Université Pierre et Marie Curie

Mémoire d'Habilitation à Diriger des Recherches

(Spécialité : Mathématiques)

Autour de la topologie algébrique des "espaces fonctionnels"

Grégory Ginot

Soutenue le 9 décembre 2011 devant le Jury :

- Bernhard KELLER (Université Denis Diderot)
- Patrick POLO (Université Pierre et Marie Curie)
- Geoffrey POWELL (Université Paris 13)
- Pierre SCHAPIRA (Université Pierre et Marie Curie)
- Lionel SCHWARTZ (Université Paris 13)
- Bertrand TOËN (Université Montpellier II)

Rapporteurs :

- *Kevin COSTELLO (Northwestern)*
- *Dennis SULLIVAN (Stony Brook et City University of New York)*
- *Bertrand TOËN (Université Montpellier II)*

Remerciements

Je tiens, avant toute autre chose, à remercier très chaleureusement les rapporteurs, Kevin Costello, Dennis Sullivan et Bertrand Toën, non seulement pour avoir accepté et m'avoir fait l'honneur d'écrire ces rapports, mais aussi pour les enrichissantes discussions que j'ai eues avec eux, et, enfin et surtout, pour leurs nombreux travaux qui ont, à l'évidence, considérablement inspiré et influencé les miens.

Je remercie bien évidemment aussi les (autres) membres du Jury qui ont réussi à se libérer (un peu au dernier moment il faut l'avouer), c'est à dire Bernhard Keller (qui avait déjà accepté d'écrire un rapport sur ma thèse au temps de ma jeunesse mathématique), Patrick Polo, Geoffrey Powell, Lionel Schwartz et Pierre Schapira (dont les nombreux encouragements, conseils et quelques critiques m'ont été précieux pendant ces 5 dernières années).

Merci aussi à Micheline Vigué qui a motivé et encouragé avec une bienveillance quasi-maternelle mon intérêt pour la topologie des cordes et à Gillou qui a suscité mon intérêt pour les déformations, m'a donné le goût des collaborations mathématiques et a surtout été un ami toujours présent. Je ne saurais oublier dans ces remerciements Christian Kassel qui m'a mis le pied à l'étrier (et botté les fesses pour soutenir une habilitation) mais ne pouvait malheureusement pas être présent aujourd'hui.

Mille et un mercis à Berhang, Kai, Mahmoud, Mathieu, Ping et Thomas qui ont collaboré à certains des travaux de ce mémoire et, je l'espère, à de futurs autres... C'est encore et toujours un grand plaisir d'user de la craie sur un tableau avec vous ! ! !

Un grand merci à la (grande et amicale) “famille” A.A qui a activement participé à mes quêtes chocolatées et expériences patissières, mais aussi au DMA empli de trublions fort sympathiques également ! Sans tous ces chouettes collègues, je n'aurais sûrement pas le même plaisir à venir travailler et mon cadre de travail ne serait pas aussi idéal. Parmi eux, il y en a un qui me supporte dans son bureau depuis toutes ces années et ce de manière très chaleureuse, voire tropicale ; muchos gracias à lui.

Enfin, je remercie aussi Patrick Roger, Jacques Génin, Hugues Pouget, Nathalie Robert, Didier Mathray, Carl Marletti, Quentin Bailly, Sadahoru Aoki et beaucoup beaucoup d'autres pour leurs œuvres et avoir maintenu mon taux de cacao et de glucose à un niveau ô combien satisfaisant !

Table des matières

1 Résumé des travaux présentés

Mon domaine de recherche porte sur la topologie algébrique, l'algèbre homotopique et leurs interactions avec la théorie des déformations et les champs.

Ce mémoire décrit mes travaux de recherche (sur la période 2006-2011) [1] consacrés à la topologie algébrique des "espaces fonctionnels" qui trouvent leur origine dans les travaux de Chas-Sullivan en *topologie des cordes* et ceux de Costello, Lurie sur les *théories topologiques des champs*, ainsi que sur des questions connexes relatives à la cohomologie des champs.

Le terme "espace fonctionnel" désigne un espace (topologiques ou de modules) $\mathrm{Map}(\Sigma, X)$ d'applications de Σ vers X. Les guillemets servent ici à rappeler que, la nature de Σ, X (espaces topologiques, variétés différentiables ou complexe ou algébrique ou même champs), mais aussi le type de fonction (pointées ou non, à support ou non, lisse ou continue, dérivée ou non, etc...) dépend des problèmes considérés et du cadre mathématique dans lequel on se place. Un principe général est que ces espaces ont beaucoup de *structures*, ce qui permet, non-seulement d'étudier plus efficacement les espaces source et but (qui en ont en général beaucoup moins), mais aussi de faire des ponts entre différents domaine des mathématiques.

1. ces travaux représentent la majeure partie de mes activités de recherche sur cette période, mais excluent cependant quelques autres travaux

Un premier exemple, ayant inspiré une bonne partie des recherches présentées dans ce mémoire, est la *topologie des espaces de lacets* (dite aussi *topologie des cordes*) qui a été popularisée par les travaux récents de Chas-Sullivan [CS] ayant mis en exergue les riches structures algébriques sur l'homologie (mais aussi, plus en amont, les chaînes) de l'espace des lacets libre $LM = \mathrm{Map}(S^1, M)$ d'une *variété orientée* M, ainsi que sur son quotient par l'action du cercle S^1. En particulier l'homologie de l'espace des lacets est munie d'une structure d'algèbre de Batalin-Vilkoviski, et donc d'algèbre de Gerstenhaber, (structures présentes également sur la cohomologie d'une algèbre vertex) et plus généralement définit une *théorie conforme homologique des champs.* Certaines propriétés des espaces de modules classiques, ou plus précisément des orbifolds et champs sont encodés par leurs *lacets fantômes* (c'est à dire ceux qui sont constants par passage au quotient "grossier" de l'orbifold). Ce point de vue apparait notamment en théorie géométrique des représentations. D'autre part, la cohomologie de l'espace des lacets fantômes d'un orbifold quasi-complexe est munie d'un cup-produit non-trivial, le *cup-produit orbifold*, du à Chen-Ruan [CR], étroitement relié à la cohomologie quantique.

Classiquement, les groupes d'homotopie d'un espace sont encodés par ceux de ses espaces de lacets (n-)itérés (c'est à dire d'applications *pointées* de S^n dans l'espace). Ces derniers héritent d'une structure supplémentaire (qui aident à calculer leur type d'homotopie) : celle d'une *E_n-algèbre.* La donnée d'une E_n-algèbre correspond à celles d'une algèbre différentielle graduée (homotopiquement associative) dont la multiplication est commutative modulo un opérateur d'homotopie qui est lui même commutatif à un opérateur d'homotopie près et ainsi de suite jusqu'à l'ordre n. Une autre façon de voir cela est de penser à la donnée de n structures associatives *compatibles* entre elles. Le principe de reconnaissance de May énonce que les espaces de lacet (n-)itérés sont *caractérisés* par leur structure de E_n-algèbre. Par ailleurs, le *type d'homotopie* d'un espace (nilpotent et de type fini) est contrôlé par une structure d'E_∞-algèbre sur son complexe de cochaines singulières. En caractéristique nulle, ceci est du aux travaux de Quillen et Sullivan : c'est le sujet de l'*homotopie rationnelle.* En caractéristique positive, ce résultat est du aux travaux récents de Mandell.

On peut aussi noter que l'étude locale (ou perturbative) d'une théorie des champs se ramène à l'étude de $Map_X(X, E) := \Gamma(X, E)$ l'espace des sections d'un fibré (différentiel gradué) (muni de structures additionnelles).

Il existe un *modèle algébrique* (plus maniable) de la topologie des cordes, donné par la *(co)homologie de Hochschild* qui est la théorie (co)homologique naturelle associée aux algèbres associatives. En effet, les *intégrales itérées de Chen* [Ch] et la dualité de Poincaré donnent un isomorphisme entre la cohomologie de Hochschild $HH^*(\Omega_M)$ de l'algèbre des formes différentielles Ω_M (sur une variété compacte orientée 1-connexe M) et l'homologie $H_{\bullet+\dim(M)}(LM)$ de l'espace des lacets LM. En fait, la cohomologie de Hochschild de toute algèbre associative a une structure naturelle d'algèbre de Gerstenhaber qui est importante car elle contrôle les déformations. La *conjecture de Deligne* prédit que cette structure se relève en une structure E_2 sur le complexe de cochaînes de Hochschild. La *conjecture de Deligne supérieure* (voir par exemple [KS, L-HA, F]), due à Kontsevich, énonce que les complexes de déformations des E_n-algèbres sont naturellement des E_{n+1}-algèbres.

De même, les travaux de Sullivan-Voronov [CV] suggèrent que les chaînes sur l'espace fonctionnel $\mathrm{Map}(S^n, M)$ d'une variété orientée M sont munies d'une structure E_{n+1}-naturelle ; cette version supérieure, encore mal-comprise, de la topologie des cordes est parfois appelée *topologie des membranes*. Notons aussi que l'homologie de Hochschild s'interprête comme un *espace de lacets dérivés* (ou homotopiques)[TV2] au sens de la *géométrie algébrique dérivée* à la Lurie, Toën-Vezzosi. Par ailleurs, Costello a montré que l'homologie de Hochschild des algèbres de Calabi-Yau correspond aux Théories Topologique Conforme des champs [Co2, L-TFT].

Décrivons maintenant le contenu de ce mémoire. La première partie concerne la topologie algébrique des champs différentiables suivant [1, 4, 3, 2]. En particulier on a établi un cadre général pour la *topologie des cordes des champs différentiables* (Section 4.2) qui englobe aussi bien les variétés standards que les orbifolds et classifiants des groupes de Lie, mais permet aussi de traiter le cas des lacets fantômes. Ceci permet de comparer la topologie des cordes au cup-produit orbifold (Section 4.4). Ce travail est dans la continuité des travaux de Sullivan visant à utiliser la topologie des cordes pour comprendre la "topologie algébrique des variétés" [S] car les champs différentiables sont également des objets géométriques non-singuliers (vu en tant que champs). En utilisant notre cadre et le formalisme des champs, nous avons établi que l'homologie de l'espace des lacets libres d'un champ orienté a une structure naturelle de *théorie homologique conforme des champs* (§ 4.3). Au delà de ce résultat, nous avons développé différentes techniques utiles dans un cadre plus général pour les champs : en particulier on a établi l'existence d'une *théorie bivariante* (généralisée) pour les champs topologiques, c'est à dire une théorie englobant homologie, cohomologie, dualité de Poincaré et surtout qui permet de construire efficacement différentes opérations (co)homologiques ; par exemple on obtient un formalisme maniable des morphismes de Gysin (que l'on peut notamment étudier en famille et via des tirés-en-arrière). On a de plus étudié une notion *d'orientation* pour des champs (§ 4.2.3).

Notre formalisme champêtre permet de munir l'espace des lacets libres sur un champ d'une action naturelle du cercle dont l'étude (ou de celle de l'action du champ en groupe $B\mathbb{Z}$ sur les lacets fantômes) se ramène à celle de l'action de *2-groupes* (au sens catégorique) sur un champ. Un fibré sur un 2-groupe (de Lie) est une notion qui englobe les fibrés principaux en groupes usuels, mais aussi les *gerbes* (au sens de Giraud). Ceci a été une des mes motivations pour étudier les notions de *fibrés principaux en 2-groupes sur un champ* (et pas seulement sur une variété), cf § 4.5. En particulier une caractérisation des gerbes (de groupe G) et gerbes centrales en tant que fibrés principaux sur les 2-groupes $[G \to \mathrm{Aut}(G)]$ et $[Z(G) \to 1]$ respectivement est donnée Section 4.5. Une construction d'une théorie (de nature homotopique) des *classes caractéristiques* des 2-fibrés principaux ainsi qu'une construction *géométrique* à la *Chern-Weil* pour les gerbes centrales est donnée en Section 4.6. En remarquant, que tout 2-groupe de Lie est l'espace total d'une fibration de base un classifiant de groupe de Lie et de fibre le classifiant du classifiant d'un groupe de Lie abélien, j'y décris aussi une suite spectrale permettant de calculer la cohomologie des 2-groupes, donne un morphisme à la Bott-Shulman, et relie la cohomologie du 2-groupe $[G \to \mathrm{Aut}(G)]$ à la cohomologie de $SL(n, \mathbb{Z})$.

La deuxième partie du mémoire est consacrée à (des généralisations de) la (co)homologie de Hochschild, vue comme espace fonctionnel (dérivé) et ses applications à la topologie des cordes et des membranes. Cette partie s'appuie sur les articles [5, 6, 7, 8, GTZ3]. En particulier, motivé par le fait que le complexe des cochaînes singulières d'un espace n'est pas qu'une simple algèbre associative mais peut être muni d'une structure *d'algèbre commutative et associative homotopique*, j'ai étudié la (co)homologie de Hochschild (et de Harrison) de ces algèbres homotopiques pour les appliquer à la topologie des cordes d'un espace à dualité de Poincaré (Section 5.2). Ceci m'a permis de munir la topologie des cordes d'une *décomposition de Hodge* (cf § 5.2.2), ou de manière équivalente d'actions d'opérations d'Adams compatibles avec la structure d'algèbre de Batalin-Vilkoviski usuelle (§ 5.2.3).

