^n$ has the natural action of $W \le GL_n(p)$; or*

(d) *$p = 3$, $W \cong \boldsymbol{ST_{12}} \cong GL_2(3)$, and $M \cong (\mathbb{F}_3)^2$ has the natural action of $W$; or*

(e) *$p = 5$, $W \cong \boldsymbol{ST_{31}}$ (an extension of the form $(C_4 \circ 2^{1+4}).\Sigma_6$), and $M$ is the unique faithful $\mathbb{F}_5W$-module with $\mathrm{rk}(M) = 4$.*

*Proof.* Let $S \in \mathrm{Syl}_p(G)$ be such that $A \trianglelefteq S$ is a large abelian subgroup. Thus $A = C_S(\Omega_2(A))$, and $\Omega_5(A)$ is homocyclic of exponent $p^5$.

By definition, and since $G$ is a finite group of Lie type in characteristic $r$ for some prime $r \ne p$, there is a simple algebraic group $\overline{G}$ over $\overline{\mathbb{F}}_r$, and an algebraic endomorphism $\sigma \in \mathrm{End}(\overline{G})$, such that $C_{\overline{G}}(\sigma)$ is finite (the group of elements fixed by $\sigma$) and $G \cong O^{r'}(C_{\overline{G}}(\sigma))$.

See, e.g., [GLS3, § 2.2] for more detail. From now on, we identify $G$ with $O^{r'}(C_{\overline{G}}(\sigma))$. By [GLS3, Theorem 4.10.2(a,b)], there is a maximal torus $\overline{T} \le \overline{G}$ (denoted $\overline{T}_w$ there) and a Sylow $p$-subgroup $S \in \mathrm{Syl}_p(G)$ such that $\sigma(\overline{T}) = \overline{T}$ and $S \le N_{\overline{G}}(\overline{T})$, and such that if we set $S_T = S \cap \overline{T}$, then $S/S_T$ is isomorphic to a subgroup of the Weyl group of $\overline{G}$. (The relation $S_T = S \cap \overline{T}$ isn't stated explicitly in that theorem, but it appears in its proof.)

Thus $\mathrm{Aut}_S(S_T)$ consists of the restrictions to $S_T$ of a finite group of automorphisms of a discrete $p$-torus (the $p$-power torsion in $\overline{T}$). Hence it acts faithfully on $\Omega_2(S_T)$ by Lemma 6.8, and so $C_S(\Omega_2(S_T)) = C_S(\overline{T}) = S_T$. The hypotheses of Lemma 6.7 thus apply with $A$ and $S_T$ in the roles of $A_1$ and $A_2$ (and $m = 2$). So $A = S_T$ by that lemma.

Assume $G^*$ is a finite group whose $p$-fusion system is isomorphic to that of $G$. Choose a fusion preserving isomophism $\rho\colon S \xrightarrow{\cong} S^*$ for some $S^* \in \mathrm{Syl}_p(G^*)$, and set $A^* = \rho(A) \trianglelefteq S^*$. Then $A^* = C_{S^*}(\Omega_2(A^*))$ and $\Omega_4(A^*)$ is homocyclic of exponent $p^4$, and $\mathrm{Aut}_{G^*}(A^*) \cong \mathrm{Aut}_G(A)$. So the proposition holds for $G$ if it holds for $G^*$.

We first handle two special cases: certain Chevalley groups and Steinberg groups. Afterwards, we deal with the remaining cases: first when $p = 2$ and then when $p$ is odd.

**Case 1:** Assume $G \cong \mathbb{G}(q)$, where $q \equiv 1 \pmod{p^k}$ (and $q$ is a power of $r$). By [BMO2, Lemma 6.1], we can choose $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_r)$ and $\sigma$ such that [BMO2, Hypotheses 5.1, case (III.1)] holds. So by [BMO2, Lemma 5.3], $\mathrm{Aut}_G(A) = \mathrm{Aut}_{\overline{G}}(A)$ is isomorphic to the Weyl group $\mathbb{G}$, and so $(\mathrm{Aut}_G(A), \Omega_k(A)) \cong (\mathrm{Aut}_{\overline{G}}(\overline{T}), \Omega_k(O_p(\overline{T})))$.

Let $\mathbf{G}$ be a maximal compact subgroup of $\mathbb{G}(\mathbb{C})$, let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$, and let $O_p(\mathbf{T}) \le \mathbf{T}$ be the subgroup of elements of $p$-power order. Then $(\mathrm{Aut}_{\mathbf{G}}(\mathbf{T}), O_p(\mathbf{T})) \cong (\mathrm{Aut}_{\overline{G}}(\overline{T}), O_p(\overline{T}))$, by Theorem 4.2, and hence $(\mathrm{Aut}_{\mathbf{G}}(\mathbf{T}), \Omega_k(O_p(\mathbf{T}))) \cong (\mathrm{Aut}_G(A), \Omega_k(A))$.

**Case 2:** Now assume $G \cong {}^2\mathbb{G}(q)$, where $\mathbb{G} \cong A_n$, $D_n$, or $E_6$, and $q \equiv 1 \pmod{p^k}$ (and $q$ is a power of $r$). Again by [BMO2, Lemma 6.1], we can choose $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_r)$ and $\sigma$ such that [BMO2, Hypotheses 5.1, case (III.1)] holds. So by [BMO2, Lemma 5.3], $\mathrm{Aut}_G(A) = \mathrm{Aut}_{W_0}(A)$, where $W_0 = C_{W(\overline{G})}(\tau)$, and where $\tau \in \mathrm{Aut}(\overline{G})$ is a graph automorphism of order 2 that permutes the root groups for positive roots. Also, $\mathrm{Aut}_G(A)$ acts faithfully on $\Omega_k(A)$ by Lemma 6.8, and $\Omega_k(A) = \Omega_k(C_{\overline{T}}(\tau))$ since $q \equiv 1 \pmod{p^k}$.

By [Car, § 13.3], $W_0$ (denoted $W^1$ in [Car]) is isomorphic to the Weyl group of $\mathbb{H}$, where $\mathbb{H} \cong B_{n/2}$ or $C_{(n+1)/2}$, $B_{n-1}$, or $F_4$ when $\overline{G}$ has type $A_n$ ($n$ even or odd), $D_n$, or $E_6$, respectively. Furthermore, the arguments in [Car, § 13.3] show that the pair $(W_0, C_{\overline{T}}(\tau))$ is isomorphic to the Weyl group and maximal torus of $\mathbb{H}(\overline{\mathbb{F}}_r)$. So if we let $\mathbf{H}$ be a maximal compact subgroup of $\mathbb{H}(\mathbb{C})$ and let $\mathbf{T}_0 \le \mathbf{H}$ be a maximal torus, then $(\mathrm{Aut}_G(A), \Omega_k(A))$ is isomorphic to $(\mathrm{Aut}_{\mathbf{H}}(\mathbf{T}_0), \Omega_k(O_p(\mathbf{T}_0)))$ by Theorem 4.2.

**General case when $p = 2$:** By [BMO2, Table 0.1] and the remark just before the table, the 2-fusion system of $G$ is isomorphic to that of a group $G^*$, where either $G^*$ is one of the Chevalley or Steinberg groups considered in Cases 1 and 2, or $G^* \cong G_2(3)$ or ${}^2G_2(q)$. But neither $G_2(3)$ nor ${}^2G_2(q)$ satisfies the hypotheses of the proposition, since neither contains elements of order $2^4 = 16$.

**General case when $p$ is odd:** By [BMO2, Proposition 1.10], the $p$-fusion system of $G$ is isomorphic to that of a group $G^*$, where either $G^*$ is one of the groups handled in Cases 1 and 2, or it is one of the following groups:

**$G^*\cong PSL_n(q)$ or $P\Omega_{2n}^{\pm}(q)$, where $n\ge p$ and $q\not\equiv 0,1 \pmod p$.** By [BMO2, Lemma 6.5], we can choose a $\sigma$-setup for $G^*$ such that [BMO2, Hypotheses 5.1, case (III.3)] holds. In particular, $\mathrm{Aut}_{G^*}(A)=\mathrm{Aut}_{W_0}(A)$, where $W_0$ is a certain subgroup of the Weyl group of $P\Omega_{2n}$ described, together with its action on a maximal torus, in [BMO2, Table 6.1]. By that table, there are $\mu\ge 1$ and $\kappa\ge p$ such that $\mathrm{rk}(A)=\kappa$, and $\mathrm{Aut}_{W_0}(A)$ has index at most 2 in a group $\mathrm{Aut}_{W_0^*}(A)\cong C_{2\mu}\wr\Sigma_\kappa\cong G(2\mu,1,\kappa)$. More precisely, from the proof of that lemma, one sees that $\mathrm{Aut}_{W_0}(A)\cong G(2\mu,1,\kappa)$ or $G(2\mu,2,\kappa)$.

**$G^*\cong {}^2F_4(q)$, where $p=3$.** By [Ma, Section 1.2], $G^*$ contains a maximal subgroup $H\cong(C_{q+1}\times C_{q+1})\rtimes GL_2(3)$, and it has index prime to 3 in $G^*$. So we can take $A=O_3(H)$. Then $H=N_{G^*}(A)$ since it is maximal, and thus $\mathrm{Aut}_{G^*}(A)\cong GL_2(3)$ with its natural action on $\Omega_1(A)\cong(\mathbb{F}_3)^2$.

**$G^*\cong {}^3D_4(q)$, where $p=3$.** By [GL, 10-1(4)], $S$ is a semidirect product $A\rtimes C_3$, where $A\cong C_{3^a}\times C_{3^{a+1}}$ for $a=v_3(q^2-1)$. Thus $\mathrm{Aut}_{G^*}(A)$ is isomorphic to a subgroup of $C_2\times\Sigma_3$, which we identify with the group of upper triangular matrices in $GL_2(3)$. Also, by the main theorem in [Kl], $G^*$ contains a maximal subgroup $H\cong(C_{q^2+q+1}\circ SL_3(q)).\Sigma_3$ (if $q\equiv 1 \pmod 3$) or $H\cong(C_{q^2-q+1}\circ SL_3(q)).\Sigma_3$ (if $q\equiv 2 \pmod 3$), in either case of index prime to 3, and using these we see that $|N_{G^*}(S)/S|=4$ and hence $\mathrm{Aut}_{G^*}(A)\cong C_2\times\Sigma_3\cong W(G_2)$. Since this is the normalizer of a Sylow 3-subgroup in $GL_2(3)$, its action on $\Omega_1(A)\cong(\mathbb{F}_3)^2$ is unique up to isomorphism. (The 3-fusion system of $G^*$ is described in detail in case (a.ii) of [O1, Theorem 2.8].)

**$G^*\cong E_8(q)$, where $p=5$ and $q\equiv\pm 2 \pmod 5$.** In this case, $\mathrm{Aut}_{G^*}(A)\cong \boldsymbol{ST_{31}}$ by [BMO2, Lemma 6.7] and its proof. More precisely, a subgroup $W_1\trianglelefteq\mathrm{Aut}_{G^*}(A)$ was defined in that proof and shown to be isomorphic to $C_4\circ 2^{1+4}$, and $\mathrm{Aut}_{G^*}(A)/W_1$ was shown to be isomorphic to $\Sigma_6$.

Let $\mathrm{Out}_0(W_1)$ be the subgroup of elements that are the identity on $Z(W_1)$; then $\mathrm{Out}_0(W_1)\cong Sp_4(2)\cong\Sigma_6$. Since $W_1$ is irreducible in $GL_4(\overline{\mathbb{F}}_5)$, we have $C_{GL_4(5)}(W_1)\cong\mathbb{F}_5^\times$, and hence $\mathrm{Aut}_{G^*}(A)\cong N_{GL_4(5)}(W_1)$.

Since $GL_4(5)$ has only one conjugacy class of subgroups isomorphic to $W_1$, this shows that it also contains only one class of extensions of the form $(C_4\circ 2^{1+4}).\Sigma_6$. On the other hand, Shephard [Sh, p. 275] constructed explicit matrices in $GL_4(\mathbb{C})$ representing reflections that generate $\boldsymbol{ST_{31}}$, and those matrices have entries in $\mathbb{Z}[\frac{1}{2},i]$ and hence reduce to matrices over $\mathbb{F}_5$. Thus $\boldsymbol{ST_{31}}$ embeds in $GL_4(5)$, so $\mathrm{Aut}_{G^*}(A)\cong\boldsymbol{ST_{31}}$. See also Aguadé's paper [Ag, §6] for more discussion. □

The following corollary combines the three main propositions in this section.

**Corollary 6.11.** *Fix a prime $p$, and set $k=2$ if $p=2$, or $k=1$ if $p$ is odd. If $G$ is a known finite simple group and has a large abelian $p$-subgroup $A\le G$, then $G$ is a group of Lie type in characteristic different from $p$, and $(\mathrm{Aut}_G(A),\Omega_k(A))\cong(W,M)$ where $(W,M)$ is one of the modules listed in cases (a)–(e) of Proposition 6.10.*

*Proof.* Let $S\in\mathrm{Syl}_p(G)$ be such that $A\trianglelefteq S$ is a large abelian subgroup. Then $C_S(\Omega_2(A))=A$ and $\Omega_5(A)$ is homocyclic of exponent $p^5$, so by Proposition 6.3(b), $\underline{\mathbf{ex}}_p(G)=\underline{\mathbf{ex}}(S)\ge p^4$. Hence by Proposition 6.4 and since $G$ is a known simple group, $G$ must be of Lie type in characteristic different from $p$. The conclusion now follows from Proposition 6.10. □

The following lemma will be useful when applying Proposition 6.10 in the proof of Theorem 9.3.

**Lemma 6.12.** *Fix an odd prime $p$, and let $(W, M)$ be one of the pairs that appears in cases* (a)–(e) *of Proposition 6.10. Then $M$ is a simple $\mathbb{F}_pW$-module,* except *in case* (a) *when $W = \mathrm{Aut}_G(T)$ and $M = \mathrm{Hom}(C_p, T)$, and either $G \cong PSU(n)$ and $p \mid n$, or $G \cong G_2$ or $E_6$ and $p = 3$.*

*Proof.* We refer to the five cases (a)–(e) listed in the statement of Proposition 6.10. In cases (b) and (c), we have $W \cong G(m, k, n)$ for some $k \mid m \mid (p-1)$ and some $n \geq p$, and $M$ is the natural $n$-dimensional $\mathbb{F}_pW$-module. Let $W_0 \trianglelefteq W$ be the subgroup of diagonal matrices (in the notation of 6.9); then $M|_{W_0}$ splits in a unique way as a direct sum of 1-dimensional submodules, and these summands are permuted transitively by $W/W_0 \cong \Sigma_n$. So $M$ is simple in this case.

In case (d), we have $W \cong GL_2(3)$, and $M \cong (\mathbb{F}_3)^2$ is clearly simple. In case (e), there is a subgroup $H \cong 2^{1+4}_+$ of index 2 in $O_2(W)$ whose action on $M$ is generated by the diagonal matrices $-\mathrm{Id}$, $\mathrm{diag}(1, 1, -1, -1)$, and $\mathrm{diag}(1, -1, 1, -1)$, together with the permutation matrices for the permutations $(1\,2)(3\,4)$ and $(1\,3)(2\,4)$, and $M|_H$ is irreducible. So $M$ is also irreducible as an $\mathbb{F}_5W$-module.