Dans la Section 5.3 on construit des des généralisations des *intégrales itérées de Chen pour tout espace fonctionnel*. Cette construction est basée sur la *(co)homologie de Hochschild supérieure* développée entre autres dans [P] et dans mes travaux [6, 7, 8]. Cette théorie associe *fonctoriellement* à tout espace (simplicial) X et algèbre commutative A, une algèbre commutative (différentielle graduée) $CH_X(A)$ (ou un complexe $CH^X(A)$ en cohomologie). La fonctorialité de ces théories, leur invariance homotopique et la flexibilité des résolutions obtenues, fait que cette théorie est parfaitement adaptée pour travailler au niveau des *complexes de chaines à homotopie près* (dans le cadre des (∞-)catégories dérivées), voir § 5.3.1 et § 5.4.1. Ces principes s'appliquent facilement à la topologie des cordes (et des membranes). Par exemple, dans la Section 5.3.3, on construit et étudie le *produit surfacique*, un analogue pour les surfaces du produit de Chas-Sullivan pour les lacets en combinant le point de vue algébrique et topologique. On donne (§ 5.4) une *caractérisation axiomatique* de l'homologie de Hochschild supérieure, similaire dans l'esprit aux axiomes d'Eilenberg-Steenrod, mais où *l'axiome d'excision* est remplacé par un axiome de localité similaire à celui des théories topologiques des champs au sens d'Atiyah-Segal. En fait, la théorie de Hochschild supérieure peut être vue comme la limite pour n allant vers l'infini d'une théorie cohomologique définie par Lurie, appelée *homologie chirale topologique* qui provient des théories des champs topologiques étendues (cf § 5.4). Dans ce dernier cadre, si on se restreint à travailler avec des variétés de dimension n et des plongements (à la place de tous les espaces topologiques et applications continues), on peut alors utiliser les E_n-algèbres à la place des algèbres commutatives. On relie (§ 5.5) l'homologie chirale de Lurie et les *algèbres de factorisation* à la Costello (qui sont la structure décrivant les observables (quantiques) de théories perturbatives des champs), puis on relie la cohomologie de Hochschild $CH^{S^n}(A)$ au dessus des sphères aux déformations en E_n-algèbres. On prouve ensuite (§ 5.6.3), pour les algèbres commutatives différentielles graduées, une *variante relative de la conjecture de Deligne supérieure* à savoir que les complexes de cochaines relatifs de Hochschild au dessus de S^n associés à un morphisme $f : A \to B$ d'algèbres sont naturellement munis d'une structure de E_n-algèbre. Ce résultat (paru dans [6]) se réinterprète, à la lumière des travaux plus récents de Lurie [L-HA], comme le centralisateur de f (dans la catégorie des E_n-algèbres). Appliqué à $f = id$, ceci donne une démonstration des conjectures de Deligne supérieures (pour les algèbres commutatives vues comme E_n-algèbres). Finalement, on applique les techniques précédentes aux décompositions de Hodge des cohomologies de Hochschild supérieures (§ 5.6.4) et surtout pour donner un modèle algébrique, au *niveau des chaînes* de la *topologie des membranes* (cf Section 5.6.5).

2 Liste des travaux présentés

La bibliographie présentée ci-dessous regroupe les articles reprenant mes travaux exposés dans ce mémoire. Ils sont numérotés de 1 à 8. Une seconde bibliographie se trouve à la fin de ce mémoire et réfère (par des lettres) les autres articles cités. Mes travaux en cours [GiNo, GTZ3] seront également évoqués.

Références

[1] K. Behrend, G. Ginot, B. Noohi, P. Xu, *String topology for loop stacks*, C. R. Acad. Sci. Paris, Ser. I **344** (2007).

[2] G. Ginot, M. Stiénon, *G-gerbes, Principal 2-group bundles and Characteristic Classes* (29 pages), Preprint arXiv :0801.1238 (SOUMIS)

[3] G. Ginot, P. Xu, *Cohomology of Lie 2-groups* avec P. Xu, L'Enseignement Mathématique **55** (2009) no2, 1–24.

[4] K. Behrend, G. Ginot, B. Noohi, P. Xu, *String topology for stacks*, (160 pages) à paraître dans Astérisque - SMF.

[5] G. Ginot, *On the Hochschild and Harrison (co)homology of C_∞-algebras and applications to string topology*, in *Deformation spaces*, Aspects Math., E40 (2010), 1–51.

[6] G. Ginot, *Higher Order Hochschild Cohomology*, C. R. Acad. Sci. Paris, Ser. I **346** (2008)

[7] G. Ginot, T. Tradler, M. Zeinalian, *A Chen model for mapping spaces and the surface product*, Ann. Scient. Éc. Norm. Supér. (4) **43**,(2010), 811– 881.

[8] G. Ginot, T. Tradler, M. Zeinalian, *Derived Higher Hochschild Homology, Topological Chiral Homology and Factorization Algebras*, (59 pages), Preprint arXiv :1011.6483 (SOUMIS).

3 Introduction and Overview

This text is based on (most of) my recent work which were motivated by a desire to understand algebraic topology of "mapping spaces" and its current development as initiated by the work of Chas-Sullivan[CS, S] on *string topology* and Costello and Lurie on *topological field theories*. This work also raised some interest and side questions on algebraic topology of stacks. The relevant research was spread from a period from 2006 to 2011, and the relevant papers [2] are listed, *with numbers*, on the first bibliography, see Section 2. Other references cited in the text will be refereed to using letters and can be found at the end of the *mémoire*.

2. which excludes a couple of other papers from this period on only loosely related questions

A basic principle, is the following: the loop space (of a manifold) inherits a lot of "algebraic structure" which allows to study the manifold and many geometric information or invariant relevant to it. This idea is a close relative of the ideas giving rise to invariant of manifolds from field theories. I have been following this principle not only in the framework of string topology, but also in two different generalizations. On one side, one can replace manifolds by other kind of geometric objects arising naturally in algebraic topology or geometry; in particular stacks. On the other side, one can replace loops by *derived* (or *homotopical*) variants (for instance Hochschild or even chiral homology), or by using different (from the circle) spaces as the source of mapping spaces (as in Brane Topology and Quantum Field Theories).

Let us be now more precise about the specific content of this text. The first part is about algebraic topology of differentiable stacks and string topology. It is based on the papers [1, 3, 2] and the monograph [4]. Many interesting geometric objects in (algebraic or differential) geometry or mathematical physics are *not* manifolds. There are, for instance, orbifolds, classifying spaces of compact Lie groups, or, more generally, global quotients of a manifold by a Lie group. All these examples belong to a common natural generalization of smooth manifolds: differentiable stacks [BX].

One important feature of differentiable stacks is that they are *non-singular*, when viewed as stacks (even though their associated coarse spaces are typically singular). For this reason, one can still do differential geometry with them and further, algebraic topology of stacks behaves much like for manifolds. In [1, 4] we develop a framework for string topology of (oriented) stacks. *String topology* is a term coined by Chas-Sullivan [CS] to describe the rich algebraic structure on the homology of the free loop manifold $\mathrm{L}M = \mathrm{Map}(S^1, M)$ of an oriented manifold M. The algebraic structure in question is induced by geometric operations on loops such as gluing or pinching of loops. These structure can be described in terms of *homological conformal field theory*, that is, inherit spaces of applications parametrized by the homology of mapping class groups. The most studied operations are those given by a Batalin-Vilkovisky (**BV** for short) algebra structure and a Frobenius algebra structure. They are known to be related to many subjects in mathematics and in particular mathematical physics [S, CFP, AZ, CV] and also have many interesting algebraic analogues related to deformation quantization of associative algebras [5, TZ, KS]. To develop string topology operations for stacks (Section 4.3), we solve three issues, of independent interest for stacks (see Section 4.2): namely we study mapping stacks endowed with the correct topology and functoriality properties, we gave a notion of oriented stacks generalizing the notion of manifolds and more important, we construct a *bivariant theory for topological stacks*. This is a theory encompassing homology and cohomology as well as all standard operations and a flexible framework for Gysin maps. This theory allows us to define Gysin maps in families which is key to our string topology operations. Further, new kinds of interesting "loops" are naturally arising when studying stacks. Indeed, several fine invariants [4, CR, JKK] of a stack $\mathfrak{X}$ are obtained by considering its associated *inertia stack* $\Lambda\mathfrak{X}$ (also called the stack of hidden or ghost loops) which, roughly, is the stack of automorphisms of the underlying stack $\mathfrak{X}$. They can somehow be thought of as loops inside the stack which vanish on the associated coarse space. In particular, we apply our bivariant theory to study (Section 4.4) an intersection product for almost complex orbifolds which is Poincaré dual to the orbifold cup-product of Chen-Ruan [CS] as well as to study some string topology operations for the inertia stack $\Lambda\mathfrak{X}$ associated to an oriented stack $\mathfrak{X}$.

In particular, our framework (§ 4.2) allows to treat on an equal footing free loops and hidden loops, but also string topology for manifolds, orbifolds or classifying space of Lie groups using similar geometric arguments see § 4.3.

The free loop stack $\mathrm{L}\mathfrak{X}$ has an natural S^1-action and the inertia stack has an action of the quotient stack $\mathcal{B}\mathbb{Z} = [*/\mathbb{Z}]$ which is a *group stack* or more simply a (Lie) 2-group. Similarly, as a group stack, S^1 is equivalent to the (2-stack associated to the) 2-group $[\mathbb{Z} \to \mathbb{R}]$ which describes the "group structure" of the quotient stack $[\mathbb{R}/\mathbb{Z}]$. One of the original motivations of the work of Chas-Sullivan [CS] was to study the S^1-*equivariant* homology of the loop space LM of a manifold recovering and generalizing a natural Lie bialgebra structure given by Goldman bracket [Go] and Turaev cobracket [T] on loops on surfaces. This motivates us to study 2-group actions on stacks (or Lie groupoids) and will be studied in [GiNo]. Nevertheless, there are far more reason to be interested in 2-groups actions since they arise naturally in mathematical physics, for instance, in higher gauge theory [BS, BCSS], to describe the parallel transport of strings [MP, ACJ, BS] as well as studying gerbes [ACJ, G, 2]. In Section 4.5, following [2], we defined a notion of Principal 2-group bundles over a Lie groupoid (or differentiable stack) and proved that G-gerbes[G, Br] (are the same as principal 2-group bundles over $[G \to \mathrm{Aut}(G)]$. Similarly, G-bound gerbes (or central gerbes) are the same as G-gerbes whose structure 2-group reduces to the 2-group $[Z(G) \to 1]$, where $Z(G)$ is the center of G. Our definition of principal 2-group bundle yields immediately a theory of characteristic classes for principal 2-group bundles (see Section 4.6) which generalizes the construction of characteristic classes of a principal bundle over manifold as pullback of cohomology classes of the classifying space of G. Following [2], we also gave a *geometric* construction of characteristic classes for central gerbes using connection and curvature data (in the spirit of Chern-Weil theory). Then we prove that the two construction agrees. Finally, following our paper [3], to complete the study of characteristic classes, we investigate the cohomology of Lie 2-groups using a spectral sequence. The idea, roughly, is that a Lie 2-group is the total space of a fibration whose base is the classifying space of a Lie group while its fibers are the classifying space of the classifying space of an abelian Lie group. In particular we gave Bott-Shulman type map, explicit computations for some 2-group and relate the cohomology of the 2-group $[G \to \mathrm{Aut}(G)]$ associated to a gerbe to the cohomology of $SL_n(\mathbb{Z})$.

The second part of the text, based on the papers [5, 6, 7, 8, GTZ3], deals with *Hochschild (co)homology theory* and its generalization which can be seen as *derived mapping spaces*. We are in particular interested to the applications to string topology and brane topology which were our main motivation to develop higher Hochschild theory (after [P]) in [6, 7, 8, GTZ3]. Hochschild (co)homology theory is the "natural" (co)homology theory of (associative) algebras which, for instance, controls deformations of commutative algebras into associative algebras.

It is heavily used in deformation quantization, algebraic and non-commutative geometry or algebraic topology. For instance Hochschild (co)homology is an *algebraic model* for string topology operations. Indeed, the Hochschild cohomology $HH^\bullet(C^*(M), C^*(M)$ of the cochain algebra of a closed manifold M is isomorphic to $H_{\bullet+\dim(M)}(LM)$ as a Gerstenhaber algebra and also carries a **BV**-structure. Such an isomorphism can be induced by Chen *iterated integrals* (and Poincaré duality). There are other evidences that Hochschild (co)homology (or cochain complex) theory is closely related to the circle (or loop space). For instance, the Connes operator giving rise to the cyclic homology theory defines an action of $H_\bullet(S^1)$ on Hochschild chains $CH^{st}(A)$ of any algebra A. Further, Hochschild chains of Calabi-Yau (A_∞-)categories is the evaluation on a circle of a Topological Conformal Field Theory [Co2]. The latter observation also holds for extended Topological Field Theories [L-TFT] and also yields an natural "homotopy/derived" circle action on Hochschild chains, also see [TV2]. For *commutative algebras*, Hochschild (co)homology has additional structure given by the Adams operations, which in turns yield a Hodge decomposition [Lo2].

Note that the cochain algebra can be made into a *homotopy commutative* algebra. This was a motivation to study Hochschild (and other natural (co)homology theory) for homotopy commutative algebra (or more precisely C_∞-algebras) in [5]. For instance, we proved that their Hochschild (co)homology inherit Adams operations and Hodge decomposition which are compatible with string topology operations in the case of singular cochains of Poincaré duality spaces, see § 5.2. The aforementioned [L-TFT, Co2, CoGw] relationship between string topology operations, loop spaces and topological field theories can be pushed forward. In fact, recently, several concepts integrating (higher) categories of spaces or manifolds with those of algebras of different types have arisen. *Higher Hochschild homology*, introduced by Pirashvili [P] and also developed in our work [6, 7, 8], is a kind of *limit* of these ideas when the dimension of the TFT goes to infinity. In contrast with most others generalizations, higher Hochschild chains are defined over *any* (simplicial set model of a) space and not only (stratified) manifolds. However, this forces us to restrict our attention to CDGAs or at best E_∞-algebras. From an algebraic topology perspective, this restriction is not a big issue since the cochain complex $C^*(X)$ is indeed an E_∞-algebra (and has commutative models in characteristic zero).

Higher Hochschild (co)homology is modeled over spaces in the same way the usual Hochschild (co)homology is modeled on circles. More precisely, we define in Section 5.3 a rule which associate to any space X, commutative algebra A and A-module M, homology groups $HH_X(A, M)$ and in fact chain complexes $CH_X(A, M)$ *functorial in every argument*, such that for $X = S^1$, one recovers the usual Hochschild homology. The functoriality with respect to spaces is a key feature which allows us to derive algebraic operations on the higher Hochschild chain complexes from maps of topological spaces. For instance Adams operations studied in Section 5.2 can be interpreted as induced by *continuous* but *not smooth* functions. This was actually the main motivation of Pirashvili [P]. Similarly we applied the naturality of the functor in both arguments to study string (and higher dimensional generalizations) topology as we will explain in Section 5.6.

In Section 5.3, we explain how to generalize the classical Chen iterated integrals from loop and path spaces to all mapping spaces and how to apply it to study the *surface* product, a string topology type product for surfaces in place of loops. In Section 5.4, following [8], we explain how an analogue of the *excision axiom* holds for higher Hochschild chains and allows to compare this theory to *chiral homology* in the sense of Lurie [L-HA, L-VI]. The latter is an homology theory for E_n-algebras (or algebras over the little dimension n-cubes operad) closely related to extended Field Theories [L-TFT] which was inspired by the work of Beilinson-Drinfeld [BD]. In Section 5.5, we compare chiral homology and factorization algebras (the algebraic structure introduced to describe Quantum Field Theories also inspired by [BD]). In Section 5.6, we then apply the higher Hochschild formalism to get operations such as the *wedge product* and a description of E_n-centralizers of commutative algebras map . The latter is a relative version of *Higher Deligne conjecture* stating that the Hochschild cochains over the n-sphere S^n of a map of algebras is naturally an E_n-algebra that we originally proved in this form in [6] before the beautiful notion of centralizers was available in [L-VI]. Following an idea of Lurie [L-HA], it can be applied to prove the higher Deligne conjecture, see Section 5.6.3. Finally, in Section 5.6.5 we apply our techniques to get *chain level* models for Brane topology operations, *i.e.*, an natural E_{n+1}-algebra structure on $C_*(\mathrm{Map}(S^n, M))$.

To sum-up, the *philosophy* is that Hochschild complexes $CH^X(A, A)$ should be thought of as some kind of functions on a "mapping space" from X to some "derived space " and the *gain* is algebraic structures/operators induced by maps of spaces as well as algebraic models for mapping spaces and new invariants for spaces and algebras.

4 Algebraic topology of Differentiable Stacks and String Topology

We refer to [4, BX, No2] for details on topological and geometric stacks. The most important thing to keep in mind is that the Examples 4.6 below are differentiable (or topological) stacks.

Remark 4.1 We will mainly consider two categories of stacks (which are all subcategories of stacks over Top). Namely, we will consider *topological stacks* (Definition 4.3), which are the stacks on which we can do algebraic topology and the category of *differentiable stacks* (Definition 4.5) on which we can do geometry.

4.1 Brief introduction to Differentiable and Topological stacks

Unless otherwise stated, by a stack we mean a stack over the site Top of compactly generated topological spaces (with the standard Grothendieck topology). This means, a stack is a category $\mathfrak{X}$ fibered in groupoids over Top satisfying the *descent condition* (see [4, Appendix A]). Alternatively, we can think of $\mathfrak{X}$ as a *presheaf of groupoids* which satisfies the descent condition. Roughly, this means, that for each topological space T, we are given a (discrete) groupoid $\mathfrak{X}(T)$ called the *fiber* *of* $\mathfrak{X}$ *over* T (or the groupoids of T*-points* of $\mathfrak{X}$). The descent condition is a kind of (categorical) "sheaf condition" which implies that the groupoid $\mathfrak{X}(T)$ can be reconstructed from its restrictions to the groupoids $\mathfrak{X}(U_i)$ when $\{U_i\}$ is an open cover of T. In particular given objects $X_i \in \mathfrak{X}(U_i)$, isomorphisms $\phi_{ij} : X_i|_{U_i \cap U_j} \to X_j|_{U_i \cap U_j}$ in $\mathfrak{X}(U_i \cap U_j)$ satisfying the "cocycle condition" $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$, there is a unique (up to unique isomorphism) object $X \in \mathfrak{X}(T)$ with isomorphisms $\phi_i : X|_{U_i} \to X_i$ such that $\phi_{ij} \circ \phi_i = \phi_j$ (the data of the ϕ_{ij} with the cocycle condition is called a *gluing data*).

We now list basic properties of stacks over Top. Stacks over Top form a 2-category in which 2-morphisms are invertible. Therefore, given two stacks $\mathfrak{X}$ and $\mathfrak{Y}$, we have the *groupoid* $\mathrm{Hom}(\mathfrak{Y}, \mathfrak{X})$ of morphisms between them. By the Yoneda lemma, there is a canonical equivalence $\mathfrak{X}(T) \cong \mathrm{Hom}(T, \mathfrak{X})$. In particular, the category of topological spaces embeds fully faithfully in the 2-category of stacks.

This embedding preserves the closed cartesian structure on Top, hence fiber products. It also admits a left adjoint. That is, to every stack $\mathfrak{X}$ one can associate a topological space, together with a natural map $\pi \colon \mathfrak{X} \to \mathfrak{X}_{mod}$ which is universal among maps from $\mathfrak{X}$ to topological spaces. The space $\mathfrak{X}_{mod}$ is called the *coarse moduli space* of $\mathfrak{X}$ and it should be thought of as the "underlying space" of $\mathfrak{X}$.

A morphism $f \colon \mathfrak{X} \to \mathfrak{Y}$ of stacks is called **representable** if for every map $T \to \mathfrak{Y}$ from a topological space T, the fiber product $T \times_{\mathfrak{Y}} \mathfrak{X}$ is a topological space. This is, roughly speaking, saying that the fibers of f are topological spaces.

Any property **P** of morphisms of topological spaces which is invariant under base change can be defined for an arbitrary representable morphism of stacks. More precisely, we say that a representable morphism $f \colon \mathfrak{X} \to \mathfrak{Y}$ is **P**, if for every map $T \to \mathfrak{Y}$ from a topological space T, the base extension $f_T \colon T \times_{\mathfrak{Y}} \mathfrak{X} \to T$ is **P** as a map of topological spaces; see ([No2], Section 4.1).

This way we can talk about *embeddings (closed, open, locally closed, or arbitrary) of stacks, proper morphisms, finite morphisms*, and so on.

Remark 4.2 Stacks over Top are not well-suited to do algebraic topology since they can be far away from topological spaces, and, in particular are not necessarily "approximated" by (simplicial) topological spaces. To fix this, we introduce the convenient subcategory of topological stacks, which, for instance have a well defined homotopy type, see Lemma 4.7.

Definition 4.3 A **topological** stack is a stack $\mathfrak{X}$ over Top which admits an *atlas*, that is a representable epimorphism $p \colon X \to \mathfrak{X}$ from a topological space X.

Note that any topological groupoid defines a stack over Top. Indeed, let $\Gamma = \Gamma_1 \rightrightarrows \Gamma_0$, we define a stack over Top, denoted, $[\Gamma_0/\Gamma_1]$, to be the stack of torsors for the groupoid $X_1 \rightrightarrows X_0$, that is its T-points $[\Gamma_0/\Gamma_1](T)$ is the groupoid of principal Γ-bundles[3] over T. This stack is a topological stack. In fact

Lemma 4.4 ([No2]) *A stack* $\mathfrak{X}$ *is a topological stack iff it is equivalent to the quotient stack* $[X_0/X_1]$ *of a topological groupoid* $X_1 \rightrightarrows X_0$.

3. principal bundles are always assumed to have local sections

The groupoid $X_1 \rightrightarrows X_0$ is recovered from the atlas $p\colon X \to \mathfrak{X}$ by setting $X_0 := X$ and $X_1 := X \times_{\mathfrak{X}} X$. Under this correspondence between topological stacks and topological groupoids, morphisms of stacks correspond to Hilsum-Skandalis bibundles [HS] or *generalized morphisms* [MM, HS].

One can define similarly geometric stacks. For instance, substituting the category of C^∞-manifolds to Top and Lie groupoids to topological groupoids. Recall that a topological groupoid is a *Lie groupoid* (or a *differentiable groupoid*) if X_1, X_0 are manifolds, all the structures maps are smooth and, in addition, the source and target maps are subjective submersions (see [MM] for details on Lie groupoids).

Lemma and Definition 4.5 A **differentiable stack** is a stack $\mathfrak{X}$ on the category of C^∞-manifolds, which is isomorphic to the quotient stack of a Lie groupoid (or, equivalently, admits a smooth atlas, that is a representable epimorphism $p\colon M \to \mathfrak{X}$ from a manifold M).

We say that *a groupoid $X_1 \rightrightarrows X_0$ presents a stack $\mathfrak{X}$* if $\mathfrak{X}$ is isomorphic to $[X_0/X_1]$. It is clear that a differentiable stack is a topological stack and, often, we will tacitly pass from a differentiable stack to its underlying topological stack. One defines similarly (to Definition 4.5) **almost complex** stacks. The category of smooth manifolds embeds as a fully faithful subcategory of differentiable stacks. In fact, if M is a manifold, then M gives rise to the trivial Lie groupoid $M \rightrightarrows M$ which presents the underlying differentiable stack of M.

We conclude by giving key examples of stacks.

Example 4.6 Classifying spaces and quotient by a group action : an important class of examples is given by group actions. Let G be a topological group acting on a topological space X. Then we can make a quotient stack $[X/G]$ which is the "good quotient" (or homotopy quotient) of X (as opposed to the usual quotient space which may be very singular). In particular, there is a canonical map $X \to [X/G]$ which is always a G-principal bundle. The (global quotient) stack $[X/G]$ can be defined as the quotient stack associated to the following *transformation groupoid* $X \rtimes G \rightrightarrows X$ as follows: the space of objects is X and the space of arrows is $X \times G$. The source map $s\colon X \times G \to X$ is the first projection and the target map is the action $X \times G \to X$. The composition of arrows is induced from the multiplication in G.

If G is a Lie group, X is a manifold and the action is smooth, then $[X/G]$ is a differentiable stack (since the transformation groupoid is a Lie groupoid).

In particular if $X = pt$ is a point, we have the differentiable stack $[pt/G]$ which is also denoted $\mathcal{B}G$. Its groupoid of T-points $\mathcal{B}G(T) \cong \hom(T, \mathcal{B}G)$ is the groupoid of *principal G-bundles over T*.

Quotient of a Lie groupoid : by Lemma 4.4 and Definition 4.5, any Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ determines a differentiable stack, denoted $[\Gamma_0/\Gamma_1]$. This stack should be thought as the "good quotient" of the manifold Γ_0 by the equivalence relation given by Γ_1 (two points are equivalent if they are related by an arrow). Two Lie groupoids gives rise to isomorphic quotient stack if they are *Morita equivalent*, see [MM, BX, 4]. Roughly this means that there is an equivalence of categories which is *locally* induced by smooth functors. For instance, for a manifold M, the trivial Lie groupoid $M \rightrightarrows M$ is always Morita equivalent to the Cech groupoid $\coprod U_{ij} \rightrightarrows \coprod U_i$ where $\{U_i\}$ is an open cover of M (here $U_{ij} := U_i \cap U_j$ and the source and target maps are given by the obvious inclusions $U_{ij} \hookrightarrow U_i$, $U_{ij} \hookrightarrow U_j$).

(smooth) Orbifolds : An orbifold, by definition, is a differentiable stack which can be covered by open substacks of the form $[X/G]$, with G a finite group. Although every orbifold $\mathfrak{X}$ is locally the quotient stack $[X/G]$ of a finite group action, this may not be the case globally, i.e., $\mathfrak{X}$ may not be *good* (though every reduced orbifold can be realized by a global quotient by a Lie group acting with finite stabilizers on a manifold). Orbifolds arise frequently in studying moduli problems (for instance moduli of various kind of Riemann surfaces) and are much closer to manifolds than the general stacks. The simplest example of a differentiable stack which is not an orbifold is given by the quotient stack $[pt/G]$ when G is *not* a finite group.

Lemma and Definition 4.7 A *classifying space* for a stack $\mathfrak{X}$ is a topological space X together with a morphism $\varphi\colon X \to \mathfrak{X}$ which is a *universal weak equivalence.* The latter means that, for every map $T \to \mathfrak{X}$ from a topological space T, the base extension $\varphi_T\colon X_T \to T$ is a weak equivalence of topological spaces.

Any *topological stack* $\mathfrak{X}$ has a classifying space $X \to \mathfrak{X}$, which can further be chosen to be an *atlas* (Definition 4.3).

If $X_0 \rightrightarrows \mathfrak{X}_1$ is a (topological) groupoid presenting $\mathfrak{X}$, then an explicit construction of a classifying space for $\mathfrak{X}$ is given by the classifying space of the underlying topological category of $X_\bullet$; that is, the geometric realization of the simplicial space $\{X_n := X_1 \times_{X_0} \cdots \times_{X_0} X_1\}_{n\in\mathbb{N}}$ (i.e. X_n is the subspace of n-composable arrows) given by the nerve of the category.

The classifying space is unique up to a unique (in the weak homotopy category) weak equivalence. Thus, it allows to define *homotopy theoretic* information/invariant on $\mathfrak{X}$. For example, to define the **relative homology** of a pair $\mathfrak{A} \subset \mathfrak{X}$, we choose a classifying space $\varphi\colon X \to \mathfrak{X}$ and define $H_\bullet(\mathfrak{X},\mathfrak{A}) := H_\bullet(X, \varphi^{-1}\mathfrak{A})$. The fact that φ is a universal weak equivalence guarantees that this is well defined up to a canonical isomorphism. This can be extended to any (co)homology theory, and the latter thus defined on topological stacks will maintain all natural properties that it had on spaces. For example, it will be homotopy invariant (in particular, it will not distinguish 2-isomorphic morphisms), it will satisfy excision, it will maintain all the products (cap, cup, etc.) that it had on spaces, and so on.

Example 4.8 In the case where $\mathfrak{X}$ is the quotient stack $[M/G]$ of a group action, the Borel construction $M \times_G EG$ is a classifying space for $\mathfrak{X}$. Here, EG is the universal G-bundle in the sense of Milnor. Moreover, the homology $H_\bullet([M/G])$ of the stack $\mathfrak{X} \cong [M/G]$ is the G-equivariant homology $H^G_\bullet(M)$ of the pair M (as defined via the Borel construction).

4.2 A framework for string topology of a stack

The aim of our paper [4] was to establish a general machinery allowing string topology for differentiable stacks. This machinery allows us to treat on an equal footing free loops in stacks and *hidden* loops. Further, it also allows to study string topology for classifying spaces and general orbifolds similarly to manifolds. At the time where a first draft of [4] was written, only some classes of examples of string topology operations for orbifolds or classifying spaces had been studied, using very different techniques, for instance, see [LUX, CM, GLSU].

In the realm of stacks we have to solve three issues: find a good notion of mapping stack, develop an efficient machinery of Gysin maps and a good generalization of oriented manifolds. We now recall how to tackle them, following [1, 4].

4.2.1 Mapping stacks, Free and Hidden loops of a stack

Forgetting about topology, it is rather straightforward to define mapping stacks as (abstract) stacks of stack morphisms. Indeed, given stacks $\mathfrak{X}$ and $\mathfrak{Y}$ over Top (§ 4.1), the inner hom between them, called the *mapping stack* $\mathrm{Map}(\mathfrak{Y},\mathfrak{X})$, is defined by defining its groupoid of T-points to be $\mathrm{Hom}(\mathfrak{Y}\times T,\mathfrak{X})$:

$$\mathrm{Map}(\mathfrak{Y},\mathfrak{X})(T)=\mathrm{Hom}(\mathfrak{Y}\times T,\mathfrak{X}),$$

where Hom denotes the *groupoid* of stack morphisms. This is easily seen to be a stack.

The *issue* here is to endow the above mapping stack with a *topological structure*, that is to make it a *topological stack*. This will require some assumptions, see Theorem 4.10. We also need the mapping stack to behave well enough with respect to pushouts in order to get geometric operations on loops: for instance, a key (and trivial for spaces) point in string topology is the identification $\mathrm{Map}(S^1\vee S^1,\mathfrak{X})\cong \mathrm{L}\mathfrak{X}\times_{\mathfrak{X}}\mathrm{L}\mathfrak{X}$. Here lies a serious issue in the realm of stacks since the embedding of topological spaces in topological stacks does *not* preserve pushouts in general. Thus extra care has to be taken in finding the correct class of topological stacks to work with as we will shortly see (Definition 4.12 and Proposition 4.14).