Now assume we are in case (a), where $W \cong \mathrm{Aut}_G(T)$ and $M$ is the $p$-torsion in $T$ for some simple compact connected Lie group $G$ with maximal torus $T$. If $G$ has type $A_{n-1}$, so $W \cong \Sigma_n$ and $M \cong (\mathbb{F}_p)^{n-1}$, then $M$ is simple except when $p \mid n$ (see [Jm, Example 5.1]). If $G$ has type $B_n$, $C_n$, or $D_n$, then $W \cong G(2, 1, n)$ or $G(2, 2, n)$, and $M$ is simple by a similar argument to that used in cases (b) and (c). If $G$ has type $F_4$, then $W$ contains a subgroup isomorphic to $G(2, 1, 4)$ (with the natural action on $M \cong (\mathbb{F}_p)^4$), and hence $M$ is simple. If $G$ is of type $E_n$ for $n = 6, 7, 8$, then by the character tables in [Atlas] (when $p \nmid |W|$) and [BrAtl] (when $p \mid |W|$), we have that $M$ is a simple $\mathbb{F}_pW$-module in all cases except when $G$ has type $E_6$ and $p = 3$. □

## 7. Realizability of fusion systems over discrete $p$-toral groups

We are now ready to prove our main nonrealizability results. Theorem 7.4 (a slightly more general version of Theorem C) says that saturated fusion systems over infinite discrete $p$-toral groups satisfying certain conditions are not sequentially realizable. In Proposition 7.5, we give a criterion for showing that certain fusion systems are not realized by linear torsion groups in characteristic 0. All of the results in this section (except Lemmas 7.1 and 7.2) depend on the classification of finite simple groups.

We first note that a quotient of a sequentially realizable fusion system (see Definition 1.16) is again sequentially realizable.

**Lemma 7.1.** *Let $\mathcal{F}$ be a sequentially realizable fusion system over a discrete $p$-toral group $S$, and let $Q \trianglelefteq S$ be a finite subgroup that is weakly closed in $\mathcal{F}$. Then $\mathcal{F}/Q$ is also sequentially realizable. If $T \trianglelefteq S$ is the identity component of $S$ and $Q \leq T$, then $C_{S/Q}(T/Q) = C_S(T)/Q$ and $\mathrm{Aut}_{\mathcal{F}/Q}(T/Q) \cong \mathrm{Aut}_{\mathcal{F}}(T)$.*

*Proof.* Since $\mathcal{F}$ is sequentially realizable, there is an increasing sequence $\mathcal{F}_1 \leq \mathcal{F}_2 \leq \ldots$ of realizable fusion subsystems of $\mathcal{F}$ over finite subgroups $S_1 \leq S_2 \leq \ldots$ of $S$, such that $S = \bigcup_{i=1}^{\infty} S_i$ and $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$. Since $Q$ is finite, we have $Q \leq S_k$ for some $k \geq 1$, and by removing the first $(k-1)$ terms in the sequence, we can arrange that $Q \leq S_i$, and hence $Q$ is weakly closed in $\mathcal{F}_i$, for all $i \geq 1$.

For each $i$, since $\mathcal{F}_i$ is realizable, there is a finite group $G_i$ with $S_i \in \mathrm{Syl}_p(G_i)$ and $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$. Then $\mathcal{F}_i/Q = \mathcal{F}_{S_i/Q}(N_{G_i}(Q)/Q)$ by Lemma 1.17, where $S_i/Q \in \mathrm{Syl}_p(N_{G_i}(Q)/Q)$, so $\mathcal{F}_i/Q$ is realizable. Also, $\mathcal{F}/Q = \bigcup_{i=1}^{\infty} \mathcal{F}_i/Q$, so $\mathcal{F}/Q$ is sequentially realizable.

If $Q \le T$ where $T$ is the identity component of $S$, then by definition and since $Q$ is weakly closed in $\mathcal{F}$, $\mathrm{Aut}_{\mathcal{F}/Q}(T/Q)$ is the image of the natural homomorphism $\chi\colon \mathrm{Aut}_{\mathcal{F}}(T) \longrightarrow \mathrm{Aut}(T/Q)$. Let $m$ be such that $Q$ has exponent $p^m$. If $\alpha \in \mathrm{Ker}(\chi)$, then $\alpha(t) \in tQ$ for each $t \in T$, so $\alpha(t^{p^m}) = t^{p^m}$, and $\alpha = \mathrm{Id}_T$ since $T$ is infinitely divisible. Thus $\mathrm{Ker}(\chi) = 1$, and so $\mathrm{Aut}_{\mathcal{F}/Q}(T/Q) \cong \mathrm{Aut}_{\mathcal{F}}(T)$ and $C_{S/Q}(T/Q) = C_S(T)/Q$. □

The following elementary lemma will also be needed.

**Lemma 7.2.** *Let $T$ be a discrete $p$-torus, and let $G \le \mathrm{Aut}(T)$ be a finite group of automorphisms of $T$. Set $\ell = v_p(|G|)$; thus $p^\ell \mid |G|$ but $p^{\ell+1} \nmid |G|$. Then either $C_T(G)$ is infinite, or it has exponent at most $p^\ell$.*

*Proof.* Set $P_1 = C_T(G)$ for short, and set

$$P_2 = \bigl\{x \in T \bigm| \textstyle\prod_{g\in G} g(x) = 1\bigr\}.$$

For each $x \in T$, we have $\prod_{g\in G} g(x) \in P_1$. Also, $x(g(x))^{-1} \in P_2$ for each $g \in G$ since $\prod_{h\in G} h(x) \cdot \prod_{h\in G}(hg(x))^{-1} = 1$. So

$$x^{|G|} = \bigl(\textstyle\prod_{g\in G} gx\bigr) \cdot \bigl(\prod_{g\in G} x(gx)^{-1}\bigr) \in P_1P_2,$$

and $T = P_1P_2$ since it is infinitely divisible.

If $P_1$ is finite, then $P_2$ has finite index in $T$, and hence $P_2 = T$, again since it is infinitely divisible. Hence for each $x \in P_1 = C_T(G)$, we have $x^{|G|} = \prod_{g\in G} gx = 1$ by definition of $P_2$, and so $x^{p^\ell} = 1$ since $T$ has no nonidentity elements of order prime to $p$. □

We next recall the definition of the *generalized Fitting subgroup* $F^*(G)$ of a finite group $G$. A subgroup $H \le G$ is subnormal (denoted $H \trianglelefteq\trianglelefteq G$) if there is a sequence of subgroups $H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$ each normal in the following one. A component of $G$ is a subnormal subgroup $C \trianglelefteq\trianglelefteq G$ that is quasisimple (i.e., $C$ is perfect and $C/Z(C)$ is simple). All components of $G$ commute with each other and with the subgroups $O_p(G)$ for all primes $p$. Thus $F^*(G)$, the subgroup generated by all components of $G$ and all subgroups $O_p(G)$ for primes $p$, is a central product of these groups. One of the important properties of $F^*(G)$ is that it is centric in $G$; i.e., $C_G(F^*(G)) \le F^*(G)$. We refer to [A2, §31] for more details and proofs of these statements, and also to [AKO, A.11–A.13] for a shorter summary.

**Lemma 7.3.** *Let $\mathcal{F}$ be a saturated fusion system over an infinite discrete $p$-toral group $S$, and let $T \trianglelefteq S$ be the identity component. Assume*

- (i) *$S > T$ and $C_S(T) = T$, and*
- (ii) *$\mathfrak{minsc}(\mathcal{F}) = S$.*

*Assume also that $\mathcal{F}$ is sequentially realizable, and let $\mathcal{F}_1 \le \mathcal{F}_2 \le \mathcal{F}_3 \le \cdots$ be realizable fusion subsystems of $\mathcal{F}$ over finite subgroups $S_1 \le S_2 \le \cdots$ of $S$ such that $S = \bigcup_{i=1}^\infty S_i$ and $\mathcal{F} = \bigcup_{j=1}^\infty \mathcal{F}_i$. If $G_1, G_2, G_3, \ldots$ are finite groups such that for each $i \ge 1$,*

$$S_i \in \mathrm{Syl}_p(G_i), \qquad O_{p'}(G_i) = 1, \qquad \text{and} \qquad \mathcal{F}_i = \mathcal{F}_{S_i}(G_i),$$

*then for $n$ large enough, the generalized Fitting subgroup $F^*(G_n)$ is a product of nonabelian simple groups of Lie type in characteristic different from $p$ that are pairwise conjugate in $G_n$.*

*Proof.* Set $T_i = T \cap S_i$ and $Q_i = \mathfrak{minsc}(\mathcal{F}_i) \le S_i$ for each $i$. Then $\bigcup_{i=1}^\infty Q_i = \mathfrak{minsc}(\mathcal{F}) = S$ by Lemma 5.4 and (ii). So we can choose $m \ge 1$ such that for all $n \ge m$, we have

$$TQ_n = S, \qquad Q_n \ge \Omega_5(T), \qquad \text{and} \qquad C_S(Q_n \cap T) = C_S(T) = T. \tag{7-1}$$

For the rest of the proof, we fix $n$ such that $n \geq m$ and hence (7-1) holds. If $O_p(G_n) \neq 1$, then $\Omega_1(Z(O_p(G_n)))$ is nontrivial and strongly closed in $\mathcal{F}_n$, hence contains $Q_n = \mathfrak{minsc}(\mathcal{F}_n)$, contradicting (7-1). Thus $O_p(G_n) = 1$, and since $O_{p'}(G_n) = 1$ by assumption, the generalized Fitting subgroup $F^*(G_n)$ is a product of nonabelian simple groups. Set $H = F^*(G_n)$ and let $H_1, \ldots, H_\ell \leq H$ be its simple factors; thus

$$H = F^*(G_n) = H_1 \times \cdots \times H_\ell.$$

Let $\mathrm{pr}_j \colon H \longrightarrow H_j$ be projection to the $j$-th factor.

Set $U = H \cap S_n = H \cap S \in \mathrm{Syl}_p(H)$, and set $U_j = U \cap H_j \in \mathrm{Syl}_p(H_j)$ for each $j$. Thus $U = U_1 \times \cdots \times U_\ell$. Also, $U_j \neq 1$ for each $j$ since otherwise $H_j \leq O_{p'}(G_n) = 1$. Finally, $U \geq Q_n = \mathfrak{minsc}(\mathcal{F}_n)$ since $U$ is strongly closed in $\mathcal{F}_n$.

Set $A = U \cap T = H \cap T$, and set $A_j = U_j \cap T = H_j \cap T$ for each $1 \leq j \leq \ell$. Then $C_U(A) \leq U \cap C_S(Q_n \cap T) = A$ by (7-1), so $A$ is a maximal abelian subgroup of $U$. Since $A \leq \mathrm{pr}_1(A) \times \cdots \times \mathrm{pr}_\ell(A) \leq U$, this implies that $A_j = \mathrm{pr}_j(A)$ for each $j$, and hence that $A = A_1 \times \cdots \times A_\ell$. If $A_j = 1$ for some $j$, then $U_j \leq C_U(A) = A$, so $U_j = 1$ which we just saw is impossible. Thus each factor $A_j$ is nontrivial.

For each $j$, since $1 \neq A_j \leq T$, we have $\Omega_1(T) \cap A_j \neq 1$. If the simple factors $H_j$ are not permuted transitively by $\mathrm{Aut}_{G_n}(H)$, then there is a nonempty proper subset $J \subsetneqq \{1, \ldots, \ell\}$ such that $\langle H_j \,|\, j \in J \rangle \trianglelefteq G_n$. Then $\langle U_j \,|\, j \in J \rangle \in \mathcal{F}_n^{\mathfrak{sc}}$, hence contains $Q_n$ but does not contain $\Omega_1(T)$. This contradicts (7-1), and we conclude that the $H_j$ are permuted transitively by $\mathrm{Aut}_{G_n}(H)$ and hence are pairwise conjugate in $G_n$.

Now,

- $A \trianglelefteq U$ since $T \trianglelefteq S$ and $A = U \cap T$;
- $\Omega_5(A)$ is homocyclic of exponent $p^5$ since $\Omega_5(T) \leq Q_n \leq U$ by (7-1) and $A = T \cap U$; and
- $C_U(\Omega_2(A)) = A$ by Lemma 6.8 and since $\mathrm{Aut}_U(A)$ is the restriction of an action on $T$.

Hence for each $j = 1, \ldots, \ell$, we have $A_j \trianglelefteq U_j$, $\Omega_5(A_j)$ is homocyclic of exponent $p^5$, and $C_{U_j}(\Omega_2(A_j)) = A_j$. So $A_j$ is a large abelian $p$-subgroup of $H_j$, and by Corollary 6.11 and the classification of finite simple groups, each simple factor $H_j$ is of Lie type in characteristic different from $p$. □

We are now ready to show that under certain hypotheses on a fusion system $\mathcal{F}$, if it is sequentially realizable, then the automizer of a maximal torus contains one of the groups $W$ that appears in cases (a)–(e) of Proposition 6.10 as a normal subgroup of index prime to $p$.

**Theorem 7.4.** *Let $\mathcal{F}$ be a saturated fusion system over an infinite discrete $p$-toral group $S$, and let $T \trianglelefteq S$ be the identity component. Assume*

(i) *$S > T$ and $C_S(T) = T$;*

(ii) *no proper subgroup $R < S$ containing $T$ is strongly closed in $\mathcal{F}$; and*

(iii) *no infinite proper subgroup of $T$ is invariant under the action of $\mathrm{Aut}_{\mathcal{F}}(T)$.*

*Assume also that $\mathcal{F}$ is sequentially realizable. Set $k = 2$ if $p = 2$, or $k = 1$ if $p$ is odd. Then there is a normal subgroup $H \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(T)$ of index prime to $p$ such that for some $\ell \geq 1$, $(H, \Omega_k(T)) \cong (W^\ell, (M')^{\oplus \ell})$, where $(W, M)$ is one of the pairs listed in Proposition 6.10(a–e) and $M'$ is an $\mathbb{Z}/p^k W$-module with the same composition factors as $M$. If $\ell > 1$, then the $\ell$ factors are permuted transitively by $\mathrm{Aut}_{\mathcal{F}}(T)$. If* (i) *and* (ii) *hold and no infinite proper subgroup of $T$ is invariant under the action of $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(T))$, then this conclusion holds with $\ell = 1$.*

*Proof.* Let $Q \le S$ be the subgroup generated by all proper subgroups of $S$ strongly closed in $\mathcal{F}$. By Lemma 5.6(b) and points (i)–(iii), $Q$ is finite, is contained in $T$, is strongly closed in $\mathcal{F}$, and $\mathfrak{minsc}(\mathcal{F}/Q) = S/Q$. So by Lemma 7.1 and since $\mathcal{F}$ is sequentially realizable, $\mathcal{F}/Q$ is also sequentially realizable, $C_{S/Q}(T/Q) = C_S(T)/Q = T/Q$, and $\mathrm{Aut}_{\mathcal{F}/Q}(T/Q) \cong \mathrm{Aut}_{\mathcal{F}}(T)$. Also, $\mathcal{F}/Q$ is saturated by Lemma 1.18. We claim that

$$\Omega_k(T) \text{ and } \Omega_k(T/Q) \text{ have isomorphic composition factors as } \mathbb{Z}/p^k\mathrm{Aut}_{\mathcal{F}}(T)\text{-modules.} \tag{7-2}$$

Then upon replacing $\mathcal{F}$ by $\mathcal{F}/Q$, we can assume without loss of generality that $\mathfrak{minsc}(\mathcal{F}) = S$.

To prove (7-2), first note that for each $i \ge 0$, there is an exact sequence

$$0 \to \Omega_{i+1}(Q)/\Omega_i(Q) \longrightarrow \Omega_k(T/\Omega_i(Q)) \longrightarrow \Omega_k(T/\Omega_{i+1}(Q)) \xrightarrow{\rho_i} \Omega_{i+1}(Q)/\Omega_i(Q) \to 0,$$

where $\rho_i(x\Omega_{i+1}(Q)) = p^k x\Omega_i(Q)$. (Here, $\Omega_0(T) = 1$.) So $\Omega_k(T/\Omega_i(Q))$ and $\Omega_k(T/\Omega_{i+1}(Q))$ have isomorphic composition factors for each $i \ge 0$ by the Jordan-Hölder theorem, and since $Q = \Omega_i(Q)$ for $i$ large enough, this implies that $\Omega_k(T)$ and $\Omega_k(T/Q)$ have isomorphic composition factors.

Now assume that $\mathfrak{minsc}(\mathcal{F}) = S$. Fix realizable fusion subsystems $\mathcal{F}_1 \le \mathcal{F}_2 \le \dots$ of $\mathcal{F}$ over finite subgroups $S_1 \le S_2 \le \dots$ of $S$ such that $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ and $S = \bigcup_{i=1}^{\infty} S_i$. Let $G_1, G_2, \dots$ be finite groups such that for each $i \ge 1$, $O_{p'}(G_i) = 1$, $S_i \in \mathrm{Syl}_p(G_i)$, and $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$. Set $F_i^* = F^*(G_i)$, $U_i = F_i^* \cap S_i \in \mathrm{Syl}_p(F_i^*)$, $T_i = T \cap S_i$, and $A_i = T \cap U_i$. Also, set $Q_i = \mathfrak{minsc}(\mathcal{F}_i) \le S_i$ for each $i$. Note that $U_i \ge Q_i$ for each $i$ since $U_i$ is strongly closed in $\mathcal{F}_i$.

We now fix $n_0 \ge 1$ such that

(1) for all $i \ge n_0$, $F_i^*$ is a product of nonabelian simple groups of Lie type in characteristic different from $p$ that are pairwise conjugate in $G_i$;

(2) $Q_i \le Q_{i+1}$ for all $i \ge n_0$, and $\bigcup_{i=n_0}^{\infty} Q_i = \mathfrak{minsc}(\mathcal{F}) = S$;

(3) $Q_{n_0}T = S$ and $Q_{n_0} \ge \Omega_5(T)$;

(4) for all $i \ge n_0$, the subgroup $T_i$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant; and

(5) $\mathrm{Aut}_{G_i}(A) = \mathrm{Aut}_{\mathcal{F}}(A) \cong \mathrm{Aut}_{\mathcal{F}}(T)$ for each $i \ge n_0$ and each $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant subgroup $A \le T_i$ containing $T_{n_0}$, where the isomorphism $\mathrm{Aut}_{\mathcal{F}}(T) \xrightarrow{\cong} \mathrm{Aut}_{\mathcal{F}}(A)$ is induced by restricting from $T$ to $A$.

Point (1) holds for $n_0$ large enough by Lemma 7.3, point (2) by Lemma 5.4, point (3) as a consequence of (2), and points (4) and (5) by Lemma 5.2.

For each $i \ge n_0$, the subgroup $T_i = T \cap G_i$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant by (4), and hence $\mathrm{Aut}_{G_i}(T_i) = \mathrm{Aut}_{\mathcal{F}}(T_i) \cong \mathrm{Aut}_{\mathcal{F}}(T)$ by (5). Since $F_i^* \trianglelefteq G_i$, the subgroup $A_i = F_i^* \cap T_i$ is $\mathrm{Aut}_{G_i}(T_i)$-invariant, and hence is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant. By (2), we can choose $n \ge n_0$ so that $Q_n \ge T_{n_0}$; then $A_i \ge T_{n_0}$ for all $i \ge n$, and so $\mathrm{Aut}_{G_i}(A_i) = \mathrm{Aut}_{\mathcal{F}}(A_i) \cong \mathrm{Aut}_{\mathcal{F}}(T)$ by (5).

Now assume $i \ge n \ge n_0$. Thus $U_iT = S$ by (3) and since $U_i \ge Q_i \ge Q_0$. So for each $g \in N_{S_i}(A_i)$, there is $h \in U_i \cap gT$, and hence $c_g = c_h \in \mathrm{Aut}(A_i)$. Thus $\mathrm{Aut}_{S_i}(A_i) \le \mathrm{Aut}_{F_i^*}(A_i)$. Since $\mathrm{Aut}_{S_i}(A_i) \in \mathrm{Syl}_p(\mathrm{Aut}_{G_i}(A_i))$, this proves that $\mathrm{Aut}_{F_i^*}(A_i)$ is normal of index prime to $p$ in $\mathrm{Aut}_{G_i}(A_i)$. Set $H_i = \mathrm{Aut}_{F_i^*}(A_i)$, and let $H \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(T)$ be the corresponding normal subgroup of index prime to $p$ under the isomorphism $\mathrm{Aut}_{G_i}(A_i) \cong \mathrm{Aut}_{\mathcal{F}}(T)$. Thus

$$H = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(T) \,|\, \alpha|_{A_i} \in H_i\} = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(T) \,|\, \alpha|_{A_i} \in \mathrm{Aut}_{F_i^*}(A_i)\}.$$

Still assuming $i \ge n$, since $U_i \ge Q_i \ge Q_{n_0} \ge \Omega_5(T)$, we have $A_i = U_i \cap T \ge \Omega_5(T)$. So $C_{U_i}(A_i) \le C_S(\Omega_2(T)) = T$ by Lemma 6.8, and hence $C_{U_i}(A_i) = U_i \cap T = A_i$. Thus $U_i$ is nonabelian ($U_i > A_i$ since $U_iT = S$), and $A_i$ is a maximal abelian subgroup of $U_i$. Also,

$\Omega_5(A_i) = \Omega_5(T)$ is homocyclic of exponent $p^5$, and $F_i^*$ is a product of simple groups of Lie type in characteristic different from $p$ permuted transitively by $G_i$ by (1). So $A_i$ is a large abelian $p$-subgroup of $F_i^*$, and each of the simple factors of $F_i^*$ has some direct factor of $A_i$ as a large abelian $p$-subgroup. By Proposition 6.10, for one of the pairs $(W, M)$ listed in 6.10(a)–(e), we have $(H, \Omega_k(T)) \cong (H_i, \Omega_k(A_i)) \cong (W^\ell, M^{\oplus\ell})$, where $\ell$ is the number of simple factors in $F_i^*$. We have already seen that the $\ell$ factors are permuted transitively by $\mathrm{Aut}_{G_i}(A_i) \cong \mathrm{Aut}_{\mathcal{F}}(T)$.

It remains to prove the last statement. Assume there is no infinite proper subgroup of $T$ that is $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(T))$-invariant; we must show that $\ell = 1$. Assume otherwise: assume $\ell \geq 2$. Then $\mathrm{Aut}_{F_i^*}(A_i) \geq O^{p'}(\mathrm{Aut}_{G_i}(A_i))$, and we can write $\mathrm{Aut}_{\mathcal{F}}(T) = W_1 \times \cdots \times W_\ell$ and $A_i = M_1 \times \cdots \times M_\ell$, where $W_j \cong W_1$ and $M_j \cong M_1$ for all $2 \leq j \leq \ell$, and where $W_j$ acts trivially on $M_t$ for $t \neq j$. We can assume that $i$ was chosen so that $A_i \geq \Omega_m(T)$ where $m = v_p(|\mathrm{Aut}_{\mathcal{F}}(T)|)$; in particular, so that each $M_j$ has exponent at least $p^m$. Thus $M_2 \leq C_T(W_1) < T$, so $C_T(W_1)$ has exponent at least equal to $|\mathrm{Aut}_{\mathcal{F}}(T)| > |W_1|$, and hence is infinite by Lemma 7.2. Thus $C_T(W_1)$ is an infinite proper subgroup of $T$ invariant under the action of $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(T))$, contradicting our assumption. □

We now show that a fusion system that satisfies the same hypotheses (i)–(iii) as in Theorem 7.4 cannot be realized by a linear torsion group in characteristic zero.

Recall that $\mathrm{rk}_q(G)$ denotes the $q$-rank of a group $G$: the least upper bound for ranks of finite abelian $q$-subgroups of $G$.

**Proposition 7.5.** *Let $\mathcal{F}$ be a saturated fusion system over an infinite discrete $p$-toral group $S$, and let $T \trianglelefteq S$ be the identity component. Assume*

(i) *$S > T$ and $C_S(T) = T$;*

(ii) *no proper subgroup $R < S$ containing $T$ is strongly closed in $\mathcal{F}$; and*

(iii) *no infinite proper subgroup of $T$ is invariant under the action of $\mathrm{Aut}_{\mathcal{F}}(T)$.*

*Then if $\mathcal{F} = \mathcal{F}_S(G)$ for some locally finite group $G$ with $S \in \mathrm{Syl}_p(G)$, there is a prime $r \neq p$ such that $\mathrm{srk}_r(G) = \infty$. In particular, $\mathcal{F}$ is not isomorphic to the fusion system of any linear torsion group in characteristic 0.*

*Proof.* Assume $\mathcal{F} = \mathcal{F}_S(G)$, where $G$ is a locally finite group with $S \in \mathrm{Syl}_p(G)$. We will show that $\mathrm{srk}_r(G) = \infty$ for some prime $r \neq p$. It then follows from Lemma 3.6(a) that $G$ cannot be a subgroup of $GL_N(K)$ for any $N \geq 1$ and any field $K$ of characteristic 0.

Let $Q \trianglelefteq S$ be the subgroup generated by all proper subgroups of $S$ strongly closed in $\mathcal{F}$. By Lemma 5.6(b) and (i)–(iii), the subgroup $Q$ is finite, $Q \leq T$, $Q$ is strongly closed in $\mathcal{F}$, and $\mathfrak{minsc}(\mathcal{F}/Q) = S/Q$. By Lemma 1.17, we have $\mathcal{F}/Q = \mathcal{F}_{S/Q}(N_G(Q)/Q)$, where $S/Q \in \mathrm{Syl}_p(N_G(Q)/Q)$ since $Q$ is strongly closed. Also, $\mathcal{F}/Q$ is saturated by Lemma 1.18, and $C_{S/Q}(T/Q) = C_S(T)/Q = T/Q$ and $\mathrm{Aut}_{\mathcal{F}/Q}(T/Q) \cong \mathrm{Aut}_{\mathcal{F}}(T)$ by Lemma 7.1. So conditions (i)–(iii) also hold for $\mathcal{F}/Q$. Since $\mathrm{srk}_q(G) \geq \mathrm{srk}_q(N_G(Q)/Q)$ for every prime $q \neq p$, we can replace $\mathcal{F}$ by $\mathcal{F}/Q$, and reduce to the case where $\mathfrak{minsc}(\mathcal{F}) = S$.

By Proposition 2.4, there is an increasing sequence $G_1 \leq G_2 \leq \cdots$ of finite subgroups of $G$ such that if we set $S_i = S \cap G_i$ and $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$ for each $i$, then $S_i \in \mathrm{Syl}_p(G_i)$ for each $i$, and $S = \bigcup_{i=1}^\infty S_i$ and $\mathcal{F} = \bigcup_{i=1}^\infty \mathcal{F}_i$. Also, $\mathcal{F} = \mathcal{F}_S\bigl(\bigcup_{i=1}^\infty G_i\bigr)$ by the same proposition, so we can assume that $G$ is the union of the $G_i$.

For each $i$, set $G_i^* = G_i/O_{p'}(G_i)$. For all $j \geq i$, we have $O_{p'}(G_j) \cap G_i \trianglelefteq G_i$, and hence $O_{p'}(G_j) \cap G_i \leq O_{p'}(G_i)$. So $G_i^*$ is isomorphic to a section (subquotient) of $G_j^*$ whenever $i < j$.

For each $i$, set $H_i^* = F^*(G_i^*)$: the generalized Fitting subgroup of $G_i^*$. By Lemma 7.3, there is $n \geq 1$ such that for all $i \geq n$, $H_i^*$ is a product of simple groups of Lie type in characteristic different from $p$, where the simple factors are permuted transitively by $\mathrm{Aut}_{G_i}(H_i^*)$. Assume, for each $i \geq n$, that $H_i^*$ is isomorphic to a product of $\ell_i$ copies of the simple group ${}^{d_i}\mathbb{G}_i(q_i)$, where $q_i$ is a power of a prime $r_i$, where $\mathbb{G}_i$ is a simple group scheme over $\mathbb{Z}$, and where ${}^{d_i}\mathbb{G}_i(-)$ means the group $\mathbb{G}_i(-)$ twisted by a graph automorphism of order $d_i = 1, 2, 3$. Thus $\ell_i \cdot \mathrm{rk}({}^{d_i}\mathbb{G}_i) = \mathrm{rk}(T)$ for each $i$, where $\mathrm{rk}({}^{d_i}\mathbb{G}_i)$ is its Lie rank. Since there are only finitely many possibilities for ${}^{d_i}\mathbb{G}_i$ of any given Lie rank, there must be some pair $({}^{d_i}\mathbb{G}_i, \ell_i)$ that occurs for infinitely many indices $i \geq n$. So upon removing the other terms in the sequence, we can assume there are $\ell$, $\mathbb{G}$, and $d = 1, 2, 3$ such that $H_i^*$ is a product of $\ell$ copies of ${}^{d}\mathbb{G}(q_i)$ for each $i$. Let $\overline{W}$ be the Weyl group of $\mathbb{G}$.

We claim that

$$\text{there is a prime } r \text{ such that } r_i = r \text{ for infinitely many } i \geq n. \tag{7-3}$$

Once we have shown this, it then follows that the set $\{q_i \,|\, i \geq n\}$ contains arbitrarily large powers of $r$. But the group ${}^{d}\mathbb{G}(q_i)$ always contains a subgroup isomorphic to $(\mathbb{F}_{q_i}, +)$ (a subgroup of a root subgroup), and hence the set $\{\mathrm{rk}_r(G_i^*)\}$ of $r$-ranks of the $G_i^*$ is unbounded. So $\mathrm{srk}_r(G) = \infty$, which is what we needed to show.

To prove (7-3), let $\widehat{H}_i^* \trianglelefteq G_i^*$ be the subgroup of elements that normalize each simple factor in $H_i^*$. If (7-3) does not hold, then in particular, there are only finitely many terms for which $r_i \mid |\overline{W}|$ or $r_i \leq \mathrm{rk}(T)$. Upon removing those terms, we have $r_i \nmid |\overline{W}|$ and $r_i \nmid |G_j^*/\widehat{H}_j^*|$ for all $i, j \geq n$. (Note that $|G_j^*/\widehat{H}_j^*|$ divides $\mathrm{rk}(T)!$ since $G_j^*/\widehat{H}_j^*$ acts faithfully on the set of simple factors of $H_j^*$, and the number of factors is at most $\mathrm{rk}(T)$.)