Note that we have a natural equivalence of groupoids

$$\mathrm{Map}(\mathfrak{Y},\mathfrak{X})(*)\cong \mathrm{Hom}(\mathfrak{Y},\mathfrak{X}),$$

where $*$ is a point. In particular, the underlying set of the coarse moduli space of $\mathrm{Map}(\mathfrak{Y},\mathfrak{X})$ is the set of 2-isomorphism classes of morphisms from $\mathfrak{Y}$ to $\mathfrak{X}$. Further, the mapping stacks are functorial in both variables.

Lemma 4.9 *The mapping stacks* $\mathrm{Map}(\mathfrak{Y},\mathfrak{X})$ *are functorial in* $\mathfrak{X}$ *and* $\mathfrak{Y}$. *That is, we have natural functors* $\mathrm{Map}(\mathfrak{Y},-)\colon \mathsf{St}\to\mathsf{St}$ *and* $\mathrm{Map}(-,\mathfrak{X})\colon \mathsf{St}^{op}\to\mathsf{St}$. *Here,* St *stands for the 2-category of stacks over* Top *and* St^{op} *is the opposite category.*

Theorem 4.10 ([4] and [No1]) *Let* $\mathfrak{X}$ *and* $\mathfrak{K}$ *be topological stacks. Assume that* $\mathfrak{K}\cong[K_0/K_1]$, *where* $K_1\rightrightarrows K_0$ *is a topological groupoid with* K_0 *and* K_1 *compact. Then,the mapping stack* $\mathrm{Map}(\mathfrak{K},\mathfrak{X})$ *is a topological stack.*

In particular, thre **free loop stack** $\mathrm{L}\mathfrak{X} = \mathrm{Map}(S^1, \mathfrak{X})$ of a topological stack is a topological stack; this result was first proved in our note [1] and extended to other mapping stacks in [4]. Also, it follows from the exponential law for mapping spaces that when X and Y are spaces, then $\mathrm{Map}(Y, X)$ is representable by the usual mapping space from Y to X (endowed with the compact-open topology).

Example 4.11 The above Theorem 4.10 means that the stack $\mathrm{Map}(S^1, \mathfrak{X})$ is isomorphic to the stack of torsors over a topological groupoid. In [1], we gave an explicit presentation of this topological groupoid (assuming $\mathfrak{X}$ is Hurewicz).

In the category of topological stacks, $S^1 \vee S^1$ is *not* the pushout of two copies of S^1. For this reason, we define Hurewicz stacks which are motivated by Proposition 4.14 below.

Definition 4.12 ([4]) A *Hurewicz* stack is a topological stack which admits a presentation by a topological groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ for which the source and target maps are local Hurewicz fibration (i.e. have local homotopy lifting properties).

Example 4.13 From Definition 4.5 follows immediately that *differentiable stacks are Hurewicz.* In particular, the (non-)interested reader can replace the word Hurewicz by differentiable with respect to applications to string topology for stacks in Section 4.3.

Proposition 4.14 ([4], Proposition 1.3) *Let $A \to Y$ be a closed embedding of Hausdorff spaces, which is a local cofibration. Let $A \to Z$ be a finite proper map of Hausdorff spaces. Suppose we are given a pushout diagram in the category of topological spaces*

$$\begin{array}{ccc} A & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z \vee_A Y \end{array}$$

Then this diagram remains a pushout diagram in the (2-)category of Hurewicz stacks. In other words, for every Hurewicz topological stack $\mathfrak{X}$, the morphism

$$\mathfrak{X}(Z \vee_A Y) \longrightarrow \mathfrak{X}(Z) \times_{\mathfrak{X}(A)} \mathfrak{X}(Y)$$

is an equivalence of groupoids.

As an immediate corollary we get,

Corollary 4.15 *Let $\mathfrak{X}$ be a Hurewicz topological stack, and let $L\mathfrak{X}$ be its loop stack. Then, the diagram*

$$\begin{array}{ccc} \mathrm{Map}(8, \mathfrak{X}) & \rightarrow & L\mathfrak{X} \\ \downarrow & & \downarrow \\ L\mathfrak{X} & \longrightarrow & \mathfrak{X} \end{array}$$

is (2-)cartesian. Hence $\mathrm{Map}(S^1 \vee S^1, \mathfrak{X}) = \mathrm{Map}(8, \mathfrak{X}) \cong \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$.

4.2.2 Bivariant theory for Topological Stacks

Another *crucial* step in string topology is the existence of a canonical Gysin homomorphism $H_\bullet(\mathrm{L}M \times \mathrm{L}M) \to H_{\bullet - d}(\mathrm{L}M \times_M \mathrm{L}M)$ when M is a d-dimensional oriented manifold. In fact, the loop product is the composition

$$\begin{aligned} &H_p(\mathrm{L}M) \otimes H_q(\mathrm{L}M) \to \\ &\qquad \to H_{p+q}(\mathrm{L}M \times \mathrm{L}M) \to H_{p+q-d}(\mathrm{L}M \times_M \mathrm{L}M) \to H_{p+q-d}(\mathrm{L}M)\,, \end{aligned} \tag{4.1}$$

where the last map is obtained by gluing two loops at their base point. This map (4.1) realizes the following geometric intersection construction. Given two families of loops $\sigma : \Delta^p \to \mathrm{L}M$ and $\tau : \Delta^q \to \mathrm{L}M$; evaluation at the base-point of a loop $\mathrm{ev} : \mathrm{L}M \to M$ (given by $f \mapsto f(0)$) provides two families of points $\mathrm{ev} \circ \sigma$, $\mathrm{ev} \circ \tau$ in M. After we take evaluation at the base point, we are in the familiar situation of sub-families inside an oriented finite dimensional manifold. Assuming standard smoothness and transversality assumption in general position, the "geometric" intersection of this two families yields (dimension $p+q-\dim(M)$) families of loops with the same base points, that is, a (linear combination of) dimension $p+q-\dim(M)$ family of "figure eight" $\Delta^{p+q-\dim(M)} \to \mathrm{Map}(8, M)$.

Roughly speaking the Gysin map is pulled-back from the usual Gysin map (relative to the diagonal embedding) for M along the evaluation map $\mathrm{ev} : \mathrm{L}M \to M$. Indeed, the free loop manifold can be endowed with a structure of Banach manifold such that the evaluation map $\mathrm{ev} : \mathrm{L}M \to M$ is a surjective submersion. The pullback along $\mathrm{ev} \times \mathrm{ev}$ of a tubular neighborhood of the diagonal $M \to M \times M$ in $M \times M$ yields a normal bundle of codimension $d = \dim(M)$ for the embedding $\mathrm{L}M \times_M \mathrm{L}M \to \mathrm{L}M$. The Gysin map can then be constructed using a standard argument on Thom isomorphism and Thom collapse.

The above pattern is quite general: most of string topology operations are build using a similar intersection pattern for families of loops and families of evaluation maps $\mathrm{ev}_t : f \mapsto f(t)$ (for $t \in S^1$) and most identities relating these operations relied on compatibilities between different Gysin maps and pushforward maps.

This approach does *not* have a straightforward generalization to stacks. For instance, the free loop stack of a differentiable stack is not a Banach stack in general, and neither is the inertia stack. In order to obtain a flexible theory of Gysin maps, in [4] we constructed a **bivariant theory** in the sense of Fulton-MacPherson [FMcP] for topological stacks, *whose underlying homology theory is singular homology.* Roughly, a bivariant theory associates ($\mathbb{Z}$-graded) groups $H^\bullet(f)$ to any map $\mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ endowed with three class of operations: *pullbacks, pushforwards and products* satisfying many compatibilities axioms (see [FMcP, 4] and below). It is an efficient tool encompassing homology and cohomology into an unified framework as well as many (co)homological operations, in particular Gysin homomorphisms. Further, *the Gysin maps of a bivariant theory can be pulled-back along any maps* and are automatically compatible with pullback, pushforward, cup and cap-products (see [FMcP]). Actually, our bivariant theory is somewhat weaker than that of Fulton-MacPherson, in that products are not always defined; but this point is harmful with respect to string topology. Our bivariant theory applies in particular to *all* orbifolds and thus manifolds.

The idea behind the definition of our bivariant theory goes back to Spanier-Whitehead duality. Let X be a compact space embedded in $\mathbb{R}^n$ for some n. Then, there is an isomorphism $H_i(X) \cong H^{n-i}(\mathbb{R}^n, \mathbb{R}^n - X)$. Recall also that the Thom class of an oriented vector bundle $E \to Y$ is a generator of $H^{\dim(E)}(E, E-Y, \mathbb{Z})$. These two pictures can be put easily together: if $f : X \to Y$ is a continuous map that factors as a map $X \xrightarrow{\widetilde{f}} E \xrightarrow{\pi} Y$ where $E \xrightarrow{\pi} Y$ is an oriented metric vector bundle and $\widetilde{f}$ is an embedding inside the unit disk of E, then, one can define cohomology groups $H^{\dim(E)+i}(E, E - \widetilde{f}(X))$. For instance, if $f : X \to Y$ is an oriented embedding, then one can take $E = Y$ and the Thom class of the embedding (yielding the Gysin map) is (induced by) a class in $H^{\dim(Y)-\dim(X)}(Y, Y - X)$.

This motivates to define bivariant cohomology classes $\alpha \in H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$ as classes lying in $H^{\dim(\mathfrak{E})+\bullet}(\mathfrak{E}, \mathfrak{E} - \widetilde{f}(\mathfrak{X}))$ for any possible factorization of f. Let us now be more precise. We start by the following definition.

Definition 4.16 Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphisms of topological stacks and $\mathfrak{E}$ a metrizable vector bundle over $\mathfrak{Y}$. A lifting $i\colon \mathfrak{X} \to \mathfrak{E}$ of f,

$$\begin{array}{ccc} & & \mathfrak{E} \\ & \overset{i}{\nearrow} & \downarrow \\ \mathfrak{X} & \xrightarrow[f]{} & \mathfrak{Y} \end{array}$$

is called **bounded** if there is a choice of metric on $\mathfrak{E}$ such that i factors through the unit disk bundle of $\mathfrak{E}$. A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of topological stacks is called **bounded proper** if there exists a metrizable orientable vector bundle $\mathfrak{E}$ on $\mathfrak{Y}$ and a bounded lifting i as above such that i is a closed embedding.

To a morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of topological stacks, we associate a category $\mathsf{C}(f)$ as follows. The objects of $\mathsf{C}(f)$ are morphism $a\colon \mathfrak{K} \to \mathfrak{X}$ such that $fa\colon \mathfrak{K} \to \mathfrak{Y}$ is bounded proper (Definition 4.16). A morphism in $\mathsf{C}(f)$ between $a\colon \mathfrak{K} \to \mathfrak{X}$ and $b\colon \mathfrak{L} \to \mathfrak{X}$ is a homotopy class (relative to $\mathfrak{X}$) of morphisms $g\colon \mathfrak{K} \to \mathfrak{L}$ over $\mathfrak{X}$.

Definition 4.17 We define the *bivariant singular cohomology* of an arbitrary morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ to be the $\mathbb{Z}$-graded abelian group

$$H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) = \varinjlim_{\mathsf{C}(f)} H^{\bullet+\operatorname{rk}\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}).$$

The homomorphisms in the direct limit of Definition 4.17 are defined as follows. Consider a morphism $\varphi\colon \mathfrak{K} \to \mathfrak{K}'$ in $\mathsf{C}(f)$. From this we will construct a natural graded pushforward homomorphism $\varphi_*\colon H^{\bullet+m}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \to H^{\bullet+n}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}')$, where $m = \operatorname{rk}\mathfrak{E}$ and $n = \operatorname{rk}\mathfrak{E}'$.

Let $\mathfrak{F} = \mathfrak{E} \oplus \mathfrak{E}'$ with the sum orientation. Let $p\colon \mathfrak{E}' \to \mathfrak{Y}$ be the projection map. Then, $p^*(\mathfrak{E})$ is an oriented vector bundle over $\mathfrak{E}'$. Note that the projection map $\pi\colon p^*(\mathfrak{E}) \to \mathfrak{E}'$ is naturally isomorphic to the second projection map $\mathfrak{F} = \mathfrak{E} \oplus \mathfrak{E}' \to \mathfrak{E}'$; this allows us to view $\mathfrak{F}$ as an oriented vector bundle of rank m over $\mathfrak{E}'$. Let $\mathfrak{D} \subseteq \mathfrak{F}$ be the unit disc bundle. It follows from the assumptions that $\mathfrak{K} \subseteq \mathfrak{D}$, hence also $\mathfrak{K} \subseteq \mathcal{L} := \pi^{-1}(\mathfrak{K}') \cap \mathfrak{D}$. The restriction homomorphism

$$\varphi_*\colon H^{\bullet+m+n}(\mathfrak{F}, \mathfrak{F} - \mathfrak{K}) \to H^{\bullet+m+n}(\mathfrak{F}, \mathfrak{F} - \mathcal{L}) \cong H^{\bullet+n}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}'), \tag{4.2}$$

induced by the inclusion of pairs $(\mathfrak{F}, \mathfrak{F} - \mathcal{L}) \to (\mathfrak{F}, \mathfrak{F} - \mathfrak{K})$ is the desired pushforward homomorphism. Here, the last isomorphism in (4.2) comes from [4, Proposition 4.6]. Note also:

Proposition 4.18 ([4], Lemma 6.5) *Assume we are given two different factorizations $(i, \mathfrak{E})$ and $(i', \mathfrak{E}')$ for $f\colon \mathfrak{X} \to \mathfrak{Y}$. Then, there is a canonical isomorphism $H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{X}) \cong H^{\bullet+\mathrm{rk}\,\mathfrak{E}'}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{X})$.*

We also show in [4] that the map φ_* is independent of the homotopy class of φ which, with Proposition 4.18, implies

Lemma 4.19 *$H^\bullet(f)$ is independent of all choices involved in its definition. Further, when $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a closed embedding, then the bivariant group*

$$H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \cong H^\bullet(\mathfrak{Y}, \mathfrak{Y} - \mathfrak{X})$$

coincides with relative cohomology.

Bivariant theory comes along with three kind of operations: pullbacks, pushfowards and products (or composition of morphisms).

Pullbacks : Consider a (2-)cartesian diagram of topological stacks

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{Y}' \\ \downarrow & & \downarrow h \\ \mathfrak{X} & \xrightarrow[f]{} & \mathfrak{Y} \end{array}$$

We define the pullback $h^*\colon H^\bullet(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \to H^\bullet(\mathfrak{X}' \xrightarrow{f'} \mathfrak{Y}')$ as follows.