For each pair $i < j$, we already saw that $G_i^*$, and hence each simple factor of $H_i^*$, is isomorphic to a section (subquotient) of $G_j^*$. Hence each simple factor is isomorphic to a section of $H_j^*$, $\widehat{H}_j^*/H_j^*$, or $G_j^*/\widehat{H}_j^*$: the second is impossible since $\widehat{H}_j^*/H_j^*$ is solvable (since the outer automorphism group of each simple factor is solvable), and the last since $r_i \nmid |G_j^*/\widehat{H}_j^*|$. Hence for each $i < j$, each simple factor of $H_i^*$ is isomorphic to a section of a simple factor of $H_j^*$.

By [GLS3, Theorem 4.10.2] and since $r_i \nmid |\overline{W}|$, the Sylow $r_i$-subgroups of $H_j^*$ are abelian for all $j > i$ such that $r_j \neq r_i$. Hence the Sylow $r_i$-subgroup of $H_i^*$ is abelian for all $i$, and $H_i^*$ is a product of copies of $PSL_2(q_i)$. So $PSL_2(q_i)$ is isomorphic to a subquotient of $PSL_2(q_j)$ for all $i < j$, which for $q_i \geq 7$ is possible only when $r_i = r_j$ (see, e.g., [GLS3, Theorem 6.5.1]). □

Proposition 7.5 will be applied in the next section to fusion systems of connected $p$-compact groups, and in particular, of connected compact Lie groups.

## 8. Fusion systems of $p$-compact groups

A $p$-compact group consists of a triple $(X, BX, i)$, where $X$ and $BX$ are spaces and $i\colon X \longrightarrow \Omega(BX)$ is a homotopy equivalence, and such that $BX$ is $p$-complete in the sense of Bousfield and Kan [BK] and $H^*(X; \mathbb{F}_p)$ is finite. Usually, we just say that $X$ is a $p$-compact group and $BX$ is its classifying space. For example, if $G$ is a compact Lie group such that $\pi_0(G)$ is a $p$-group, then $G_p^\wedge \simeq \Omega(BG_p^\wedge)$ is a $p$-compact group with classifying space $BG_p^\wedge$.

A $p$-subgroup of a $p$-compact group $X$ is a pair $(P, f)$, where $P$ is a discrete $p$-toral group and $f\colon BP \longrightarrow BX$ is a pointed map that does not factor (up to homotopy) through any quotient group $P/Q$ for $1 \neq Q \trianglelefteq P$. (By [DW2, Theorems 7.2–7.3], this is equivalent to the definition of a $p$-subgroup of $X$ given in [DW2, § 3] and used in [BLO3, § 10].) By a theorem of Dwyer and Wilkerson (see [DW4, Propositions 2.10 and 2.14] or [BLO3, Proposition 10.1(a)]), every $p$-compact group $X$ contains a Sylow $p$-subgroup: a $p$-subgroup $(S,f)$ of $X$ such that every other $p$-subgroup of $X$ factors through $f$ up to homotopy.

Let $X$ be a $p$-compact group with $(S,f)\in\mathrm{Syl}_p(X)$. In [BLO3, Definition 10.2], we give an explicit construction of a fusion system $\mathcal{F}_{S,f}(X)$ over $S$, defined via maps to $BX$. If $T$ is the identity component of $S$, we refer to $T$ (or $(T,f|_{BT})$) as the maximal torus of $X$ and to $\mathrm{Aut}_{\mathcal{F}_{S,f}(X)}(T)$ as its Weyl group.

Recall that $\mathbb{Z}_p$ denotes the ring of $p$-adic integers and $\mathbb{Q}_p=\mathbb{Z}_p[1/p]$ its field of fractions. Note that $\mathbb{Q}_p/\mathbb{Z}_p\cong\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}\cong\mathbb{Z}/p^\infty$.

**Lemma 8.1.** *Let $T$ be a discrete $p$-torus of rank $r$. Then*

(a) $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p,T)\cong(\mathbb{Z}_p)^r$ *and* $\mathrm{Hom}(\mathbb{Q}_p,T)\cong(\mathbb{Q}_p)^r$*, and the inclusion of* $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p,T)$ *into* $\mathrm{Hom}(\mathbb{Q}_p,T)$ *induces an isomorphism* $\mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p,T)\cong\mathrm{Hom}(\mathbb{Q}_p,T)$*; and*

(b) *there is a natural isomorphism of* $\mathbb{Z}_p\mathrm{Aut}(T)$*-modules* $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p,T)\xrightarrow{\cong}\pi_2(BT^\wedge_p)$.

*Proof.* **(a)** It suffices to prove this when $r=1$; i.e., when $T=\mathbb{Q}_p/\mathbb{Z}_p$. Consider the homomorphisms

$$\mathbb{Q}_p\xrightarrow{\ \psi\ }\mathrm{Hom}(\mathbb{Q}_p,\mathbb{Q}_p/\mathbb{Z}_p)\xrightarrow{\ \mathrm{ev}_1\ }\mathbb{Q}_p/\mathbb{Z}_p$$

defined by setting $\psi(x)(y)=xy+\mathbb{Z}_p$ and $\mathrm{ev}_1(\rho)=\rho(1)$. One easily seens that $\psi$ is injective, and $\mathrm{ev}_1$ is surjective with kernel $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p,\mathbb{Q}_p/\mathbb{Z}_p)$. If $\rho\in\mathrm{Hom}(\mathbb{Q}_p,\mathbb{Q}_p/\mathbb{Z}_p)$, then for each $n\ge1$, we can choose elements $x_n\in\mathbb{Q}_p$ such that $\rho(p^{-n})=x_n+\mathbb{Z}_p$; the sequence $\{p^nx_n\}_{n\ge0}$ converges in the $p$-adic topology to some $x\in\mathbb{Q}_p$, and $\rho=\psi(x)$. So $\psi$ is surjective.

Thus $\psi$ is an isomorphism, and $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p,\mathbb{Q}_p/\mathbb{Z}_p)=\mathrm{Ker}(\mathrm{ev}_1)\cong\mathrm{Ker}(\mathrm{ev}_1\circ\psi)$. Since $\mathrm{ev}_1\circ\psi$ is the natural surjection of $\mathbb{Q}_p$ onto $\mathbb{Q}_p/\mathbb{Z}_p$, this proves that

$$\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p,\mathbb{Q}_p/\mathbb{Z}_p)\cong\mathbb{Z}_p\qquad\text{and}\qquad\mathrm{Hom}(\mathbb{Q}_p,\mathbb{Q}_p/\mathbb{Z}_p)\cong\mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p,\mathbb{Q}_p/\mathbb{Z}_p).$$

**(b)** This is a special case of [BK, VI.5.1], applied with $X=BT$. □

In the next lemma, we use these $\mathbb{Q}_p$-vector spaces $\mathrm{Hom}(\mathbb{Q}_p,T)$ to provide a useful criterion when verifying condition (iii) in Theorem 7.4.

**Lemma 8.2.** *Let $T$ be a discrete $p$-torus, and let $G$ be a finite group of automorphisms of $T$. Then there is an infinite proper $G$-invariant subgroup $T_0<T$ if and only if the $\mathbb{Q}_pG$-module $\mathrm{Hom}(\mathbb{Q}_p,T)$ is reducible.*

*Proof.* Set $V=\mathrm{Hom}(\mathbb{Q}_p,T)$ for short, regarded as a $\mathbb{Q}_pG$-module. Assume $P<T$ is an infinite proper $G$-invariant subgroup of $T$, and let $T_0\le P$ be its identity component. Then $T_0$ is an infinite discrete $p$-torus and is also $G$-invariant. Set $V_0=\mathrm{Hom}(\mathbb{Q}_p,T_0)\le V$. Then $0<\mathrm{rk}(T_0)<\mathrm{rk}(T)$ implies that $0<\dim_{\mathbb{Q}_p}(V_0)<\dim_{\mathbb{Q}_p}(V)$, so $V_0$ is a proper nontrivial $\mathbb{Q}_pG$-submodule of $V$, and $V$ is reducible.

Conversely, assume $V$ is reducible, and let $V_0<V$ be a proper nontrivial submodule. Let $\mathrm{ev}_1\colon V=\mathrm{Hom}(\mathbb{Q}_p,T)\longrightarrow T$ be the homomorphism $\mathrm{ev}_1(v)=v(1)$, and set $T_0=\mathrm{ev}_1(V_0)$. The image under $\mathrm{ev}_1$ of each 1-dimensional subspace of $V$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p\cong\mathbb{Z}/p^\infty$, and from this it follows that $T_0$ is a discrete $p$-torus where $\mathrm{rk}(T_0)=\dim_{\mathbb{Q}_p}(V_0)<\dim_{\mathbb{Q}_p}(V)=\mathrm{rk}(T)$. Thus $T_0$ is an infinite, $G$-invariant, proper subgroup of $T$. □

We showed in [BLO3, Theorem 10.7] that if $X$ is a $p$-compact group and $(S,f)\in\mathrm{Syl}_p(X)$, then $\mathcal{F}_{S,f}(X)$ is saturated, it has an associated centric linking system $\mathcal{L}^c_{S,f}(X)$, and $|\mathcal{L}^c_{S,f}(X)|^\wedge_p \simeq BX$. We also showed that the fusion system of $X$ is determined by the homotopy type of $BX$, as is made explicit in the following lemma.

**Lemma 8.3.** *Let $X$ and $Y$ be $p$-compact groups. If $BX \simeq BY$, then for $(S,f) \in \mathrm{Syl}_p(X)$ and $(U,g) \in \mathrm{Syl}_p(Y)$ we have $\mathcal{F}_{S,f}(X) \cong \mathcal{F}_{U,g}(Y)$. If $p$ is odd and $X$ and $Y$ are connected, then this is the case if the Weyl groups of $X$ and $Y$ are isomorphic and have isomorphic actions on their maximal tori.*

*Proof.* By [BLO3, Theorem 10.7], there are centric linking systems $\mathcal{L}^c_{S,f}(X)$ and $\mathcal{L}^c_{U,g}(Y)$, associated to $\mathcal{F}_{S,f}(X)$ and $\mathcal{F}_{U,g}(Y)$, such that

$$|\mathcal{L}^c_{S,f}(X)|^\wedge_p \simeq BX^\wedge_p \simeq BY^\wedge_p \simeq |\mathcal{L}^c_{U,g}(Y)|^\wedge_p.$$

Hence $\mathcal{F}_{S,f}(X) \cong \mathcal{F}_{U,g}(Y)$ by [BLO3, Theorem 7.4]. This proves the first statement. The second follows from [AGMV, Theorem 1.1], together with Lemma 8.1(b) which says that the actions of the Weyl groups on $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, T_X)$ and $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, T_Y)$, where $T_X$ and $T_Y$ are maximal discrete $p$-tori in $X$ and $Y$, are isomorphic to the actions on $L_X$ and $L_Y$ used in [AGMV]. □

The next lemma is similar, and is our means of showing that the fusion systems of certain $p$-compact groups are realized by linear torsion groups.

**Lemma 8.4.** *Fix a prime $p$, a $p$-compact group $X$, and a linear torsion group $\Gamma$ in characteristic different from $p$ such that $BX \simeq B\Gamma^\wedge_p$. Then for $(S,f) \in \mathrm{Syl}_p(X)$ and $S_\Gamma \in \mathrm{Syl}_p(\Gamma)$, we have $\mathcal{F}_{S,f}(X) \cong \mathcal{F}_{S_\Gamma}(\Gamma)$.*

*Proof.* By [BLO3, Theorems 8.10 and 10.7], there are centric linking systems $\mathcal{L}^c_{S,f}(X)$ and $\mathcal{L}^c_{S_\Gamma}(\Gamma)$, associated to $\mathcal{F}_{S,f}(X)$ and $\mathcal{F}_{S_\Gamma}(\Gamma)$, respectively, such that

$$|\mathcal{L}^c_{S,f}(X)|^\wedge_p \simeq BX^\wedge_p \simeq B\Gamma^\wedge_p \simeq |\mathcal{L}^c_{S_\Gamma}(\Gamma)|^\wedge_p.$$

So $\mathcal{F}_{S,f}(X) \cong \mathcal{F}_{S^*}(\Gamma)$ by [BLO3, Theorem 7.4]. □

The next lemma proves some of the properties of fusion systems of $p$-compact groups needed to apply Theorem 7.4 and similar results. Recall that a connected $p$-compact group $X$ with maximal torus $T$ and Weyl group $W$ is *simple* if $Z(X) = 1$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \pi_2(BT^\wedge_p)$ is irreducible as a $\mathbb{Q}_pW$-module (see [DW3, Definition 1.2]).

**Lemma 8.5.** *Let $X$ be a connected $p$-compact group, fix $(S,f) \in \mathrm{Syl}_p(X)$, and set $\mathcal{F} = \mathcal{F}_{S,f}(X)$. Let $T \trianglelefteq S$ be the identity component. Then*

(a) *each $s \in S$ is $\mathcal{F}$-conjugate to an element in $T$;*

(b) *$C_S(T) = T$, and no proper subgroup of $S$ containing $T$ is strongly closed in $\mathcal{F}$; and*

(c) *if $X$ is simple, then no infinite proper subgroup of $T$ is invariant under the action of $\mathrm{Aut}_{\mathcal{F}}(T)$.*

*Thus whenever $X$ is simple and $p \mid |\mathrm{Aut}_{\mathcal{F}}(T)|$, conditions (i), (ii), and (iii) in Theorem 7.4 all hold.*

*Proof.* The last statement follows immediately from (b) and (c), and since $p \mid |\mathrm{Aut}_{\mathcal{F}}(T)|$ implies $S > T$.

**(a)** By assumption, $f\colon BS \longrightarrow BX$ is a Sylow $p$-subgroup of $X$. Set $f_0 = f|_{BT}\colon BT \longrightarrow BX$.

Fix $s \in S \smallsetminus T$, and let $m \geq 1$ be such that $|s| = p^m$. Let $\rho \in \mathrm{Hom}(\mathbb{Z}/p^m, S)$ be the homomorphism that sends the class of 1 to $s$, and set $\chi = f \circ B\rho$. By [DW2, Proposition 5.6] and since $X$ is connected, $\chi$ extends to maps defined on $B\mathbb{Z}/p^n$ for all $n > m$, and hence to a map $\widehat{\chi}\colon BA \longrightarrow BX$ where $A \cong (S^1)^\wedge_p$ has discrete approximation $A_\infty \cong \mathbb{Z}/p^\infty$.

By [DW2, Proposition 8.11], there is a pointed map $\widehat{\tau}\colon BA \longrightarrow BT^\wedge_p$ such that $\widehat{\chi} \simeq f_0 \circ \widehat{\tau}$. By [DW4, Proposition 3.2], there is $\tau \in \mathrm{Hom}(A_\infty, T)$ such that $\widehat{\tau} \simeq B\tau$. So $f \circ B\rho = \widehat{\chi}|_{B\mathbb{Z}/p^m} \simeq f_0 \circ B(\tau|_{\mathbb{Z}/p^m})$, and by definition of the fusion system $\mathcal{F}$, this means that the homomorphisms $\rho \in \mathrm{Hom}(\mathbb{Z}/p^m, S)$ and $\tau|_{\mathbb{Z}/p^m} \in \mathrm{Hom}(\mathbb{Z}/p^m, T)$ are $\mathcal{F}$-conjugate. So $s = \rho(1)$ is $\mathcal{F}$-conjugate to $\tau(1) \in T$.