Pullback along h induces a functor $h^*\colon \mathsf{C}(f) \to \mathsf{C}(f')$, $\mathfrak{K} \mapsto h^*\mathfrak{K} := \mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{K}$. Furthermore, we have a natural homomorphism

$$H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \to H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(h^*\mathfrak{E}, h^*\mathfrak{E} - h^*\mathfrak{K})$$

induced by the map of pairs $(h^*\mathfrak{E}, h^*\mathfrak{E} - h^*\mathfrak{K}) \to (\mathfrak{E}, \mathfrak{E} - \mathfrak{K})$. Using Proposition 4.18, this induces the desired homomorphism of colimits

$$h^*\colon \varinjlim_{\mathsf{C}(f)} H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \to \varinjlim_{\mathsf{C}(f')} H^{\bullet+\mathrm{rk}\,\mathfrak{E}'}(\mathfrak{E}', \mathfrak{E}' - \mathfrak{K}').$$

Pushforwards : Let $h\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of topological stacks. There is a pushforward homomorphism $h_*\colon H^\bullet(\mathfrak{X} \xrightarrow{g\circ h} \mathfrak{Z}) \to H^\bullet(\mathfrak{Y} \xrightarrow{g} \mathfrak{Z})$ defined as follows. There is a natural functor $\mathsf{C}(g\circ h) \to \mathsf{C}(g)$, which sends $a\colon \mathfrak{K} \to \mathfrak{X}$ to $h \circ a\colon \mathfrak{K} \to \mathfrak{Y}$. A factorization for $(g \circ h) \circ a$ gives a factorization for $g \circ (h \circ a)$ in a trivial manner:

$$\begin{array}{ccc} \mathfrak{K} & \xhookrightarrow{i} & \mathfrak{E} \\ {\scriptstyle a}\downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow[f]{} & \mathfrak{Z} \end{array} \qquad \mapsto \qquad \begin{array}{ccc} \mathfrak{K} & \xhookrightarrow{i} & \mathfrak{E} \\ {\scriptstyle ha}\downarrow & & \downarrow \\ \mathfrak{Y} & \xrightarrow[g]{} & \mathfrak{Z} \end{array}$$

Hence Proposition 4.18 induces the desired homomorphism

$$h_*\colon \varinjlim_{\mathsf{C}(f)} H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}) \to \varinjlim_{\mathsf{C}(g)} H^{\bullet+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}).$$

Products : Unfortunately, we are not able to define product $H^i(\mathfrak{X} \xrightarrow{f} \mathfrak{Y}) \otimes H^j(\mathfrak{Y} \xrightarrow{g} \mathfrak{Z}) \to H^{i+j}(\mathfrak{X} \xrightarrow{g\circ f} \mathfrak{Z})$ for arbitrary pairs of composable morphisms f and g. However, under an extra assumption on g this will be possible.

Definition 4.20 A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of topological stacks is called **adequate** if in the cofiltered category $\mathsf{C}(f)$ the subcategory $\mathsf{C}_{sp}(f)$ is cofinal, where the subcategory $\mathsf{C}_{sp}(f)$ consists of $a\colon \mathfrak{K} \to \mathfrak{X}$ such that $f \circ a\colon \mathfrak{K} \to \mathfrak{Y}$ satisfies that every orientable metrizable vector bundle $\mathfrak{E}$ on $\mathfrak{X}$ is a direct summand of $(f \circ a)^*(\mathfrak{E}')$ for some orientable metrizable vector bundle $\mathfrak{E}'$ on $\mathfrak{Y}$. If the identity $id : \mathfrak{X} \to \mathfrak{X}$ itself is in $\mathsf{C}_{sp}(f)$, we say that f is **strongly adequate** [4].

The above definition 4.20 is kinda hard to grasp; fortunately we are only interested in the following examples (see [4]):

Example 4.21

1. A morphism $f\colon \mathfrak{X} \to Y$ in which Y is a paracompact topological space is adequate.
2. Let $\mathfrak{X}$ be a topological stack such that the diagonal $\Delta\colon \mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ is bounded proper. Then Δ is strongly adequate. The same holds for iterated diagonal $\Delta^{(n)}\colon \mathfrak{X} \to \mathfrak{X}^n$.
3. Let X, Y be compact G-manifolds (with G compact) and $f\colon X \to Y$ be a G-equivariant map. Then the induced map of stacks $[f/G]\colon [X/G] \to [Y/G]$ is strongly adequate.
4. If $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ are strongly adequate, then $g \circ f\colon \mathfrak{X} \to \mathfrak{Z}$ is strongly adequate (the same property does *not* hold for adequate morphisms in general).

4. this property actually means that f is strongly proper in the sense of [4]

Using the cup-product of (relative) cohomology classes we get

Lemma 4.22 ([4], Section 7.4) *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ be morphisms of topological stacks, and assume g is adequate. Then there is a graded linear map $H^i(f) \otimes H^j(g) \to H^{i+j}(g \circ f)$, $(\alpha, \beta) \mapsto \alpha \cdot \beta$ which is further associative.*

Theorem 4.23 ([4]) *The singular homology and cohomology theory for topological stacks fits inside a (generalized) Fulton-MacPherson bivariant theory given by Definition 4.17.*

The content of the Theorem is that the product (when defined), pullback and push-forward operations satisfy various natural compatibility axioms (we refer to [FMcP] for a detailled (lengthy) list) and further that there are canonical isomorphisms

$$H^n(\mathfrak{X}) \cong H^n(\mathfrak{X} \xrightarrow{\mathrm{id}} \mathfrak{X}) = \varinjlim_{\mathsf{C}(\mathrm{id}_{\mathfrak{X}})} H^{n+\mathrm{rk}\,\mathfrak{E}}(\mathfrak{E}, \mathfrak{E} - \mathfrak{K}),$$

which follows from Lemma 4.19, as well as canonical isomorphisms

$$H_n(\mathfrak{X}) = H^{-n}(\mathfrak{X} \to pt)$$

where the left hand-side is the singular homology of $\mathfrak{X}$. Here the isomorphism follows since $\mathsf{C}(\mathfrak{X})$ is the category whose object are pairs (E, K) where E is a Euclidean space of dimension e and K is a compact subspace of E together with a map $K \to \mathfrak{X}$. Thus

$$H^{-n}(\mathfrak{X} \to pt) = \varinjlim_{\mathsf{C}(\mathfrak{X})} H^{e-n}(E, E - K) \cong \varinjlim_{K \to \mathfrak{X}} H_n(K).$$

where the colimit is taken over the category of all maps $K \to \mathfrak{X}$ with K a compact topological space that is embeddable in some Euclidean space.

An immediate corollary is that we get a very flexible theory of Gysin maps. Indeed, fix a class $\theta \in H^i(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$. Let $u : \mathfrak{Y}' \to \mathfrak{Y}$ be an arbitrary morphism of topological stacks and $\mathfrak{X}' = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ the base change given by the cartesian square:

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f'} & \mathfrak{Y}' \\ \downarrow & & \downarrow u \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}. \end{array} \tag{4.3}$$

Then θ determines **Gysin homomorphisms**

$$\theta^! \colon H_j(\mathfrak{Y}') \to H_{j-i}(\mathfrak{X}')$$

and

$$\theta_! \colon H^j(\mathfrak{X}') \to H^{j+i}(\mathfrak{Y}').$$

For the cohomology Gysin map, we need to assume that f' is adequate. These homomorphisms are defined by

$$\theta^!(a) = \big(u^*(\theta)\big) \cdot a, \quad \text{for} \ \ a \in H_j(\mathfrak{Y}') = H^{-j}(\mathfrak{Y}' \to pt),$$

and

$$\theta_!(b) = f'_*\big(b \cdot u^*(\theta)\big), \quad \text{for} \ \ b \in H^j(\mathfrak{X}') = H^j(\mathfrak{X}' \xrightarrow{\text{id}} \mathfrak{X}').$$

The homology Gysin map is defined because the map $\mathfrak{X}' \to *$ is adequate (see Example 4.21).

4.2.3 Oriented stacks

Oriented stacks are the stacks over which we are able to do string topology. *Examples of oriented stacks* include: oriented manifolds, oriented orbifolds, and quotients of oriented manifolds by compact Lie groups (if the action is orientation preserving and of finite orbit type). There are some subtleties to address here since, unlike manifolds or orbifolds, differentiable stacks do not have a tangent topological stack (but rather a tangent *complex*). The idea, which is classical in topology and intersection theory, is to define a stack to be oriented iff its diagonal has a Thom class. Let us start with a general definition of oriented morphisms of stacks.

Definition 4.24 We say that a representable morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of topological stacks is *normally nonsingular* (nns for short), if there exist vector bundles $\mathfrak{F}$ and $\mathfrak{E}$ over the stacks $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, and a commutative diagram

$$\begin{array}{ccc} \mathfrak{F} & \xhookrightarrow{\ i\ } & \mathfrak{E} \\ {\scriptstyle s}\big\uparrow & & \big\downarrow{\scriptstyle p} \\ \mathfrak{X} & \xrightarrow{\ f\ } & \mathfrak{Y} \end{array}$$

where s is the zero section of the vector bundle $\mathfrak{F} \to \mathfrak{X}$, and i is an open immersion. A normally nonsingular f is called *oriented* if $\mathfrak{F}$ and $\mathfrak{E}$ are oriented vector bundles (i.e. have Thom classes). The integer $c = \operatorname{rk}\mathfrak{E} - \operatorname{rk}\mathfrak{F}$ depends only on f and is called the *codimension* of f.

The embedding $\mathfrak{F} \to \mathfrak{E}$ plays the role of a *tubular neighborhood*.

Example 4.25 ([4]) Let G be a compact Lie group, and X and Y smooth G-manifolds, with $\mathfrak{X} = [X/G]$ and $\mathfrak{Y} = [Y/G]$ the corresponding quotient stacks. Assume further that X is of finite orbit type. Then, for every G-equivariant smooth map $X \to Y$, the induced morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of quotient stacks is normally nonsingular.

The key property of oriented normally nonsingular morphism is that they give rise to *canonical bivariant classes* and thus canonical Gysin homomorphisms.

Proposition 4.26 ([4]) *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a (strongly) adequate morphism of topological stacks equipped with an oriented normally nonsingular diagram. Then, f has a canonical (strong) orientation class $\theta_f \in H^{\mathrm{codim}(f)}(f)$, that is a class such that for every $g\colon \mathfrak{Z} \to \mathfrak{X}$, multiplication by θ_f is an isomorphism $H^i(\mathfrak{Z} \xrightarrow{g\circ f} \mathfrak{Y}) \xrightarrow{\sim} H^{i+\mathrm{codim}(f)}(\mathfrak{X} \xrightarrow{f} \mathfrak{Y})$.*

The orientation class θ_f is essentially the Thom class of the map $i \circ s : \mathfrak{X} \to \mathfrak{E}$.

Definition 4.27 A topological stack $\mathfrak{X}$ is (strongly) *orientable* if the diagonal map $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is (strongly) orientable in the sense of Definition 4.24.

It follows from Proposition 4.26 that an oriented stack $\mathfrak{X}$ has a canonical orientation class $\theta \in H^{\dim(\mathfrak{X})}(\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X})$.

Further, morphisms between strongly oriented stacks have *canonical* orientations, which are *multiplicative* with respect to products of bivariant classes.

Proposition 4.28 ([4], Section 8.3) *i) Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a strongly adequate normally nonsingular morphism of topological stacks, and assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are both strongly oriented (Definition 4.27). Let $d = \dim \mathfrak{X}$ and $c = \dim \mathfrak{Y} - \dim \mathfrak{X}$. Then, there is a* unique *strong orientation class $\theta_f \in H^c(f)$ which satisfies the equality $\theta_f \cdot \theta_{\mathfrak{Y}} = (-1)^{cd}\theta_{\mathfrak{X}} \cdot (\theta_f \times \theta_f)$*

ii) Assume $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ are strongly adequate normally nonsingular morphisms of strongly oriented topological stacks. Let $\theta_f \in H^c(f)$, and $\theta_g \in H^d(g)$, be the strong orientations constructed in i) *above. Then, $g \circ f$ is a strongly adequate normally nonsingular. Furthermore, $\theta_f \cdot \theta_g = \theta_{g\circ f}$.*

Two main examples of oriented stacks were studied in our monograph [4]:

Corollary 4.29 *Let $\mathfrak{X}$ be a stack that is equivalent to the quotient stack $[X/G]$ of smooth orientation preserving action of a compact Lie group G on a smooth oriented manifold X having finitely generated homology groups. Then, the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is naturally (strongly) oriented. In particular, the diagonal of the classifying stack $\mathcal{B}G$ of a compact Lie group G is naturally (strongly) oriented.*

Proposition 4.30 *Let $\mathfrak{X}$ be a paracompact* orbifold *whose tangent bundle is oriented. Then the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is strongly oriented and in particular, $\mathfrak{X}$ is naturally oriented.*

By Proposition 4.26, when the map $f\colon \mathfrak{X} \to \mathfrak{Y}$ in Diagram (4.3) is strongly oriented, it has a canonical strong orientation θ_f. In this case, we have a canonical Gysin morphism (see (4.3))

$$f^! := (\theta_f)^!\colon H_\bullet(\mathfrak{Y}') \to H_{\bullet-c}(\mathfrak{X}'), \tag{4.4}$$

where c is the codimension of f. We collect two (of manies) standard properties of these Gysin morphisms which follow from Theorem 4.23.

1. **Functoriality.** Assume given a commutative diagram of cartesian squares

$$\begin{array}{ccccc} \mathfrak{X}' & \longrightarrow & \mathfrak{Y}' & \longrightarrow & \mathfrak{Z}' \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} & \xrightarrow{g} & \mathfrak{Z} \end{array} \tag{4.5}$$

   with $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$ strongly oriented of codimensions c and d, respectively. Then, the induced Gysin morphisms $f^!\colon H_\bullet(\mathfrak{Y}') \to H_{\bullet - c}(\mathfrak{X}')$ and $g^!\colon H_\bullet(\mathfrak{Z}') \to H_{\bullet - d}(\mathfrak{Y}')$ satisfy the functoriality identity

$$(g \circ f)^! = f^! \circ g^!.$$

2. **Naturality.** Assume given a a commutative diagram of cartesian squares

$$\begin{array}{ccc} \mathfrak{X}'' & \longrightarrow & \mathfrak{Y}'' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ \mathfrak{X}' & \longrightarrow & \mathfrak{Y}' \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array} \tag{4.6}$$

   with f strongly oriented. Then, the induced Gysin morphisms satisfy

$$v_* \circ f^! = f^! \circ u_*.$$

4.3 String topology operations for oriented stacks

We finally explain how to derive string topology operations for stacks using the framework of Section 4.2. In this section we *assume that* $\mathfrak{X}$ *is a Hurewicz (for instance differentiable), oriented stack.* A key point is that, by Proposition 4.28, the iterated diagonals $\mathfrak{X} \to \mathfrak{X} \times \cdots \times \mathfrak{X}$ are strongly oriented so that any cartesian square

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \mathfrak{Z} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{\text{iterated diagonal}} & \mathfrak{X}^n \end{array}$$

defines a canonical Gysin map (see (4.4)) $H_\bullet(\mathfrak{Z}) \to H_{\bullet - (n-1)\dim(\mathfrak{X})}(\mathfrak{Y})$.