**(b)** Fix an element $x \in C_S(T)$. By (a), $\langle x\rangle$ is $\mathcal{F}$-conjugate to a subgroup of $T$, so by [BLO3, Lemma 2.4(a)], it is conjugate to a subgroup of $T$ fully centralized in $\mathcal{F}$. So fix $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, T)$ such that $\varphi(\langle x\rangle) \le T$ and is fully centralized in $\mathcal{F}$. Then $\varphi$ extends to some $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), S)$ by the extension axiom. But $T \le C_S(x)$ and $\overline{\varphi}(T) = T$, so $x \in T$ since $\overline{\varphi}(x) \in T$. Thus $C_S(T) = T$.

If $R$ is strongly closed in $\mathcal{F}$ and $T \le R \le S$, then $R = S$ since each element of $S$ is $\mathcal{F}$-conjugate to an element of $R$.

**(c)** Set $L_X = \pi_2(BT^\wedge_p)$ (following the notation of Dwyer and Wilkerson [DW3]). By the definition in [DW3, § 1] of a simple connected $p$-compact group, $\mathbb{Q} \otimes L_X$ is a simple $\mathbb{Q}_p\mathrm{Aut}_{\mathcal{F}}(T)$-module whenever $X$ is simple. So $\mathrm{Hom}(\mathbb{Q}_p, T)$ is a simple $\mathbb{Q}_p\mathrm{Aut}_{\mathcal{F}}(T)$-module by Lemma 8.1(a,b), and there is no infinite proper $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant subgroup of $T$ by Lemma 8.2. □

The following notation for certain infinite algebraic extensions of $\mathbb{F}_q$ will be useful when stating our main theorem.

**Notation 8.6.** *When $q$ is a prime power and $\mathscr{P}$ is a set of primes, then $\mathbb{F}_{q\langle\mathscr{P}\rangle}$ denotes the union of all finite fields $\mathbb{F}_{q^a}$ for $a \ge 1$ divisible only by primes in $\mathscr{P}$. When $0 \ne m \in \mathbb{N}$, we also write $\mathbb{F}_{q\langle m\rangle} = \bigcup_{i=1}^\infty \mathbb{F}_{q^{m^i}}$ and $\mathbb{F}_{q\langle m'\rangle} = \bigcup\{\mathbb{F}_{q^k} \mid \gcd(k,m) = 1\}$; i.e., the special cases of $\mathbb{F}_{q\langle\mathscr{P}\rangle}$ where $\mathscr{P}$ is the set of primes dividing $m$ or its complement.*

Thus, for example, $\mathbb{F}_{q\langle 1\rangle} = \mathbb{F}_q$ and $\mathbb{F}_{q\langle 1'\rangle} = \overline{\mathbb{F}}_q$.

We are now ready to determine exactly which fusion systems of simple, connected $p$-compact groups are sequentially realizable and which are not.

**Theorem 8.7.** *Let $X$ be a simple, connected $p$-compact group, choose $S \in \mathrm{Syl}_p(X)$, and set $\mathcal{F} = \mathcal{F}_S(X)$. Let $T \trianglelefteq S$ be the identity component of $S$, and let $W = \mathrm{Aut}_{\mathcal{F}}(T)$ be the Weyl group.*

(a) *If $p \nmid |W|$, then $\mathcal{F}$ is* LT*-realized by the semidirect product $T \rtimes W$.*

(b) *If $W$ is the Weyl group of a simple group scheme $\mathbb{G}$ over $\mathbb{Z}$, then $\mathcal{F}$ is* LT*-realized by the algebraic group $\mathbb{G}(\overline{\mathbb{F}}_q)$ for each prime $q \ne p$.*

(c) *If $p \mid |W|$ and $W$ is not the Weyl group of a simple algebraic group scheme, then $\mathcal{F}$ is either* LT*-realizable or not sequentially realizable, as described in Table 8.1.*

*Proof.* For the proofs that each simple, connected $p$-compact group has the form listed in (a) or (b) or is one of the entries in Table 8.1, we refer to [AGMV, Theorem 1.1] or [AG, Theorem 1.1], as well as the classification by Clark and Ewing [CE] of finite reflection groups over $\mathbb{Z}_p$. Also, when $p$ is odd, a simple, connected $p$-compact group is uniquely determined by the Weyl group and its action on the maximal torus by [AGMV, Theorem 1.1] again.

**(a)** If $p \nmid |W|$, then $S = T \trianglelefteq \mathcal{F}$ by Theorem 1.15 (Alperin's fusion theorem), and so $\mathcal{F}$ is the fusion system of $T \rtimes W$.

| $p$ | $W$ | conditions or comments | $\mathrm{rk}(T)$ | realized by |
|---|---|---|---|---|
| $p$ | $G(m,1,n)$ | $3 \le m \mid (p-1),\ n \ge p$ | $n$ | $GL_{mn}(\mathbb{F}_{q_1^{\langle p \rangle}})$ |
| $p$ | $G(2m,2,n)$ | $2 \le m \mid (p-1)/2,\ n \ge p$ | $n$ | $PSO_{2mn}^{(-1)^n}(\mathbb{F}_{q_2^{\langle p \rangle}})$ |
| $p$ | $G(m,k,n)$ | $1 \ne k \mid m \mid (p-1),\ k \ge 3,\ n \ge p$ | $n$ | not seq. real. |
| 3 | $\boldsymbol{ST_{12}}$ | $W \cong GL_2(3)$ | 2 | ${}^2F_4(\mathbb{F}_{2^{\langle 3 \rangle}})$ |
| 2 | $\boldsymbol{ST_{24}}$ | $W \cong C_2 \times SL_3(2)$ | 3 | not seq. real. |
| 5 | $\boldsymbol{ST_{29}}$ | $W \cong (C_4 \circ Q_8 \circ Q_8).\Sigma_5$ | 4 | not seq. real. |
| 5 | $\boldsymbol{ST_{31}}$ | $W \cong (C_4 \circ Q_8 \circ Q_8).\Sigma_6$ | 4 | $E_8(\mathbb{F}_{q_3^{\langle 5 \rangle}})$ |
| 7 | $\boldsymbol{ST_{34}}$ | $W \cong 6.PSU_4(3).2$ | 6 | not seq. real. |

TABLE 8.1. Here, $q_1$, $q_2$, and $q_3$ are prime powers: $q_1$ and $q_2$ have order $m$ and $2m$, respectively, in $(\mathbb{Z}/p)^\times$, while $q_3 \equiv \pm 2 \pmod 5$. In the last five cases, $\boldsymbol{ST_n}$ is the reflection group of type $n$ in the Shephard-Todd list [ST, p. 301]. Also, $G \circ H$ denotes a central product of the groups $G$ and $H$, while $G.H$ denotes an extension of $G$ by $H$.

**(b)** This was shown in Theorem 4.2.

**(c)** We consider individually the entries in Table 8.1.

**Case 1:** Assume $W \cong G(m,1,n) \cong C_m \wr \Sigma_n$ for some $3 \le m \mid (p-1)$ and $n \ge p$. In particular, $p$ is odd. Fix $q_1$ of order $m$ in $(\mathbb{Z}/p)^\times$, set $K = \mathbb{F}_{q_1^{\langle m' \rangle}} \subseteq \overline{\mathbb{F}}_{q_1}$, the union of all finite extensions of $q_1$ of degree prime to $m$, and set $\Gamma = GL_{mn}(K)$. Also, set $K_0 = \mathbb{F}_{q_1^{\langle p \rangle}} \le K$ and $\Gamma_0 = GL_{mn}(K_0) \le \Gamma$. For each $r \ge 1$ prime to $m$, if $p^i \mid r$ is the largest power of $p$ dividing $r$, then the $p$-fusion systems of $GL_{mn}(q_1^{p^i})$ and of $GL_{mn}(q_1^r)$ are isomorphic by [BMO1, Theorem A(a)]. Since $\Gamma_0 = \bigcup_{i=0}^\infty GL_{mn}(q_1^{p^i})$ and $\Gamma$ is the union of the groups $GL_{mn}(q_1^r)$ for all $r$ prime to $m$, the $p$-fusion systems of $\Gamma_0$ and $\Gamma$ are also isomorphic.

By a theorem of Quillen [Q, §10], $H^*(B\Gamma;\mathbb{F}_p)$ is a polynomial algebra, and hence the loop space $\Omega(B\Gamma^\wedge_p)$ has finite cohomology (an exterior algebra). So $B\Gamma^\wedge_p$ is the classifying space of a $p$-compact group. Thus by Lemma 8.4, the $p$-fusion system of $\Gamma$ is the fusion system of a $p$-compact group. From [BMO2, Table 6.1] (applied to finite subfields of $K$), it follows that $\Gamma$ has a maximal torus $T$ of rank $n$ with Weyl group $\mathrm{Aut}_{\mathcal{F}}(T) \cong C_m \wr \Sigma_n$. So by the uniqueness statement in [AGMV, Theorem 1.1], $\mathcal{F}$ is the fusion system of $X$.

**Case 2:** Assume $W \cong G(2m,2,n)$ for some $2 \le m \mid (p-1)/2$ and $n \ge p$. Let $G = SO_{2nm}(\mathbb{C})$ be the simple complex algebraic group of type $SO_{2nm}$. Let $\tau \in \mathrm{Aut}(BG)$ be the self map induced by a graph automorphism of order 2. By [Fr, Theorem 12.2], if $q = r^m$ for some prime $r \ne p$ and $\psi^q \in \mathrm{Aut}(BG^\wedge_p)$ is the unstable Adams map of degree $q$, then $(BSO^+_{2nm}(q))^\wedge_p$ and $(BSO^-_{2nm}(q))^\wedge_p$ are the spaces of homotopy fixed points of the actions of $\psi^q$ and $\tau\psi^q$, respectively, on $BG^\wedge_p$.

Choose a prime power $q_2$ with $\mathrm{ord}_p(q_2) = 2m$. Let $\zeta$ be the primitive $2m$-th root of unity in the $p$-adic integers $\mathbb{Z}_p$ such that $\zeta \equiv q_2 \pmod p$. Set $q_0 = \zeta^{-1}q_2 \in \mathbb{Z}_p$; thus $q_2 = \zeta q_0$ and $q_0 \equiv 1 \pmod p$. Set $\Gamma = \langle \tau^n\psi^\zeta \rangle$. Since $\tau$ has order 2 and commutes with $\psi^\zeta$ up to homotopy by [JMO, Corollary 3.5], $\Gamma$ is cyclic of order $2m$ as a subgroup of $\mathrm{Out}(BG^\wedge_p)$ in all cases. By [BM, Theorem B], the homotopy action of $\Gamma \cong C_{2m}$ on $BG$ lifts to an actual action (i.e., a homomorphism $\Gamma \longrightarrow \mathrm{Aut}(BG)$). So for each $i \geq 0$,

$$(BSO_{2nm}^{(-1)^n}(q_2^{p^i}))^\wedge_p \simeq (BSO_{2nm}(\mathbb{C})^\wedge_p)^{h\tau^n\psi^{q_2^{p^i}}} \simeq \big((BSO_{2nm}(\mathbb{C})^\wedge_p)^{h\Gamma}\big)(q_0^{p^i}) \tag{8-1}$$

where the second equivalence follows from [BM, Theorem E] and since $q_2^{p^i} = \zeta q_0^{p^i}$ (recall $\zeta^{2m} = 1$ and $2m \mid (p-1)$). Here, $BSO_{2nm}^{(-1)^n}$ means $BSO_{2nm}^{+}$ if $n$ is even and $BSO_{2nm}^{-}$ if $n$ is odd. When $X$ is a $p$-compact group, we follow the notation in [BM] and let $BX(q_2)$ denote the homotopy fixed set of an unstable Adams operation of degree $q_2$ acting on $BX$.

Set $BY = (BSO_{2nm}(\mathbb{C})^\wedge_p)^{h\Gamma}$ for short. Then (8-1) implies that

$$(BSO_{2nm}^{(-1)^n}(\mathbb{F}_{q_2^{\langle p\rangle}}))^\wedge_p \simeq \operatorname*{hocolim}_{i\geq 0} BY(q_0^{p^i}).$$

By [BM, Theorem B], $BY$ is the classifying space of a connected $p$-compact group. By the same theorem and since $H^*(BSO(2nm);\mathbb{F}_p)$ is polynomial, $H^*(BY;\mathbb{F}_p)$ is also polynomial. So by [BM, Theorem F], for each $i \geq 1$, $H^*(BY(q_0^{p^i});\mathbb{F}_p)$ is isomorphic as a graded ring to the tensor product of $H^*(BY;\mathbb{F}_p)$ with an exterior algebra, where the polynomial generators are higher Bocksteins of the exterior generators (different higher Bocksteins depending on $i$). Hence the natural maps from $BY(q_0^{p^i})$ to $BY(q_0^{p^{i+1}})$ induce isomorphisms on the polynomial parts of the cohomology rings and trivial maps on the exterior parts. The natural map from $\mathrm{hocolim}(BY(q_0^{p^i}))$ to $BY$ thus induces an isomorphism

$$\begin{aligned} H^*(BY;\mathbb{F}_p) \cong \operatorname*{holim}_{i\geq 0} H^*(BY(q_0^{p^i});\mathbb{F}_p) &\cong H^*\big(\operatorname*{hocolim}_{i\geq 0}(BY(q_0^{p^i}));\mathbb{F}_p\big) \\ &\cong H^*(BSO_{2nm}^{(-1)^n}(\mathbb{F}_{q_2^{\langle p\rangle}});\mathbb{F}_p), \end{aligned}$$

and so $BSO_{2nm}^{(-1)^n}(\mathbb{F}_{q_2^{\langle p\rangle}})^\wedge_p \simeq BY$.

Thus $BSO_{2nm}^{(-1)^n}(\mathbb{F}_{q_2^{\langle p\rangle}})^\wedge_p$ is the classifying space of a connected $p$-compact group. Let $W(Y)$ denote its Weyl group; equivalently, the Weyl group (torus automizer) of $BSO_{2nm}^{(-1)^n}(q_2^{p^i})$ for large $i$. From [BMO2, Table 6.1], we see that $W(Y) \cong C_{G(2,2,mn)}(w_0)$ where $w_0$ acts on a maximal torus in $SO_{2mn}(\overline{\mathbb{F}}_{q_2})$ by sending $(\lambda_1, \lambda_2, \ldots, \lambda_{mn})$ to

$$(\lambda_m^{-1}, \lambda_1, \ldots, \lambda_{m-1}, \lambda_{2m}^{-1}, \lambda_{m+1}, \ldots, \lambda_{2m-1}, \ldots, \lambda_{nm}^{-1}, \lambda_{(n-1)m+1}, \ldots, \lambda_{nm-1}).$$

Thus $W(Y)$ is generated by all products of evenly many permutations

$$[\lambda_{im+1}, \ldots, \lambda_{im+m-1}, \lambda_{(i+1)m}] \;\mapsto\; [\lambda_{(i+1)m}^{-1}, \lambda_{im+1}, \ldots, \lambda_{im+m-1}]$$

(of order $2m$) together with all permutations of the $n$ blocks of length $m$, and is isomorphic to $G(2m,2,n)$. So $BY \simeq BX(2m,2,n)$ by Lemma 8.3, and hence the fusion systems of $SO_{2nm}^{(-1)^n}(\mathbb{F}_{q^{\langle p\rangle}})$ and $X(2m,2,n)$ are isomorphic by Lemma 8.4.