4.3.1 BV and Frobenius algebras structures for loop stacks

We first define the loop product. By functoriality of mapping stacks (Lemma 4.9), the pinching map $\mathrm{pinch}\colon S^1 \to S^1 \vee S^1$ (identifying $\frac{1}{2}$ and 0 in

S^1) induces a "Pontrjagin" map $\mathrm{pinch}^*\colon \mathrm{Map}(8, \mathfrak{X}) \to \mathrm{L}\mathfrak{X}$. The loop product is now given by the following diagram

$$\begin{array}{ccccc}
\mathrm{L}\mathfrak{X} & \xleftarrow{\mathrm{pinch}^*} & \mathrm{Map}(8,\mathfrak{X}) \cong \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} & \longrightarrow & \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}\times\mathrm{ev}} \\
 & & \mathfrak{X} & \xrightarrow{\mathrm{diagonal}} & \mathfrak{X} \times \mathfrak{X}
\end{array}$$

in which the right square is cartesian by Corollary 4.15, hence defines a canonical Gysin homomorphism. We thus can define the *loop product* as the composition:

$$\star\colon H_\bullet(\mathrm{L}\mathfrak{X})^{\otimes 2} \to H_\bullet(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \stackrel{\mathrm{diagonal}^!}{\longrightarrow} H_{\bullet - \dim(\mathfrak{X})}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \stackrel{\mathrm{pinch}^*}{\longrightarrow} H_{\bullet - \dim(\mathfrak{X})}(\mathrm{L}\mathfrak{X}). \quad (4.7)$$

Since S^1 acts canonically on itself, functoriality of the mapping stack confers an induced S^1-action to $\mathrm{L}\mathfrak{X} = \mathrm{Map}(S^1, \mathfrak{X})$ for any topological stack $\mathfrak{X}$, which in turns, endows $H_\bullet(\mathrm{L}\mathfrak{X})$ with a degree one operator D as follows. Let $[S^1] \in H_1(S^1)$ be the fundamental class. Then a linear map $D : H_\bullet(\mathrm{L}\mathfrak{X}) \to H_{\bullet+1}(\mathrm{L}\mathfrak{X})$ is defined by the composition

$$H_\bullet(\mathrm{L}\mathfrak{X}) \stackrel{\times[S^1]}{\longrightarrow} H_{\bullet+1}(\mathrm{L}\mathfrak{X} \times S^1) \stackrel{\rho_*}{\longrightarrow} H_{\bullet+1}(\mathrm{L}\mathfrak{X}), \quad (4.8)$$

where the last arrow is induced by the action $\rho : S^1 \times \mathrm{L}\mathfrak{X} \to \mathrm{L}\mathfrak{X}$. It is immediate to check that D squares to zero.

Let us recall that a **Batalin-Vilkovisky algebra** (**BV**-algebra for short) is a graded commutative associative algebra with a degree 1 operator D such that $D(1) = 0$, $D^2 = 0$, and the following identity is satisfied:

$$D(abc) - D(ab)c - (-1)^{|a|}aD(bc) - (-1)^{(|a|+1)|b|}bD(ac) + \\ + D(a)bc + (-1)^{|a|}aD(b)c + (-1)^{|a|+|b|}abD(c) = 0. \quad (4.9)$$

In other words, D is a second-order differential operator. Using the nice properties of Gysin maps provided by our bivariant theory, we proved

Theorem 4.31 ([1, 4]) *Let $\mathfrak{X}$ be an oriented Hurewicz*[5] *stack of dimension d. Then the shifted homology $\mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) = H_{\bullet+d}(\mathrm{L}\mathfrak{X})$ admits a* **BV***-algebra structure given by the loop product* (4.7) *$\star\colon \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \otimes \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X})$ and the operator* (4.8) *$D : \mathbb{H}_\bullet(\mathrm{L}\mathfrak{X}) \to \mathbb{H}_{\bullet+1}(\mathrm{L}\mathfrak{X})$.*

For $t \in S^1$, let $\mathrm{ev}_t\colon \mathrm{L}\mathfrak{X} \to \mathfrak{X}$ be the evaluation map $f \mapsto f(t)$. Applying Theorem 4.10, since S^1 is compact and $\mathfrak{X}$ Hurewicz, we see that the topological

5. Recall that any differentiable stack is Hurewicz

stack $\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}$ also fits into a diagram

$$\begin{array}{ccccc} \mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X} & \xleftarrow{i} & \mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X} & \xrightarrow{m} & \mathrm{L}\mathfrak{X} \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0 \times \mathrm{ev}_{\frac{1}{2}}} \\ & & \mathfrak{X} & \xrightarrow{\text{diagonal}} & \mathfrak{X} \times \mathfrak{X}\,. \end{array} \tag{4.10}$$

in which the right square is cartesian. Since we assume $\mathfrak{X}$ is oriented, the right square of diagram (4.10) yields a canonical Gysin homomorphism and we can thus define the loop coproduct as the composition

$$\begin{aligned} \delta\colon H_\bullet(\mathrm{L}\mathfrak{X}) &\overset{\text{diagonal}^!}{\longrightarrow} H_{\bullet-\dim(\mathfrak{X})}(\mathrm{L}\mathfrak{X} \times_{\mathfrak{X}} \mathrm{L}\mathfrak{X}) \\ &\overset{j_*}{\longrightarrow} H_{\bullet-\dim(\mathfrak{X})}(\mathrm{L}\mathfrak{X} \times \mathrm{L}\mathfrak{X}) \cong \bigoplus_{i+j=\bullet-\dim(\mathfrak{X})} H_i(\mathrm{L}\mathfrak{X}) \otimes H_j(\mathrm{L}\mathfrak{X}). \end{aligned} \tag{4.11}$$

A simple application of naturality and functoriality of Gysin homomorphisms yields

Theorem 4.32 ([4]) *Let $\mathfrak{X}$ be an oriented Hurewicz stack. Then $(H_\bullet(\mathrm{L}\mathfrak{X}), \star, \delta)$ is a Frobenius algebra, where both operations $\star$ and δ are of degree $\dim(\mathfrak{X})$.*

For closed (oriented) manifolds, the loop product is *unital* while the loop coproduct is *not* counital. This is an instance of Poincaré duality and thus it shall *not* be expected for stacks. Indeed, for the stack $[*/G]$ representing the classifying space of a compact Lie group, the loop product is trivial while the coproduct is counital, and actually injective as we have shown in [4]. We also compute explicitly various other examples in [4] and show that if M is an oriented manifold, then the string topology operations given by Theorem 4.31 and Theorem 4.32 coincides with the standard ones in [CS, CJ, CG].

4.3.2 The homology of free loop stacks as a homological conformal field theory

The structure of **BV**-algebra and Frobenius algebra described in Theorem 4.31 and 4.32 are only a part of a larger structure of operations parametrized by the homology of some moduli space of Riemann surfaces. They actually correspond to genus zero operations (in homological degree 0 and 1).

We start by recalling some definitions of Homological Conformal Field Theories (HCFT for short). We will make strong restrictions on the type of boundary we consider (which simplify greatly the theory). We follow [Co1, Co2, Go].

We first recall that, a *complex cobordism* from a family $\coprod_{i=1}^n S^1$ of circles to another family $\coprod_{i=1}^m S^1$ of circles is a closed (non-necessarily connected) Riemann surface Σ equipped with two holomorphic embeddings (with disjoint images) $\rho_{in}\colon \coprod_{i=1}^n D^2 \hookrightarrow \Sigma$ and $\rho_{out}\colon \coprod_{i=1}^m D^2$ of closed disks. The image of ρ_{in} is called the *incoming* boundary and the image of ρ_{out} the outgoing boundary. Two complex cobordism Σ_1 and Σ_2 (from $\coprod_{i=1}^n S^1$ to $\coprod_{i=1}^m S^1$) are equivalent if there exists a biholomorphism $h\colon \Sigma_1 \stackrel{\sim}{\to} \Sigma_2$ which fixes the boundary (*i.e.* commutes with ρ_{in} and ρ_{out}).

We denote $\mathfrak{M}_{n,m}$ the moduli space of equivalences classes of complex cobordism from $\coprod_{i=1}^n S^1$ to $\coprod_{i=1}^m S^1$. The disjoint union of surfaces yields a canonical morphism

$$\mathfrak{M}(n,m)\times\mathfrak{M}(n',m')\to\mathfrak{M}(n+n',m+m').$$

Further, given $\Sigma_1\in\mathfrak{M}(\ell,n)$ and $\Sigma_2\in\mathfrak{M}(n,m)$, using the embeddings of disks

$$\Sigma_1 \hookleftarrow \coprod_{i=1}^{n} D^2 \hookrightarrow \Sigma_2,$$

we can glue Σ_2 on Σ_1 along their common boundary. We denote $\Sigma_2\circ\Sigma_1\in\mathfrak{M}_{\ell,m}$ the Riemann surface thus obtained. Applying the singular homology functor to the above operations yields linear map

$$H_\bullet(\mathfrak{M}_{n,m)})\otimes H_\bullet(\mathfrak{M}_{n',m'})\stackrel{H_\bullet(\coprod)}{\to} H_\bullet(\mathfrak{M}_{n+n',m+m'})$$

and

$$H_\bullet(\mathfrak{M}_{\ell,n})\otimes H_\bullet(\mathfrak{M}_{n,m)})\stackrel{H_\bullet(\circ)}{\longrightarrow} H_\bullet(\mathfrak{M}_{\ell,m})$$

that satisfies natural associativity and compatibility relations. It follows that the collection $\big(H_\bullet(\mathfrak{M}_{n,m})\big)_{n,m\geq 0}$ are the morphisms of a graded linear symmetric monoidal category $\mathcal{C}_{\mathfrak{M}}$ whose objects are the nonnegative integers $n\in\mathbb{N}$ and the monoidal structure is induced by $k\otimes\ell=k+\ell$ on the objects and disjoint union of surfaces on morphisms. To define *non-unital and non-counital* homological conformal field theory, we also consider $\mathcal{C}_{\mathfrak{M}}^{nu,nc}\subset\mathcal{C}_{\mathfrak{M}}$ which is the (monoidal) subcategory obtained by considering only cobordisms in $\mathfrak{M}_{n,m}$ for which every connected component has at least one ingoing *and* one outgoing boundary component.

Furthermore, since the basic operations we consider are non-trivially graded (for instance the loop product is of degree $\dim(\mathfrak{X})$), we need to plug in a notion of dimension in the definition of conformal field theories to take care of this phenomenon and encode the sign issues. There is a standard way to do this due to Costello [Co2] (also see [Go, CM]), where the grading is taken into account by a local coefficient system $\det^{\otimes\dim(\mathfrak{X})}$ on the moduli spaces $\mathfrak{M}_{n,m}$. We refer to [Co2, Go, 4] for a precise definition of this local coefficient system which is compatible with the glueing of surfaces and disjoint union. Following [Co2, Go], we have the following

Definition 4.33 A (non-unital, non-counital) *d-dimensional homological conformal field theory* is a symmetric monoidal functor from the category $\mathcal{C}^{nu,nc}_{\mathfrak{M},\det^{\otimes d}}$ to the category of graded vector spaces.

Informally, this definition simply means that an homological conformal field theory is a graded vector space A with a (graded) operation $\mu(c)\colon A^{\otimes n}\to A^{\otimes m}$ for any homology class $c\in H_\bullet(\mathfrak{M}_{n,m})$ such that $\mu(c\circ d)=\mu(c)\circ\mu(d)$ and $\mu(c\coprod d)=\mu(c)\otimes\mu(d)$.

Theorem 4.34 (see [4]) *Let $\mathfrak{X}$ be an oriented (Hurewicz[6]) stack of dimension d. There is a d-dimensional non-unital, non-counital homological conformal field theory on the homology $H_\bullet(\mathrm{L}\mathfrak{X})$ of the free loop stack which induces the* **BV***-algebra and Frobenius structure on the homology $H_\bullet(\mathrm{L}\mathfrak{X})$ given by Theorem 4.31 and Theorem 4.32.*

The proof of Theorem 4.34 in [4] relies heavily on stacks techniques and the bivariant theory. Indeed, we prove that there is a well defined topological stack $\big[\mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})/\, Diff^+_{g,n,m}\big]$ for any genus g surface $\Sigma_{g,n,m}$ with n-incoming and m-outgoing boundary components, where $Diff^+_{g,n,m}$ is the group of oriented diffeomorphisms of $\Sigma_{g,n,m}$ preserving the boundaries pointwise acting by naturality of mapping stacks. Further, this quotient fits inside a zigzag

$$\begin{aligned}[*/Diff^+_{g,n,m}] \times (\mathrm{L}\mathfrak{X})^n \leftarrow \big[\mathrm{Map}(\Sigma_{g,n,m}, \mathfrak{X})/Diff^+_{g,n,m}\big] \\ \to (\mathrm{L}\mathfrak{X})^m \times [*/Diff^+_{g,n,m}] \to (\mathrm{L}\mathfrak{X})^m. \quad (4.12)\end{aligned}$$

The homology of the left hand side is precisely a piece of $H_\bullet(\mathfrak{M}_{n,m}) \otimes H_\bullet(\mathrm{L}\mathfrak{X})^{\otimes n}$ and using the bivariant theory and orientation class of $\mathfrak{X}$, we construct a Gysin map associated to the left map in (4.12) which yield the desired operations defining the HCFT.

4.3.3 Frobenius structures for hidden loops

Besides free loops, there are other kind of interesting loops for a stack. Indeed any (topological) stack gives rise to a stack of hidden loops, called the *inertia stack* and denoted $\Lambda\mathfrak{X}$. It is the stack of pairs (x, φ) where x is an object of $\mathfrak{X}$ and φ an automorphism of x. Indeed, $\Lambda\mathfrak{X}$ fits in the (2-)cartesian diagram

$$\begin{array}{ccc} \Lambda\mathfrak{X} & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle \text{diagonal}} \\ \mathfrak{X} & \xrightarrow[\text{diagonal}]{} & \mathfrak{X} \times \mathfrak{X}\,. \end{array}$$

If $\mathfrak{X}$ is a Hurewicz topological stack then so is $\Lambda\mathfrak{X}$. However, if $\mathfrak{X}$ is differentiable, $\Lambda\mathfrak{X}$ is not necessarily differentiable.

The inertia stack $\Lambda\mathfrak{X}$ can be identified with the mapping stack $\mathrm{Map}(\mathcal{B}\mathbb{Z}, \mathfrak{X})$ (this will be treated in [GiNo]), where $\mathcal{B}\mathbb{Z}$ is the quotient stack $[*/\mathbb{Z}]$. There is a canonical map of stack $\Phi\colon \Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$ induced, by functoriality of mapping stacks by the map of stacks $S^1 \cong [\mathbb{R}/\mathbb{Z}] \to [*/\mathbb{Z}]$. This map was first described in [1].

Example 4.35 If $\mathfrak{X}$ is a topological space, then $\Lambda\mathfrak{X} \cong \mathfrak{X}$. If $\mathfrak{X} = [*/G]$, then $\Lambda\mathfrak{X} \cong [G/G]$ where G acts on itself by conjugation. If G is a connected Lie group, the map $\Phi\colon \Lambda[*/G] \to L[*/G]$ is an homotopy equivalence ([4, Lemma 17.14]).

6. Recall that any differentiable stack is Hurewicz

There is an evaluation map $\mathrm{ev}_0\colon \Lambda\mathfrak{X} \to \mathfrak{X}$, which on the level of objects sends a pair (x, φ) to x, fitting inside a commutative diagram

$$\begin{array}{ccc} \Lambda\mathfrak{X} & \xrightarrow{\Phi} & \mathrm{L}\mathfrak{X} \\ & {\scriptstyle \mathrm{ev}_0}\searrow & \downarrow{\scriptstyle \mathrm{ev}_0} \\ & & \mathfrak{X}\,. \end{array} \tag{4.13}$$

The stack $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ (of pairs of hidden loops with the same evaluation) is known as the *double inertia stack*[7] or stacks of double twisted sectors. Its objects are triples (x, φ, ψ) where x is an object of $\mathfrak{X}$ and φ and ψ are automorphisms of x. The double inertia stack is endowed with a "Pontrjagin" multiplication map $m : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ given by $m(x, \varphi, \psi) = (x, \varphi\psi)$, that is by composition of automorphisms. We thus have a pattern analogous to the loop product. Indeed, if $\mathfrak{X}$ is oriented, we define the hidden loop product as the composition

$$\begin{aligned} \star\colon H_\bullet(\Lambda\mathfrak{X}) \otimes H_\bullet(\Lambda\mathfrak{X}) \to H_\bullet(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \stackrel{\mathrm{diagonal}^!}{\longrightarrow} & H_{\bullet-\dim(\mathfrak{X})}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \\ & \stackrel{m_*}{\longrightarrow} H_{\bullet-\dim(\mathfrak{X})}(\Lambda\mathfrak{X}). \end{aligned} \tag{4.14}$$

where $\mathrm{diagonal}^!$ is the Gysin homomorphism induced by the cartesian square

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \xrightarrow{j} & \Lambda\mathfrak{X} \times \Lambda\mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0 \times \mathrm{ev}_0} \\ \mathfrak{X} & \xrightarrow{\mathrm{diagonal}} & \mathfrak{X} \times \mathfrak{X} \end{array} . \tag{4.15}$$

In [4, §12.3], we defined "another evaluation map" $\mathrm{ev}_{\frac{1}{2}} : \Lambda\mathfrak{X} \to \mathfrak{X}$ and proved

Lemma 4.36 *The stack $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ fits into a cartesian square*

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \xrightarrow{m} & \Lambda\mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_{\frac{1}{2}})} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array} .$$

Thanks to Lemma 4.36, we can define the following hidden loop coproduct:

$$\begin{aligned} \nabla\colon H_\bullet(\Lambda\mathfrak{X}) & \stackrel{m^!}{\longrightarrow} H_{\bullet-\dim(\mathfrak{X})}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \\ & \longrightarrow H_{\bullet-\dim(\mathfrak{X})}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \cong \bigoplus_{p+q=\bullet-\dim(\mathfrak{X})} H_p(\Lambda\mathfrak{X}) \otimes H_q(\Lambda\mathfrak{X}). \end{aligned} \tag{4.16}$$

7. it should however not be confused with the stack $\mathrm{Map}(\mathcal{B}(\mathbb{Z}^2), \mathfrak{X})$ which sometimes goes under the same name

Theorem 4.37 ([4], Theorem 12.10) *Let $\mathfrak{X}$ be an oriented stack of dimension d. The hidden loop product* (4.14) $\star$ *and hidden loop coproduct* (4.16) ∇ *makes the homology $H_\bullet(\Lambda\mathfrak{X})$ of the inertia stack a (nonunital, noncounital) Frobenius algebra whose operations are of degree* $\dim(\mathfrak{X})$.

Further, the map $\Phi : \Lambda\mathfrak{X} \to \mathrm{L}\mathfrak{X}$ induces a map of Frobenius algebras after taking homology.

4.4 Orbifold intersection pairing and Chen-Ruan cohomology

In this Section, we *assume $\mathfrak{X}$ is an (almost) complex orbifold* (not necessarily compact) and we consider (co)homology with coefficient in $\mathbb{C}$. Proposition 4.30 and Proposition 4.28 show that the canonical map (see diagram (4.15)) $i : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$ is oriented and thus induces a Gysin homomorphism, which, according to the *excess formula*([4, Proposition 9.5]), is different from the Gysin map induced by the cartesian square (4.15).

The *Poincaré duality homomorphism* $\mathcal{P} : H_i(\mathfrak{X}) \to H^{d-i}(\mathfrak{X})$ is well-defined for *connected* (oriented) orbifolds, see [Be]. It is the composition

$$H_i(\mathfrak{X}) \longrightarrow (H^i(\mathfrak{X}))^* \xrightarrow{\sim} (H^i_{\mathrm{DR}}(\mathfrak{X}))^* \xrightarrow{\sim} H^{d-i}_{\mathrm{DR,\,c}}(\mathfrak{X}) \xrightarrow{\text{inclusion}_*} H^{d-i}_{\mathrm{DR}}(\mathfrak{X}) \xrightarrow{\sim} H^{d-i}(\mathfrak{X}). \quad (4.17)$$

The inertia stack $\Lambda\mathfrak{X}$ has usually *many* components of varying dimension but there is however an interesting shift of grading inducing a well defined Poincaré duality homomorphism and an interesting pairing in cohomology (first studied by Chen and Ruan [CR]). We recall briefly this grading. First, the inverse map $I\colon \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ is the isomorphism defined for any object (x,φ) in $\Lambda\mathfrak{X}$, where x is an object of $\mathfrak{X}$ and φ an automorphism of X, by $I(x,\varphi) = (x,\varphi^{-1})$.

The age is a locally constant function age: $\Lambda\mathfrak{X} \to \mathbb{Q}$. If $\mathfrak{X} = [M/G]$ is a global quotient with G a finite group, then

$$\Lambda\mathfrak{X} = \Big[\Big(\coprod_{g\in G} M^g\Big)/G\Big]$$

and for $x \in M^g$, the age is equal to $\sum k_j$ if the eigenvalues of g on T_xM are $\exp(2i\pi k_j)$ with $0 \le k_j < 1$. The age does not depend on which way $\mathfrak{X}$ is considered as a global quotient. So it is well-defined on $\Lambda\mathfrak{X}$ for any arbitrary almost complex orbifold, because any such $\mathfrak{X}$ can be locally written as a global quotient $[M/G]$. Similarly, the dimension is a locally constant function dim: $\Lambda\mathfrak{X} \to \mathbb{Z}$. The age and the dimension are related by the formula (see [CR])

$$\dim = d - 2\,\mathrm{age} - 2\,\mathrm{age}\circ I$$

where $I\colon \Lambda\mathfrak{X} \to \Lambda\mathfrak{X}$ is the inverse map (as above). The **orbifold homology** of $\mathfrak{X}$ is

$$H^{\mathrm{orb}}_\bullet(\mathfrak{X}) = H_{\bullet - 2\,\mathrm{age}\circ I}(\Lambda\mathfrak{X}) = \bigoplus_{q\in\mathbb{Q}} H_{\bullet-2q}\big([\Lambda\mathfrak{X}]_{\mathrm{age}\circ I = q}\big)$$

where $[\Lambda\mathfrak{X}]_{\mathrm{age}=n}$ is the component of $\Lambda\mathfrak{X}$ for which the age is equal to n. The orbifold cohomology is $H^\bullet_{\mathrm{orb}}(\mathfrak{X}) = H^{\bullet-2\,\mathrm{age}}(\Lambda\mathfrak{X})$ (see [CR]). Note that the shift of degrees are not the same, but rather are Poincaré dual. Indeed

Lemma 4.38 ([4], § 16.1) *The Poincaré duality homomorphism*

$$H_\bullet(\Lambda\mathfrak{X}) \xrightarrow{\mathcal{P}} H^\bullet(\Lambda\mathfrak{X})$$

maps $H_i^{\mathrm{orb}}(\mathfrak{X})$ *into* $H^{d-i}_{\mathrm{orb}}(\mathfrak{X})$. *We call it the* **orbifold Poincaré duality homomorphism** $\mathcal{P}^{\mathrm{orb}}\colon H_i^{\mathrm{orb}}(\mathfrak{X}) \to H^{d-i}_{\mathrm{orb}}(\mathfrak{X})$.

Since we have a bivariant theory and the map $i : \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$ is oriented, we can *refine* the intersection pairing (induced by the Gysin map given by the cartesian square (4.15)) using the Gysin homomorphism $j^!\colon H(\Lambda\mathfrak{X}\times\Lambda\mathfrak{X}) \to H(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X})$ given by the fundamental class (Proposition 4.26) of j instead. Note that the shifting degrees in $j^!$ varies on different components (unlike for the Gysin map induced by the square (4.15)). Due to the excess bundle formula ([4]), one no longer get an associative multiplication, unless one twists the definition by some obstruction classes (following an idea of Chen-Ruan [CR]). Indeed, there is an **obstruction bundle** whose construction is explained in detail in [CR, JKK]. This is a bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ denoted $\mathfrak{O}_{\mathfrak{X}}$. We denote $\mathfrak{c}_{\mathfrak{X}} = e(\mathfrak{O}_{\mathfrak{X}})$ the Euler class of $\mathfrak{O}_{\mathfrak{X}}$. The **orbifold intersection pairing** is the composition:

$$H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \longrightarrow H(\Lambda\mathfrak{X}\times\Lambda\mathfrak{X}) \xrightarrow{j^!} H(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) \xrightarrow{\cap\mathfrak{c}_{\mathfrak{X}}} H(\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}) \xrightarrow{m_*} H(\mathfrak{X}). \quad (4.18)$$

In [4, Section 16.2] we proved

Theorem 4.39 *Suppose $\mathfrak{X}$ is an almost complex orbifold of (real) dimension d.*

1. *The orbifold intersection pairing defines a bilinear pairing*

$$H_i^{\mathrm{orb}}(\mathfrak{X}) \otimes H_j^{\mathrm{orb}}(\mathfrak{X}) \xrightarrow{\Cap} H^{\mathrm{orb}}_{i+j-d}(\mathfrak{X}).$$

2. *The orbifold intersection pairing $\Cap$ is associative and graded commutative.*
3. *The orbifold Poincaré duality homomorphism $\mathcal{P}^{\mathrm{orb}}\colon H_\bullet^{\mathrm{orb}}(\mathfrak{X}) \longrightarrow H^{d-\bullet}_{\mathrm{orb}}(\mathfrak{X})$ is a homomorphism of $\mathbb{C}$-algebras, where $H^{d-\bullet}_{\mathrm{orb}}(\mathfrak{X})$ is equipped with the orbifold cup-product [CR].*

This result allows to compare the *hidden loop product* with the orbifold cup-product (or rather its Poincaré dual) for (almost) complex orbifolds. In fact, we associate to an almost complex orbifold $\mathfrak{X}$ two others vector bundles over $\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X}$. First the inverse map $I\colon \Lambda\mathfrak{X} \xrightarrow{\sim} \Lambda\mathfrak{X}$ induces the "inverse" obstruction bundle $\mathfrak{O}^{-1}_{\mathfrak{X}} = (I \times_{\mathfrak{X}} I)^*(\mathfrak{O}_{\mathfrak{X}})$. We let $\mathfrak{N}_{\mathfrak{X}}$ be the normal bundle of the regular embedding $\Lambda\mathfrak{X}\times_{\mathfrak{X}}\Lambda\mathfrak{X} \xrightarrow{m} \Lambda\mathfrak{X}$.

We can *twist* the definition of the orbifold intersection pairing as follows.

Definition 4.40 Let $\mathfrak{E}$ be a vector bundle over $\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}$ and $e(\mathfrak{E})$ be its Euler class. We define the **orbifold intersection pairing twisted by** $\mathfrak{E}$, denoted $\Cap^{\mathfrak{E}}$, to be the composition

$$H(\Lambda\mathfrak{X}) \otimes H(\Lambda\mathfrak{X}) \xrightarrow{\times} H(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \xrightarrow{j^!} \\ H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{\cap(\mathfrak{e}_{\mathfrak{X}} \cup e(\mathfrak{E}))} H(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{m_*} H(\mathfrak{X}).$$

Theorem 4.41 ([4], Section 16.2) *Let $\mathfrak{X}$ be an almost complex orbifold. Then, the hidden loop product coincides with the orbifold intersection pairing twisted by $\mathfrak{O}_{\mathfrak{X}}^{-1} \oplus \mathfrak{N}_{\mathfrak{X}}$, i.e., for any $x, y \in H_\bullet(\Lambda\mathfrak{X})$, one has*

$$x \star y = x \Cap^{\mathfrak{O}_{\mathfrak{X}}^{-1} \oplus \mathfrak{N}_{\mathfrak{X}}} y.$$

Parallel to our work, the hidden loop product for global quotient orbifolds was studied in [LUX, GLSU]. Furthermore, a nice interpretation of the hidden loop product in terms of the Chen-Ruan product of the cotangent bundle was given by González et al. [GLSU].

Several examples of complex orbifolds are studied in [4] (as well as the difference between the various products).