**Case 3:** Assume $W \cong \boldsymbol{ST_{12}}$ or $\boldsymbol{ST_{31}}$. The fusion systems of these two $p$-compact groups are described in [OR, Table 5.1].

Consider the pairs $(G,p) = ({}^2F_4(2^k),3)$ and $(E_8(q_3^k),5)$, where $q_3 \equiv \pm 2 \pmod 5$ is a prime power and $k$ is odd. In each of these cases, the Sylow $p$-subgroup $S \in \mathrm{Syl}_p(G)$ contains a homocyclic abelian subgroup $A \trianglelefteq S$ of index $p$, and information about the fusion systems $\mathcal{F}_S(G)$, including the essential subgroups other than $A$, is given in [OR, Table 4.2]. If $(G,p) = ({}^2F_4(2^k),3)$, then $A$ has rank 2 and exponent $3^e$ where $e = v_3(2^k+1) > 0$. If $(G,p) = (E_8(q_3^k),5)$, then $A$ has rank 4 and exponent $5^e$ where $e = v_5(q_3^{2k}+1) > 0$. By comparing Tables 4.2 and 5.1 in [OR], we see that $\mathcal{F}$ is the union of the fusion systems of the finite groups ${}^2F_4(2^k)$ or $E_8(q_3^k)$ taken over all odd $k$, and hence is realized by the linear torsion group ${}^2F_4(\mathbb{F}_{2^{\langle 2'\rangle}})$ or $E_8(\mathbb{F}_{q_3^{\langle 2'\rangle}})$, respectively. A similar argument shows that $\mathcal{F}$ is also realized by ${}^2F_4(\mathbb{F}_{2^{\langle 3\rangle}})$ or $E_8(\mathbb{F}_{q_3^{\langle 5\rangle}})$.

**Case 4:** Assume either $W \cong G(m,k,n)$ where $1 \neq k \mid m \mid (p-1)$, $k \geq 3$, and $n \geq p$, or $W \cong \boldsymbol{ST_{24}}$, $\boldsymbol{ST_{29}}$, or $\boldsymbol{ST_{34}}$. By Lemma 8.5, $\mathcal{F}$ satisfies conditions (i)–(iii) in Theorem 7.4.

By inspection, $W = \mathrm{Aut}_{\mathcal{F}}(T)$ does not contain a normal subgroup of index prime to $p$ isomorphic to a product of copies of any of the groups listed in points (a)–(e) of Theorem 7.4. So $\mathcal{F}$ is not sequentially realizable by that theorem. □

**Remark 8.8.** *In Theorem 8.7(c), the fusion systems of certain 3-compact groups are realized by Ree groups ${}^2F_4(K)$ for an infinite field $K$ of characteristic 2. The groups ${}^2F_4(K)$ are often defined only when $K$ is a finite extension of $\mathbb{F}_2$ of odd degree, but in fact, they were defined by Ree in [R] for each perfect field $K$ of characteristic 2 that has an automorphism $\theta \in \mathrm{Aut}(K)$ such that $\theta^2\psi = \mathrm{Id}_K$ where $\psi \in \mathrm{Aut}(K)$ is the Frobenius ($t \mapsto t^2$). In particular, this is the case for each algebraic extension of $\mathbb{F}_2$ that is a union of extensions of odd degree. Ree's construction in this general situation is also described in* [Car, Sections 12.3 and 13.4].

*More precisely, Ree defines, for each perfect field $K$ of characteristic 2, a graph automorphism $\sigma \in \mathrm{Aut}(F_4(K))$, with the property that $\sigma^2 = \widehat{\psi}$, where $\widehat{\psi} \in \mathrm{Aut}(F_4(K))$ is the field automorphism induced by $\psi$. He does this by defining $\sigma$ explicitly on the root groups, exchanging the root groups for long and short roots. If $\theta \in \mathrm{Aut}(K)$ is such that $\theta^2\psi = \mathrm{Id}_K$ and $\widehat{\theta} \in \mathrm{Aut}(F_4(K))$ is the induced field automorphism, then $(\sigma\widehat{\theta}) = \mathrm{Id}$, and he defines ${}^2F_4(K) = C_{F_4(K)}(\sigma\widehat{\theta})$.*

So far, all of our examples of realizing fusion systems of $p$-compact groups involve linear torsion groups in characteristic $q$ for $q \neq p$. In Theorem 8.10, as well as generalizing Theorem 8.7 to arbitrary connected $p$-compact groups, we will apply Proposition 7.5 to show that in most cases, their fusion systems cannot be realized by linear torsion groups in characteristic 0.

We first need to know that the fusion system of a product of $p$-compact groups is the product of their fusion systems.

**Proposition 8.9.** *Let $X_1$ and $X_2$ be $p$-compact groups, and set $X = X_1 \times X_2$. Choose Sylow $p$-subgroups $(S_i, f_i) \in \mathrm{Syl}_p(X_i)$, and set $S = S_1 \times S_2$ and $f = f_1 \times f_2 \colon BS \longrightarrow BX$. Then $(S,f) \in \mathrm{Syl}_p(X)$, and*

$$\mathcal{F}_{S,f}(X) \cong \mathcal{F}_{S_1,f_1}(X_1) \times \mathcal{F}_{S_2,f_2}(X_2).$$

*Proof.* Set $\mathcal{F} = \mathcal{F}_{S,f}(X)$ and $\mathcal{F}_i = \mathcal{F}_{S_i,f_i}(X_i)$ for short. Let $\mathrm{pr}_i \colon S \longrightarrow S_i$ and $\rho_i \colon BX \longrightarrow BX_i$ be the projections. By a theorem of Dwyer and Wilkerson (see [BLO3, Proposition 10.1(a)] for details), a pair $(S,f)$ is in $\mathrm{Syl}_p(X)$ if and only if the homotopy fiber of $f \colon BS \longrightarrow BX$ has finite mod $p$ homology and Euler characteristic prime to $p$. Since $\mathrm{hofib}(f) \simeq \mathrm{hofib}(f_1) \times \mathrm{hofib}(f_2)$, it follows immediately that $(S,f) \in \mathrm{Syl}_p(X)$.

Set $\mathcal{L} = \mathcal{L}^c_{S,f}(X)$ and $\mathcal{L}_i = \mathcal{L}^c_{S_i,f_i}(X_i)$ in the notation of [BLO3, Theorem 10.7]. Then $BX \simeq |\mathcal{L}|^\wedge_p$ and $BX_i \simeq |\mathcal{L}_i|^\wedge_p$ by the same theorem, and so $|\mathcal{L}|^\wedge_p \simeq |\mathcal{L}_1|^\wedge_p \times |\mathcal{L}_2|^\wedge_p$. For each discrete $p$-toral group $P$, the projections $\mathrm{pr}_i$ induce a bijection $[BP, |\mathcal{L}|^\wedge_p] \cong [BP, |\mathcal{L}_1|^\wedge_p] \times [BP, |\mathcal{L}_2|^\wedge_p]$, and hence by [BLO3, Theorem 6.3(a)] a bijection

$$\mathrm{Rep}(P,\mathcal{L}) \cong \mathrm{Rep}(P,\mathcal{L}_1) \times \mathrm{Rep}(P,\mathcal{L}_2). \tag{8-2}$$

Here, $\mathrm{Rep}(P,\mathcal{L}) = \mathrm{Hom}(P,S)/{\sim}$, where $\psi \sim \psi'$ if and only if there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\psi(P), \psi'(P))$ such that $\psi' = \varphi\psi$.

Thus for each $P \le S$ and each $\varphi \in \mathrm{Hom}(P,S)$, we have $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ if and only if $[\varphi]=[P \hookrightarrow S]$ in $\mathrm{Rep}(P,\mathcal{L})$. By (8-2), this holds if and only if $[\mathrm{pr}_i\circ\varphi]=[\mathrm{pr}_i|_P]$ in $\mathrm{Rep}(P,\mathcal{L}_i)$ for $i=1,2$; i.e., if and only if $\mathrm{Ker}(\mathrm{pr}_i\circ\varphi)=\mathrm{Ker}(\mathrm{pr}_i|_P)=P\cap S_{3-i}$ and $\mathrm{pr}_i\circ\varphi=\varphi_i\circ(\mathrm{pr}_i|_P)$ for some $\varphi_i\in\mathrm{Hom}_{\mathcal{F}_i}(\mathrm{pr}_i(P),S_i)$. This is exactly the condition for $\varphi$ to be in $\mathrm{Hom}_{\mathcal{F}_1\times\mathcal{F}_2}(P,S)$, and hence $\mathcal{F}=\mathcal{F}_1\times\mathcal{F}_2$. □

We are now ready to look more generally at fusion systems of connected $p$-compact groups.

**Theorem 8.10.** *Let $X$ be a connected $p$-compact group, fix $(S,f)\in\mathrm{Syl}_p(X)$, and set $\mathcal{F}=\mathcal{F}_{S,f}(X)$.*

(a) *Set $q=2$ if $p=3$, and let $q$ be any prime whose class generates $(\mathbb{Z}/p)^\times$ if $p\ne 3$. If $\mathcal{F}$ is sequentially realizable, then $\mathcal{F}$ is realized by a torsion group that is linear over $\overline{\mathbb{F}}_q$.*

(b) *The fusion system $\mathcal{F}$ is realized by a linear torsion group in characteristic 0 if and only if the Weyl group of $X$ has order prime to $p$.*

*Proof.* By [AGMV, Theorem 1.2] or [AG, Theorem 1.1], there are a compact connected Lie group $G$, and simple connected $p$-compact groups $X_1,\dots,X_k$ (some $k\ge 0$) such that $X$ is the direct product of $G^\wedge_p=\Omega(BG^\wedge_p)$ with the $X_i$. Hence for $S_0\in\mathrm{Syl}_p(G)$ and $(S_i,f_i)\in\mathrm{Syl}_p(X_i)$, the fusion system $\mathcal{F}$ is the product of $\mathcal{F}_{S_0}(G)$ and the $\mathcal{F}_{S_i,f_i}(X_i)$ by Proposition 8.9.

**(a)** The fusion system of $G$ is realized by a linear torsion group in characteristic $q$ by Theorem 4.2. By Theorem 8.7(a,c) and Table 8.1, the fusion system of each $X_i$ is either realizable by a linear torsion group in characteristic $q$ or is not sequentially realizable. So either their product $\mathcal{F}$ is realizable by a linear torsion group in characteristic $q$; or the fusion system of one of the factors $X_i$ is not sequentially realizable, in which case $\mathcal{F}$ fails to be sequentially realizable by Proposition 2.6(b).

**(b)** Let $T\trianglelefteq S$ be the identity component of $S$, and set $W=\mathrm{Aut}_{\mathcal{F}}(T)$ (the Weyl group of $X$). If $p\nmid|W|$, then $S=T$, and by Alperin's fusion theorem, $\mathcal{F}$ is realized by an extension $\Gamma$ of $T$ by $W$. Since $W$ is finite, $\Gamma$ is a linear torsion group in characteristic 0.

Now assume that $p\mid|W|$; equivalently, that $S>T$. Let $\Gamma$ be a locally finite group such that $S\in\mathrm{Syl}_p(\Gamma)$ and $\mathcal{F}=\mathcal{F}_S(\Gamma)$. We will show that there is a prime $q\ne p$ such that $\mathrm{srk}_q(\Gamma)=\infty$, and hence that $\Gamma$ cannot be linear in characteristic 0 by Lemma 3.6(a).

Assume first that $X$ is simple. By Lemma 8.5 and since $p\mid|W|$, conditions (i), (ii), and (iii) in Proposition 7.5 all hold. So by that proposition, there is a prime $q\ne p$ such that $\mathrm{srk}_q(\Gamma)=\infty$.

Now let $X$ be arbitary (not necessarily simple). Then either $p$ divides the order of the Weyl group in one of the simple factors $X_i$ (for $1\le i\le k$), or $p$ divides the order of the Weyl group of $\Gamma$. In either case, there is a connected simple $p$-compact group $Y$ whose Weyl group has order a multiple of $p$, and whose fusion system is a quotient of the fusion system of $X$. Then for some $U\trianglelefteq S$ strongly closed in $\mathcal{F}$, $\mathcal{F}/U$ is isomorphic to the fusion system of $Y$. Then $\mathcal{F}/U\cong\mathcal{F}_{S/U}(N_\Gamma(U)/U)$ by Lemma 1.17, so by what was shown in the last paragraph, there is a prime $q\ne p$ such that $\mathrm{srk}_q(N_\Gamma(U)/U)=\infty$. Hence $\mathrm{srk}_q(\Gamma)=\infty$. □

Theorem 8.10(b) shows in particular that the fusion system of a compact connected Lie group $G$ at a prime $p$ that divides the order of the Weyl group of $G$ cannot be realized by any torsion subgroup of $G$.

## 9. Other examples

By analogy with groups, a saturated fusion system is *simple* if it contains no proper nontrivial normal fusion subsystems (see [AKO, Definition I.6.1]). In Section 5 of [OR], the authors described all simple fusion systems over nonabelian infinite discrete $p$-toral groups containing a discrete $p$-torus with index $p$. With the help of Theorem 7.4, we can now determine in all cases exactly which of those fusion systems are sequentially realizable and which are not. We will see that in fact, most of them are not sequentially realizable. The only exceptions are those that are fusion systems of $p$-compact groups and were shown in Section 8 to be sequentially realizable.

Before going into details of the examples, we recall a few of the definitions used in [OR]. Let $Z_2(G)$ denote the second term in the upper central series of a group $G$; thus $Z_2(G)/Z(G) = Z(G/Z(G))$.

**Notation 9.1.** *Let $p$ be an odd prime. Assume $\mathcal{F}$ is a saturated fusion system over a nonabelian discrete $p$-toral group $S$ whose identity component $T$ has index $p$ in $S$. Assume also that $T \ntrianglelefteq \mathcal{F}$; then $S$ splits over $T$ by* [OR, Corollary 2.6].

- *Let $\mathcal{H}$ be the set of all $Z(S)\langle x\rangle$ for $x \in S \smallsetminus T$.*
- *Let $\mathcal{B}$ be the set of all $Z_2(S)\langle x\rangle$ for $x \in S \smallsetminus T$.*
- *Let $\mathbf{E}_\mathcal{F}$ be the set of all $\mathcal{F}$-essential subgroups (see Definition 1.12). By* [OR, Lemma 2.2], *$\mathbf{E}_\mathcal{F} \subseteq \{T\} \cup \mathcal{H} \cup \mathcal{B}$.*

If $\mathrm{rk}(T) = p-1$, then by Proposition A.4 and Lemma A.5(a,c) in [OR], $T \cong \mathbb{Q}_p(\zeta)/\mathbb{Z}_p[\zeta]$ as $\mathbb{Z}[S/T]$-modules, where $\zeta$ is a primitive $p$-th root of unity in $\overline{\mathbb{Q}_p}$ and a generator of $S/T$ acts via multiplication by $\zeta$. Hence $Z(S) = C_S(T)$ has order $p$ in this case, and so $\mathrm{Aut}^\vee_\mathcal{F}(T) = N_{\mathrm{Aut}_\mathcal{F}(T)}(\mathrm{Aut}_S(T))$ in the terminology of [OR, Theorem B]. See also [OR, Notation 2.9].

**Lemma 9.2.** *Let $\mathcal{F}$ be a saturated fusion system over an infinite nonabelian discrete $p$-toral group $S$ whose identity component $T \trianglelefteq S$ has index $p$ in $S$. Assume also that $O_p(\mathcal{F}) = 1$. Then there is no infinite proper $\mathrm{Aut}_\mathcal{F}(T)$-invariant subgroup of $T$.*

*Proof.* Set $G = \mathrm{Aut}_\mathcal{F}(T)$ for short. Set $V = \mathrm{Hom}(\mathbb{Q}_p, T)$, regarded as a $\mathbb{Q}_pG$-module, and let $\mathrm{ev}_1\colon V \longrightarrow T$ be the surjective homomorphism evaluation at $1 \in \mathbb{Q}_p$. By Lemma 8.2, it suffices to show that the $V$ is irreducible.

Set $U = \mathrm{Aut}_S(T) \in \mathrm{Syl}_p(G)$. By [OR, Lemma 5.3] and since $O_p(\mathcal{F}) = 1$, $T/C_T(U)$ is a discrete $p$-torus of rank $p-1$. Hence $\dim_{\mathbb{Q}_p}(V/C_V(U)) = p-1$, and $V/C_V(U)$ is irreducible as a $\mathbb{Q}_pU$-module. So if $V = V_1 \oplus V_2$ where the $V_i$ are $\mathbb{Q}_pG$-submodules, then one of the factors, say $V_2$, has trivial action of $U$, and hence trivial action of $O^{p'}(G)$. But then $\mathrm{ev}_1(V_2)$ is an infinite $G$-invariant subgroup of $T$ on which $O^{p'}(G)$ acts trivially, which is impossible by [OR, Lemma 2.7] and since $O_p(\mathcal{F}) = 1$. □

We are now ready to examine the realizability of such fusion systems.

**Theorem 9.3.** *Let $\mathcal{F}$ be a simple saturated fusion system over an infinite nonabelian discrete $p$-toral group $S$ whose identity component $T \trianglelefteq S$ has index $p$ in $S$. Then either*

(a) *$\mathcal{F}$ is isomorphic to the fusion system of a simple, connected $p$-compact group, and is* LT*-realizable or not sequentially realizable as described in Theorem 8.7; or*

(b) *$\mathcal{F}$ is not sequentially realizable.*

*Proof.* Let $\mathcal{F}$ be a simple saturated fusion system over an infinite nonabelian discrete $p$-toral group $S$ with identity component $T$ of index $p$ in $S$. Note that $C_S(T)=T$ since $S$ is nonabelian.

If $p=2$, then by [OR, Theorem 5.6], $\mathcal{F}$ is isomorphic to the fusion system of $SO(3)$ ($\mathrm{Aut}_{\mathcal{F}}(T)\cong C_2$ and $\mathrm{rk}(T)=1$) or $PSU(3)$ ($\mathrm{Aut}_{\mathcal{F}}(T)\cong\Sigma_3$ and $\mathrm{rk}(T)=2$). Both of these are LT-realizable by Theorem 4.2.

If $p$ is odd and $T\notin\mathbf{E}_{\mathcal{F}}$, then by [OR, Theorem 5.12], $\mathrm{Aut}_{\mathcal{F}}(T)\cong C_p\rtimes C_{p-1}$, $\mathrm{rk}(T)=p-1$, and $\mathbf{E}_{\mathcal{F}}=\mathcal{H}$, and there is a unique such fusion system for each odd prime $p$. When $p=3$, this is the 3-fusion system of the compact Lie group (or 3-compact group) $PSU(3)$. So assume $p\geq 5$. Condition (i) in Theorem 7.4 clearly holds, (iii) holds by Lemma 9.2, and (ii) holds by Lemma 5.6(a) and since $T\ntrianglelefteq\mathcal{F}$ (since a normal subgroup is contained in all essential subgroups and hence in each member of $\mathcal{H}$). No subgroup of index prime to $p$ in $C_p\rtimes C_{p-1}$ is among those listed in Theorem 7.4(b), and so $\mathcal{F}$ is not sequentially realizable by that theorem.

The remaining cases, where $p$ is odd and $T\in\mathbf{E}_{\mathcal{F}}$, are all described in [OR, Theorem B]. Note first that $\mathrm{rk}(T)\geq p-1$ in all cases (the minimal dimension of a faithful $\mathbb{Q}_pC_p$-module).

We recall some more notation used in [OR], when $\mathcal{F}$ is a saturated fusion system over a nonabelian discrete $p$-toral group $S$ whose identity component $T$ has index $p$ in $S$.

- Set $\Delta=(\mathbb{Z}/p)^\times\times(\mathbb{Z}/p)^\times$. For each $i\in\mathbb{Z}$, set $\Delta_i=\{(r,r^i)\,|\,r\in(\mathbb{Z}/p)^\times\}\leq\Delta$.
- Define $\mu\colon \mathrm{Aut}(S)\longrightarrow\Delta$ by setting $\mu(\alpha)=(r,s)$ if $\alpha(x)\in x^rT$ for $x\in S\smallsetminus T$ and $\alpha(g)=g^s$ for $g\in Z(S)\cap[S,S]$. (In all cases, $|Z(S)\cap[S,S]|=p$ by [OR, Lemma 2.4].)
- Define $\mu_{\mathcal{F}}\colon N_{\mathrm{Aut}_{\mathcal{F}}(T)}(\mathrm{Aut}_S(T))\longrightarrow\Delta$ by setting $\mu_{\mathcal{F}}(\beta)=\mu(\alpha)$ for some $\alpha\in\mathrm{Aut}(S)$ such that $\alpha|_T=\beta$. (Such an $\alpha$ exists by the extension axiom.)
- Set $\mathrm{Aut}^\vee_{\mathcal{F}}(T)=\bigl\{\beta\in N_{\mathrm{Aut}_{\mathcal{F}}(T)}(\mathrm{Aut}_S(T))\,\big|\,[\beta,Z(S)]\leq Z(S)\cap[S,S]\bigr\}$.

By Theorem B in [OR], a simple fusion system $\mathcal{F}$ that realizes a given pair $(\mathrm{Aut}_{\mathcal{F}}(T),T)$ is determined up to isomorphism by $\mathbf{E}_{\mathcal{F}}$, where $\mathbf{E}_{\mathcal{F}}=\{T\}\cup\mathcal{H}$ or $\{T\}\cup\mathcal{B}$ in all cases. Set $W=\mathrm{Aut}_{\mathcal{F}}(T)$ and $W^\vee=\mathrm{Aut}^\vee_{\mathcal{F}}(T)$ for short. Then the following implications hold by [OR, Theorem B] again:

$$\begin{aligned}\mathrm{rk}(T)=p-1,\ \mathbf{E}_{\mathcal{F}}=\{T\}\cup\mathcal{B}\ &\Longrightarrow\ \mu_{\mathcal{F}}(W^\vee)\geq\Delta_0 \text{ and } W=O^{p'}(W)\mu_{\mathcal{F}}^{-1}(\Delta_0)\\ \mathrm{rk}(T)=p-1,\ \mathbf{E}_{\mathcal{F}}=\{T\}\cup\mathcal{H}\ &\Longrightarrow\ \mu_{\mathcal{F}}(W^\vee)\geq\Delta_{-1} \text{ and } W=O^{p'}(W)\mu_{\mathcal{F}}^{-1}(\Delta_{-1}).\end{aligned}\tag{9-1}$$

$$\mathrm{rk}(T)\geq p\quad\Longrightarrow\quad\mathbf{E}_{\mathcal{F}}=\{T\}\cup\mathcal{B},\ \mu_{\mathcal{F}}(W^\vee)=\Delta_0,\ \text{and } W=O^{p'}(W)\cdot W^\vee.\tag{9-2}$$

Thus there are up to isomorphism at most two simple fusion systems realizing $(\mathrm{Aut}_{\mathcal{F}}(T),T)$ if $\mathrm{rk}(T)=p-1$, and at most one such system if $\mathrm{rk}(T)\geq p$.

Condition (i) in Theorem 7.4 ($S>T$ and $C_S(T)=T$) holds in all cases since $|S/T|=p$ and $S$ is nonabelian. Condition (ii) ($T$ is not strongly closed in $\mathcal{F}$) holds since $\mathbf{E}_{\mathcal{F}}\nsubseteq\{T\}$, and (iii) holds by Lemma 9.2. Thus by Theorem 7.4, $\mathcal{F}$ can be sequentially realizable only if there is a normal subgroup $H\trianglelefteq\mathrm{Aut}_{\mathcal{F}}(T)$ of index prime to $p$ isomorphic to one of the groups listed in that theorem, and whose action on $\Omega_1(T)$ has the same composition factors as the $\mathbb{F}_pH$-module $M$ listed in Proposition 6.10.

Assume first that $\mathrm{rk}(T)\geq p$. Thus we are in one of cases (a)–(c) in Proposition 6.10, so there is a connected simple $p$-compact group $X$ with maximal discrete $p$-torus $T^*$ and Weyl group $H$. Also, by Lemma 6.12, the $\mathbb{F}_pH$-module $M$ is simple, and hence $\Omega_1(T^*)\cong M\cong\Omega_1(T)$ as $\mathbb{F}_pH$-modules. Let $\mathcal{E}$ be the fusion system of $X$, and set $H^\vee=H\cap W^\vee=\mathrm{Aut}^\vee_{\mathcal{E}}(T)$. By [OR, Theorem B(b)], we have $T^*\cong T$ as $\mathbb{Z}_pH$-modules.

By [COS, Lemma 2.5(a)], $\mathrm{Ker}(\mu_{\mathcal{F}}|_{W^\vee}) = \mathrm{Aut}_S(T) \le H^\vee$. Since $\mu_{\mathcal{F}}(W^\vee) = \Delta_0 = \mu_{\mathcal{E}}(H^\vee)$ by (9-2), this proves that $H^\vee = W^\vee$, and hence that $H = W$ by (9-2) again (and since $O^{p'}(H) = O^{p'}(W)$). We already saw that the fusion system is determined by $(W, T)$, and hence $\mathcal{F} \cong \mathcal{E}$. Thus $\mathcal{F}$ is always the fusion system of a connected simple $p$-compact group in this case.

We are left with the cases where $\mathrm{rk}(T) = p - 1$ and $H \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(T)$ is one of the groups listed in Theorem 7.4. Note that for such $\mathcal{F}$, $Z(S) \le [S, S]$ and hence $\mathrm{Aut}_{\mathcal{F}}^\vee(T) = N_{\mathrm{Aut}_{\mathcal{F}}(T)}(\mathrm{Aut}_S(T))$. We list these cases in Table 9.1.

| $p$ | $\mathrm{Aut}_{\mathcal{F}}(T)$ | $\mathrm{rk}(T)$ | $\mathbf{E}_{\mathcal{F}}$ | seq. realizable? |
|---|---|---|---|---|
| $p \ge 5$ | $\Sigma_p$ | $p{-}1$ | $\{T\} \cup \mathcal{H}$ | $PSL_p(\overline{\mathbb{F}}_q)$ |
| 3 | $\boldsymbol{ST_{12}} \cong GL_2(3)$ | 2 | $\{T\} \cup \mathcal{B}$ | ${}^2F_4(\mathbb{F}_{2^{\langle 3 \rangle}})$ |
| 3 | $\boldsymbol{ST_{12}} \cong GL_2(3)$ | 2 | $\{T\} \cup \mathcal{H}$ | not seq. real. |
| 5 | $\boldsymbol{ST_{31}} \cong (C_4 \circ 2^{1+4}).\Sigma_6$ | 4 | $\{T\} \cup \mathcal{B}$ | $E_8(\mathbb{F}_{2^{\langle 5 \rangle}})$ |
| 5 | $\boldsymbol{ST_{31}} \cong (C_4 \circ 2^{1+4}).\Sigma_6$ | 4 | $\{T\} \cup \mathcal{H}$ | not seq. real. |

Table 9.1. In each case, either the entry in the last column either is a linear torsion group that realizes the fusion system $\mathcal{F}$ determined by the information in the other columns, or $\mathcal{F}$ is not sequentially realizable. In Case 1, $q$ is an arbitrary prime different from $p$. See Notation 8.6 for the definition of the fields $\mathbb{F}_{2^{\langle 3 \rangle}}$ and $\mathbb{F}_{2^{\langle 5 \rangle}}$.

**$\mathrm{Aut}_{\mathcal{F}}(T)$ contains $\Sigma_p$ for $p \ge 5$.** Again set $W = \mathrm{Aut}_{\mathcal{F}}(T)$, and assume that $H \trianglelefteq W$ has index prime to $p$, that $H \cong \Sigma_p$, and that the action of $H$ on $\Omega_1(T)$ has the same composition factors as that of the Weyl group of $PSU(p)$ on the $p$-torsion in its maximal torus. Let $M \cong (\mathbb{F}_p)^p$ be the module with permutation action of $H$, and let $M_1 < M_{p-1} < M$ be the $\mathbb{F}_p H$-submodules of dimension 1 and $p - 1$, respectively. (Thus $M_1 = C_M(H)$ and $M_{p-1} = [H, M]$.) By [OR, Table 6.1], $\Omega_1(T)$ is isomorphic to $M_{p-1}$ or $M/M_1$ as $\mathbb{F}_p H$-modules. By direct computation,

$$\Omega_1(T) \cong M_{p-1} \implies \mu_{\mathcal{F}}(H) = \Delta_0 \qquad \text{and} \qquad \Omega_1(T) \cong M/M_1 \implies \mu_{\mathcal{F}}(H) = \Delta_{-1}.$$

By [OR, Theorem B] and since $M_{p-1}$ contains a 1-dimensional $\mathbb{F}_p W$-submodule, $\mathbf{E}_{\mathcal{F}} = \{T\} \cup \mathcal{H}$ if $\Omega_1(T) \cong M_{p-1}$.

If $\Omega_1(T) \cong M_{p-1}$ and $\mathbf{E}_{\mathcal{F}} = \{T\} \cup \mathcal{H}$ or $\Omega_1(T) \cong M/M_1$ and $\mathbf{E}_{\mathcal{F}} = \{T\} \cup \mathcal{B}$, then the first condition on each line in (9-1) implies that $W = H \times C_{p-1}$. But then $O^{p'}(W)\mu_{\mathcal{F}}^{-1}(\Delta_t)$ ($t = 0, -1$) has index 2 in $W$ in each case, so the second condition fails to hold. So these cases cannot occur, and we are left with the case where $\Omega_1(T) \cong M/M_1$ and $\mathbf{E}_{\mathcal{F}} = \{T\} \cup \mathcal{H}$, where $\mathcal{F}$ is the fusion system of the compact Lie group $PSU(p)$, and is sequentially realizable by Theorem 4.2 and Proposition 3.3.

**$\mathrm{Aut}_{\mathcal{F}}(T)$ contains $\boldsymbol{ST_{12}}$ or $\boldsymbol{ST_{31}}$.** Set $W = \mathrm{Aut}_{\mathcal{F}}(T)$, and let $H \trianglelefteq W$ be such that either $H \cong \boldsymbol{ST_{12}}$, $p = 3$, and $\mathrm{rk}(T) = 2$; or $H \cong \boldsymbol{ST_{31}}$, $p = 5$, and $\mathrm{rk}(T) = 4$. Recall (Lemma 6.8) that $W$ acts faithfully on $\Omega_1(T)$. So $H = W$ if $H \cong \boldsymbol{ST_{12}}$. If $H \cong \boldsymbol{ST_{31}}$, then since $\boldsymbol{ST_{31}}$ is the normalizer in $GL_4(5)$ of $4 \circ 2^{1+4}$, as shown in the proof of Proposition 6.10, we also have $W = H$ in this case.

In both cases, $\mu_{\mathcal{F}}(\mathrm{Aut}^{\vee}_{\mathcal{F}}(T)) = \Delta$. If $\mathbf{E}_{\mathcal{F}} = \{T\} \cup \mathcal{B}$, then $\mathcal{F}$ is the fusion system of a $p$-compact group by Theorem B and Table 5.1 in [OR]. By Theorem 8.7, $\mathcal{F}$ is realized by the group ${}^2F_4(\mathbb{F}_{2^{\langle 3\rangle}})$ or $E_8(\mathbb{F}_{2^{\langle 5\rangle}})$ as listed in Table 9.1.

Now assume $\mathrm{Aut}_{\mathcal{F}}(T) \cong \boldsymbol{ST_{12}}$ or $\boldsymbol{ST_{31}}$ and $\mathbf{E}_{\mathcal{F}} = \{T\} \cup \mathcal{H}$; we must show that $\mathcal{F}$ is not sequentially realizable. Assume otherwise, and let $\mathcal{F}_1 \le \mathcal{F}_2 \le \cdots$ be an increasing sequence of realizable fusion subsystems of $\mathcal{F}$ over finite subgroups $S_1 \le S_2 \le \cdots$ of $S$ such that $S = \bigcup_{i=1}^{\infty} S_i$ and $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$. Let $G_1, G_2, \ldots$ be finite groups such that $S_i \in \mathrm{Syl}_p(G_i)$ and $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$ for each $i \ge 1$. Set $T_i = T \cap S_i$.

By Lemma 5.2, there is $n$ such that $S_n \not\le T$ and $S_n \ge \Omega_2(T)$, and also $T_n \ntrianglelefteq \mathcal{F}_n$ and $\mathrm{Aut}_{\mathcal{F}_n}(T_n) \cong \mathrm{Aut}_{\mathcal{F}}(T)$. Thus $T_n = \Omega_k(T)$ for some $k \ge 2$ since $\mathrm{Aut}_{\mathcal{F}}(T)$ acts irreducibly on $\Omega_1(T)$. By assumption, $\mathbf{E}_{\mathcal{F}} \smallsetminus \{T\} = \mathcal{H}$: the set of subgroups of $S$ not contained in $T$ and isomorphic to $C_p \times C_p$. Hence no extraspecial subgroups of order $p^3$ can be essential in $\mathcal{F}_n$, so $\mathbf{E}_{\mathcal{F}_n} \subseteq \{T_n\} \cup \mathcal{H}$. Also, $\mathbf{E}_{\mathcal{F}_n} \supsetneqq \{T_n\}$ since $T_n \ntrianglelefteq \mathcal{F}_n$.

We now check that $\mathcal{F}_n$ is reduced using the criteria in [COS, Lemma 2.7].

- By point (a) in that lemma, $O_p(\mathcal{F}) = 1$ if there are no nontrivial $\mathrm{Aut}_{\mathcal{F}_n}(T_n)$-invariant subgroups of $Z(S_n)$, and this holds since $\mathrm{Aut}_{\mathcal{F}_n}(T_n)$ acts irreducibly on $\Omega_1(T_n) > Z(S_n)$.
- By point (b) in the lemma, $O^p(\mathcal{F}_n) = \mathcal{F}_n$ since $[\mathrm{Aut}_{\mathcal{F}_n}(T_n), T_n] = T_n$ — again since the action is irreducible.
- By point (c.iii) in the lemma and since $\mathbf{E}_{\mathcal{F}_n} \smallsetminus \{T_n\} \subseteq \mathcal{H}$, $O^{p'}(\mathcal{F}_n) = \mathcal{F}_n$ if
$$\mathrm{Aut}_{\mathcal{F}_n}(S_n) = \big\langle \mathrm{Aut}_{\mathcal{F}_n}(S_n) \cap \mu^{-1}(\Delta_{-1}), \mathrm{Aut}^{(T_n)}_{\mathcal{F}_n}(S_n) \big\rangle, \tag{9-3}$$
where $\mu$ and $\Delta_{-1}$ are defined above and
$$\mathrm{Aut}^{(T_n)}_{\mathcal{F}_n}(S_n) = \big\{\alpha \in \mathrm{Aut}_{\mathcal{F}_n}(S_n) \,\big|\, \alpha|_A \in O^{p'}(\mathrm{Aut}_{\mathcal{F}_n}(T_n))\big\}.$$
One checks in each case that $\mu(O^{p'}(\mathrm{Aut}_{\mathcal{F}_n}(T_n))) = \{(1,s) \,|\, s \in (\mathbb{Z}/p)^\times\} < \Delta$. Since this subgroup together with $\Delta_{-1} = \{(r, r^{-1}) \,|\, r \in (\mathbb{Z}/p)^\times\}$ generates $\Delta$, condition (9-3) does hold, and hence $O^{p'}(\mathcal{F}_n) = \mathcal{F}_n$.

Thus $\mathcal{F}_n$ is reduced, and we are in the situation of case (d.iii) in [COS, Theorem 2.8]. But by Table 2.2 in that paper, such a fusion system is not realizable, contradicting the original assumption that $\mathcal{F}$ is sequentially realizable. □

Note in particular the cases in Theorem 9.3 where $\mathrm{Aut}_{\mathcal{F}}(T) \cong \boldsymbol{ST_{12}}$ $(p = 3)$ or $\boldsymbol{ST_{31}}$ $(p = 5)$. Each of these groups is realized by two different simple fusion systems: one which is realized by a $p$-compact group and by a linear torsion group, and another which is not sequentially realizable.

There are many simple fusion systems over discrete $p$-toral groups $S$ whose identity component $T$ has index $p$ in $S$ that are described by [OR, Theorem B] and are not sequentially realizable. One can get an idea of the many possibilities for $\mathrm{Aut}_{\mathcal{F}}(T)$ by looking at Table 6.1 in [OR], and at Examples 6.2 and 6.3 in the same paper.

## References


[Ag] J. Aguadé, Constructing modular classifying spaces, Israel Jour. Math. 66 (1989), 23–40

[AG] K. Andersen & J. Grodal, The classification of 2-compact groups, Journal Amer. Math. Soc. 22 (2009), 387–436

[AGMV] K. Andersen, J. Grodal, J. Møller, & A. Viruel, The classification of $p$-compact groups for $p$ odd, Annals of Math. 167 (2008), 95–210

[A1] M. Aschbacher, Overgroups of Sylow subgroups in sporadic groups, Memoirs Amer. Math. Soc. 343 (1986)

[A2] M. Aschbacher, Finite Group Theory, Cambridge Univ. Press (1986).

[AKO] M. Aschbacher, R. Kessar, & B. Oliver, Fusion systems in algebra and topology, Cambridge Univ. Press (2011)

[BGM] M. Barratt, V. Gugenheim, & J. Moore, On semisimplicial fibre-bundles, Amer. J. Math. 81 (1959), 639–657

[Bb2-3] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 2 et 3, Hermann (1972)

[Bb4-6] N. Bourbaki, Groupes et algèbres de Lie, Chapitres 4, 5, et 6, Hermann (1968)

[Bb9] N. Bourbaki, Groupes et algèbres de Lie, Chapitre 9, Hermann (1982)

[BK] P. Bousfield & D. Kan, Homotopy limits, completions, and localizations, Lecture notes in math. 304, Springer-Verlag (1972)

[BD] T. Bröcker & T. tom Dieck, Representations of compact Lie groups, Springer-Verlag (1985)

[BLO1] C. Broto, R. Levi, & B. Oliver, Homotopy equivalences of $p$-completed classifying spaces of finite groups, Inventiones math. 151 (2003), 611–664

[BLO2] C. Broto, R. Levi, & B. Oliver, The homotopy theory of fusion systems, J. Amer. Math. Soc. 16 (2003), 779–856

[BLO3] C. Broto, R. Levi, & B. Oliver, Discrete models for the $p$-local homotopy theory of compact Lie groups and $p$-compact groups, Geometry & Topology 11 (2007) 315–427

[BLO6] C. Broto, R. Levi, & B. Oliver, An algebraic model for finite loop spaces, Algebraic & Geometric Topology, 14 (2014), 2915–2981

[BM] C. Broto & J. Møller, Chevalley $p$-local finite groups, Algebr. Geom. Topology 7 (2007), 1809–1919

[BMO1] C. Broto, J. Møller, & B. Oliver, Equivalences between fusion systems of finite groups of Lie type, Jour. Amer. Math. Soc. 25 (2012), 1–20

[BMO2] C. Broto, J. Møller, & B. Oliver, Automorphisms of fusion systems of finite simple groups of Lie type, Memoirs Amer. Math. Soc. 1267 (2019), 1–117

[Br] K. Brown, Cohomology of groups, Springer-Verlag (1982)

[Car] R. Carter, Simple groups of Lie type, John Wiley & Sons (1972)

[CE] A. Clark & J. Ewing, The realization of polynomial algebras as cohomology rings, Pacific J. Math. 50 (1974), 425–434

[Atlas] J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson, ATLAS of finite groups, Clarendon Press (1985)

[Cr] D. Craven, The theory of fusion systems, Cambridge Univ. Press (2011)

[COS] D. Craven, B. Oliver, & J. Semeraro, Reduced fusion systems over $p$-groups with abelian subgroup of index $p$: II, Advances in Mathematics 322 (2017), 201–268

[DW2] W. Dwyer & C. Wilkerson, Homotopy fixed-point methods for Lie groups and finite loop spaces, Annals of Math. 139 (1994), 395–442

[DW3] W. Dwyer & C. Wilkerson, Product splittings of $p$-compact groups, Fund. Math. 147 (1995), 279–300

[DW4] W. Dwyer & C. Wilkerson, The center of a $p$-compact group, The Čech centennial, Contemporary Math. 181 (1995), 119–157

[Fr] E. Friedlander, Étale homotopy of simplicial schemes, Annals of Mathematics Studies, vol. 104, Princeton University Press (1982).

[FM] E. Friedlander & G. Mislin, Cohomology of classifying spaces of complex Lie groups and related discrete groups, Comment. Math. Helv. 59 (1984), 347–361

[FP] E. Friedlander & B. Parshall, Etale cohomology of reductive groups, Algebraic $K$-theory, Evanston, 1980, Lecture notes in math. 854, Springer-Verlag (1981)

[Gz] A. Gonzalez, Finite approximations of $p$-local compact groups, Geom. & Topol. 20 (2016), 2923–2995

[G] D. Gorenstein, Finite groups, Harper & Row (1968)

[GL] D. Gorenstein & R. Lyons, The local structure of finite groups of charateristic 2 type, Memoirs Amer. Math. Soc. 276 (1983)

[GLS3] D. Gorenstein, R. Lyons, & R. Solomon, The classification of the finite simple groups, nr. 3, Amer. Math. Soc. surveys and monogr. 40 #3 (1997)

[Ho] G. Hochschild, The structure of Lie groups, Holden-Day (1965)

[JMO] S. Jackowski, J. McClure, & B. Oliver, Self homotopy equivalences of classifying spaces of compact connected Lie groups, Fundamenta math. 147 (1995), 99–126

[Jm] G. D. James, The representation theory of the symmetric groups, Springer-Verlag (1978)

[Jn] Z. Janko, Some new simple groups of finite order, I, Symposia mathematica I (1968), 25–64

[BrAtl] C. Jansen, K. Lux, R. Parker, & R. Wilson, An atlas of Brauer characters, Oxford Univ. Press (1995)

[KW] O. H. Kegel & B. A. F. Wehrfritz, Locally finite groups. North-Holland Mathematical Library, Vol. 3. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973

[KMS1] R. Kessar, G. Malle, & J. Semeraro, Weights for compact connected Lie groups, manuscripta math. 174 (2024), 1059–1073

[KMS2] R. Kessar, G. Malle, & J. Semeraro, Weight conjectures for $\ell$-compact groups and spetses, Ann. Sci. Ecole Norm. Sup. 57 (2024), 841–894

[Kl] P. Kleidman, The maximal subgroups of the Steinberg triality groups ${}^3D_4(q)$ and of their automorphism groups, J. Algebra 115 (1988), 182–199

[LO] R. Levi & B. Oliver, Construction of 2-local finite groups of a type studied by Solomon and Benson, Geometry & Topology 6 (2002), 917–990

[LO2] R. Levi & B. Oliver, Correction to: Construction of 2-local finite groups of a type studied by Solomon and Benson, Geometry & Topology 9 (2005), 2395–2415

[Ma] G. Malle, The maximal subgroups of ${}^2F_4(q)$, J. Algebra 139 (1991), 52–69

[MTo] M. Mimura & H. Toda, Topology of Lie groups I and II, Transl. Math. Monographs 91, Amer. Math. Soc. (1991)

[O1] B. Oliver, Simple fusion systems over $p$-groups with abelian subgroup of index $p$: I, J. Algebra 398 (2014), 527–541

[O2] B. Oliver, Automorphisms of fusion systems of sporadic simple groups, Memoirs Amer. Math. Soc. 1267 (2019), 119–163

[OR] B. Oliver & A. Ruiz, Reduced fusion systems over $p$-groups with abelian subgroup of index $p$: III, Proceedings A of the Royal Society of Edinburgh 150 (2020), 1187–1239

[Pa] D. Parrott, A characterization of the Tits' simple group, Can. J. Math. 24 (1972), 672–685

[Pg] L. Puig, Frobenius categories, J. Algebra 303 (2006), 309–357

[Q] D. Quillen, On the cohomology and $K$-theory of the general linear groups over a finite field, Annals of Math. 96 (1972), 552–586

[R] R. Ree, A family of simple groups associated with the simple Lie algebra of type $(F_4)$, Amer. Jour. Math. 83 (1961), 401–420

[RV] A. Ruiz & A. Viruel, The classification of $p$-local finite groups over the extraspecial group of order $p^3$ and exponent $p$, Math. Z. 248 (2004), 45–65

[Se] J.-P. Serre, Représentations linéaires des groupes finis, Hermann (1967)

[Sh] G. Shephard, Abstract definitions for reflection groups, Canadian Math. Jour. 9 (1957), 273–276

[ST] G. C. Shephard & J. A. Todd, Finite unitary reflection groups, Canadian J. Math. 6 (1954), 274–304

[Sp] T. A. Springer, Linear algebraic groups, 2nd ed., Birkhäuser (1998)

[St1] R. Steinberg, Variations on a theme of Chevalley, Pacific J. Math. 9 (1959), 875–891

[St2] R. Steinberg, Générateurs, relations, et revêtements de groupes algébriques, Colloquium on algebraic groups, Brussels (1962), 113–127

[Wr] A. J. Weir, Sylow $p$-subgroups of the classical groups over finite fields with characteristic prime to $p$, Proc. Amer. Math. Soc. 6 (1955), 529–533

[Za] A. Zabrodsky, On the realization of invariant subgroups of $\pi_*X$, Trans. Amer. Math. Soc. 285 (1984), 467–496

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