4.5 Principal 2-groups bundles and gerbes

Several recent works have approached the concept of bundles with a "structure Lie 2-group" over a manifold from various perspectives (for instance [BS, ACJ, BCSS]). We are actually interested in the case of bundles over a *differentiable stack* or equivalently a *Lie groupoid* (and not just manifold). Natural examples are given by the action of $S^1 \cong [\mathbb{R}/\mathbb{Z}]$ and $\mathcal{B}\mathbb{Z} = [*/\mathbb{Z}]$ on $\mathrm{L}\mathfrak{X} = \mathrm{Map}(S^1, \mathfrak{X})$ and $\Lambda\mathfrak{X} = \mathrm{Map}(\mathcal{B}\mathbb{Z}, \mathfrak{X})$ (induced by naturality of mapping stacks, Proposition 4.9). They will be studied (as well as analogues of Goldman bracket for stacks) in more details in[GiNo] using techniques outlined in [4].

A convenient model for the kind of group stack we are interested in is given by Lie 2-groups. In this section we recall our definition (which is well suited from the point of view of stacks) of Principal bundles over a Lie 2-group and relate it to the notion of gerbes.

A **Lie 2-group** is a Lie groupoid $\Gamma_2 \rightrightarrows \Gamma_1$, whose spaces of objects Γ_1 and of morphisms Γ_2 are Lie groups and all of whose structure maps are group morphisms [8]. More generally, a **Lie 2-groupoid** is a small *strict* 2-category in which all arrows are invertible, the sets of objects, 1-arrows and 2-arrows are smooth manifolds, all structure maps are smooth and the sources and targets are surjective submersions. Another convenient model for Lie 2-groupoids is given by

8. *i.e.*, it is a groupoid in the category of groups. One can equivalently think of it as a group object in the category of groupoids

Definition 4.42 A *crossed module of groupoids* is a morphism of groupoids

$$\begin{array}{ccc} X_1 & \xrightarrow{\rho} & \Gamma_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ X_0 & \xrightarrow[=]{} & \Gamma_0 \end{array}$$

which is the identity on the base spaces (i.e. $X_0 = \Gamma_0$) and where $X_1 \rightrightarrows X_0$ is a family of groups (i.e. source and target maps coincide), together with a right action by automorphisms $(\gamma, x) \mapsto x^\gamma$ of Γ on X satisfying:

$$\rho(x^\gamma) = \gamma^{-1}\rho(x)\gamma \qquad \forall (x,\gamma) \in X_1 \times_{\Gamma_0} \Gamma_1, \tag{4.19}$$

$$x^{\rho(y)} = y^{-1}xy \qquad \forall (x,y) \in X_1 \times_{\Gamma_0} X_1. \tag{4.20}$$

Note that the equalities (4.19) and (4.20) make sense because X_1 is a family of groups. A *crossed modules of groups* is a crossed module of groupoids for which $X_0 (= \Gamma_0) = *$ is a point.

There is a well-known equivalence between Lie 2-groupoids and crossed modules of groupoids. Under this equivalence, *Lie 2-groups are mapped to crossed modules of groups* and reciprocally. It can be seen as follows. A 2-groupoid $\Gamma_2 \overset{l}{\underset{u}{\rightrightarrows}} \Gamma_1 \overset{s}{\underset{t}{\rightrightarrows}} \Gamma_0$ determines a crossed module of groupoids $(G \xrightarrow{\rho} H)$ where $H = \Gamma_1 \rightrightarrows \Gamma_0$, $G_1 = \{g \in \Gamma_2 | l(g) \in \Gamma_0 \subset \Gamma_1\}$, $\rho(g) = u(g)$ and the action of $H_1 = \Gamma_1$ on $G_1 \subset \Gamma_2$ is by conjugation. That is, if 1_h is the unit over an object h in the groupoid $\Gamma_2 \overset{l}{\underset{u}{\rightrightarrows}} \Gamma_1$, then $g^h = 1_{h^{-1}} * g * 1_h$. Conversely, given a crossed module of groupoids $X \xrightarrow{\rho} \Gamma$, one gets a Lie 2-groupoid $X_1 \ltimes \Gamma_1 \overset{l}{\underset{u}{\rightrightarrows}} \Gamma_1 \overset{s}{\underset{t}{\rightrightarrows}} \Gamma_0$, where $X_1 \ltimes \Gamma_1 \rightrightarrows \Gamma_1$ is the transformation groupoid and $X_1 \ltimes \Gamma_1 \rightrightarrows \Gamma_0$ is the semi-direct product of groupoids. More precisely, for all $x, x' \in X_1$ and $\gamma, \gamma' \in \Gamma_1$, the structures maps are defined by

$$l(x,\gamma) = \gamma, \qquad (x',\gamma') * (x,\gamma) = (x' x^{\gamma'^{-1}}, \gamma'\gamma),$$

$$u(x,\gamma) = \rho(x)\gamma, \qquad (x', \rho(x)\gamma) \star (x,\gamma) = (x'x, \gamma).$$

Notation 4.43 : in the sequel, we will denote the Lie 2-groupoid associated to the crossed module $(G \xrightarrow{\rho} H)$ by $[G \xrightarrow{\rho} H]$.

Example 4.44

1. A Lie group G yields the canonical 2-group $[1 \to G]$. This gives an embedding of the category of Lie groups in the one of Lie 2-groups.
2. If A is an *abelian* group, there is a Lie 2-group $[A \to 1]$
3. Let G be a Lie group. There is a canonical morphism $G \xrightarrow{i} \mathrm{Aut}(G)$ given by inner automorphisms which is also a crossed module. Since inner automorphisms are orientation preserving, we also have a crossed module

$G \xrightarrow{i^+} \mathrm{Aut}^+(G)$ where $\mathrm{Aut}^+(G)$ is the group of orientation preserving automorphisms. Hence we got two natural Lie 2-groups $[G \xrightarrow{i} \mathrm{Aut}(G)]$ and $[G \xrightarrow{i^+} \mathrm{Aut}^+(G)]$ associated to a Lie group.

We now explain our definition of a principal 2-group bundle. Let us explain first the idea; the data of a principal G-bundle over M (up to equivalence) is (uniquely) encoded by a *stack morphism* $M \to \mathcal{B}G$ (see Example 4.6)[9] or equivalently by a "generalized morphism" (in the sense of [HS, MM]) from the manifold M to the Lie group G(both considered as 1-groupoids). It works as follows. A principal G-bundle can be defined as a collection of transition functions $g_{ij} : U_{ij} \to G$ on the double intersections of some open covering, satisfying the cocycle condition $g_{ij}g_{jk} = g_{ik}$. These transition functions constitute a morphism of groupoids from the Čech groupoid $\coprod U_{ij} \rightrightarrows \coprod U_i$ (Example 4.6) to the Lie group $G \rightrightarrows *$. Hence we have a diagram

$$(M \rightrightarrows M) \xleftarrow{\sim} (\coprod U_{ij} \rightrightarrows \coprod U_i) \to (G \rightrightarrows *)$$

in the category of Lie groupoids whose leftward arrow is a Morita equivalence, in other words a *generalized morphism from the manifold M to the Lie group G*, which induces the desired *stack morphism* $M \to \mathcal{B}G$.

The generalization of the concept of "generalized morphism" to 2-groupoids is straightforward: a **generalized morphism** of Lie 2-groupoids $\mathbf{\Gamma} \rightsquigarrow \mathbf{\Delta}$ is a diagram $\mathbf{\Gamma} \underset{\sim}{\xleftarrow{\phi}} \mathbf{E} \xrightarrow{f} \mathbf{\Delta}$ in the category **2Gpd** of Lie 2-groupoids, where ϕ is a Morita equivalence (a "smooth" equivalence of 2-groupoids). It is sometimes useful to think of two Morita equivalent Lie 2-groupoids as two different choices of an atlas (or open cover) on the same geometric object (which is a differentiable 2-stack [Br, BX]). Hence generalized morphisms are the morphisms in the category obtained from the category of Lie 2-groups by inverting the Morita maps. We refer to our paper [2] for details. Following the scheme described above for principal bundles, we make the following

Definition 4.45 A principal (2-group) $[G \to H]$-bundle over a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ is a generalized morphism Υ from $\Gamma_1 \rightrightarrows \Gamma_0$ (seen as a Lie 2-groupoid) to the 2-group $[G \to H]$ associated to the crossed module $(G \to H)$.

Remark 4.46 Since Definition 4.45 is compatible with Morita equivalence, it can be restated in a more stacky way. A principal (2-group) $[G \to H]$-bundle over a differentiable stack $\mathfrak{X}$ is a 2-stack morphism $\Upsilon \colon \mathfrak{X} \to [G \to H]$ (where we identify the Lie 2-groupoid $[G \to H]$ with its underlying 2-stack).

Geometrically, a G**-gerbe over a differentiable stack** $\mathfrak{X}$, where G is a Lie group, can be defined as a *groupoid G-extensions modulo Morita equivalence* [G, BX, LSX]. We recall that

9. this is a *geometric* analogue of the fact that such a data is encoded by an homotopy class of map $M \to BG$, the classifying *space* of G

Definition 4.47 A Lie groupoid G-extension is a short exact sequence of Lie groupoids over the identity map on the unit space M

$$1 \to M \times G \xrightarrow{i} \widetilde{\Gamma} \xrightarrow{\phi} \Gamma \to 1 \tag{4.21}$$

Here Γ, $\widetilde{\Gamma}$ are Lie groupoids over M and $M \times G \rightrightarrows M$ is a (trivial) bundle of groups.

Theorem 4.48 ([2]) *There exists a bijection between (equivalence classes of) Lie groupoid G-extensions and (Morita equivalence classes of) $[G \to \mathrm{Aut}(G)]$-bundles over Lie groupoids (where $[G \to \mathrm{Aut}(G)]$ is the 2-group from Example 4.44).*

An important class of G-extensions is formed by the so called **central G-extensions**. They correspond to G-gerbes with *trivial band* or *G-bound* gerbes. The classical definition of a G-bound gerbe is quite technical. However, from our point of view of gerbes as 2-groups bundles, they are gerbes whose structure 2-group $[G \to \mathrm{Aut}(G)]$ reduces to the simpler 2-group $[Z(G) \to 1]$ (where $Z(G)$ stands for the center of G). Indeed, in [2], we proved

Proposition 4.49 *Let $\widetilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$ be a G-extension of a Lie groupoid Γ and let Υ be the corresponding $[G \to \mathrm{Aut}(G)]$-bundle. The extension is central if and only if the $[G \to \mathrm{Aut}(G)]$-bundle Υ reduces to a principal $[Z(G) \to 1]$-bundle, i.e., there exists a generalized morphism $Z\Upsilon \colon \Gamma \to [Z(G) \to 1]$ such that*

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Upsilon} & [G \to \mathrm{Aut}(G)] \\ & \searrow^{Z\Upsilon} & \uparrow \\ & & [Z(G) \to *] \end{array}$$

is commutative (up to Morita equivalences).

4.6 Characteristic classes and cohomology of 2-groups

As in the case of a group, associated to a Lie 2-group $\Gamma = \Gamma_2 \rightrightarrows \Gamma_1 \rightrightarrows \{*\}$, there is a simplicial manifold $N_\bullet\Gamma$, called its *(geometric) nerve* (we refer to [No3, MS, 3] for details). It is the nerve of the underlying 2-category as defined by Street. Note that $N_0\Gamma = \{*\}$, $N_1\Gamma = \Gamma_1$ and $N_2\Gamma$ consists of 2-arrows of Γ_2 fitting in a commutative square:

$$\begin{array}{ccccc} & & A_1 & & \\ & {}^{f_2}\nearrow & \Downarrow \alpha & \searrow^{f_0} & \\ A_0 & & \xrightarrow[f_1]{} & & A_2 \end{array} \tag{4.22}$$

In particular $N_2\Gamma$ is naturally a submanifold of $\Gamma_2 \times \Gamma_1 \times \Gamma_1 \times \Gamma_1$.

The nerve $N_\bullet$ defines a functor from the category of Lie 2-groups to the category of simplicial manifolds. The nerve of a Lie group considered as a Lie 2-group is isomorphic to the usual (1-)nerve. Taking the fat realization of the nerve defines a functor from Lie 2-groups to topological spaces. In particular, the homotopy and homology groups of a Lie 2-group can be defined as the homotopy and homology groups of its nerve (as in the case of usual (1-)stacks § 4.1).

Let $\mathfrak{X} = [M/G]$ be a quotient stack. Then the canonical epimorphism $M \to \mathfrak{X}$ is a fibration (and a principal G-bundle). There is a similar picture for 2-groups since $[G \xrightarrow{i} H]$ can be thought as kind of "group stack". Indeed, a Lie 2-group $[G \to H]$ induces a short exact sequence of Lie 2-groups:

$$1 \to [\ker i \to 1] \to [G \to H] \to [1 \to \mathrm{Coker}\, i] \to 1$$

which in turn induces a fibration of 2-groups (in the sense of Henriques [He]). Let us describe this more precisely. Let $(\phi, \psi) : \big(G_2 \xrightarrow{i_2} H_2\big) \to \big(G_1 \xrightarrow{i_1} H_1\big)$ be a morphism of crossed modules with ψ being a submersion. The **kernel** of the map (ϕ, ψ) is, by definition (see [No3]), the crossed module $(G_2 \xrightarrow{\widetilde{i}} H_2 \times_{H_1} G_1)$ where $\widetilde{i}$ is the natural group morphism induced by i_2 and ϕ. The $H_2 \times_{H_1} G_1$-action on G_2 is induced by the H_2-action: $g_2^{(h_2,g_1)} = g_2^{h_2}$. The structure map $H_2 \times_{H_1} G_1 \to H_2$ induces a natural crossed module morphism $(G_2 \xrightarrow{\widetilde{i}} H_2 \times_{H_1} G_1) \to \big(G_2 \xrightarrow{i_2} H_2\big)$.

Lemma 4.50 ([3], Lemma 2.2) *Let $(\phi, \psi) : \big(G_2 \xrightarrow{i_2} H_2\big) \to \big(G_1 \xrightarrow{i_1} H_1\big)$ be a morphism of crossed modules with ϕ and ψ being surjective submersions. Then $(\phi, \psi) : [G_2 \xrightarrow{i_2} H_2] \to [G_1 \xrightarrow{i_1} H_1]$ is a fibration of Lie 2-groups. The kernel of the morphism (ϕ, ψ) is equivalent to $[ker(\phi) \xrightarrow{i_2} ker(\psi)]$.*

As an immediate consequence, we get a Leray-Serre spectral sequence

Lemma 4.51 *There is a converging spectral sequence of algebras*

$$L_2^{p,q} = H^p([1 \to H/i(G)], \mathcal{H}^q([\ker(i) \to 1])) \Longrightarrow H^{p+q}([G \xrightarrow{i} H]) \qquad (4.23)$$

where $\mathcal{H}^q([\ker(i) \to 1])$ is the de Rham cohomology viewed as a local coefficient system on $[1 \to H/i(G)]$.

For any compact Lie group G with Lie algebra $\mathfrak{g}$, there are *Bott-Shulman type map* $S(\mathfrak{g}^*)^G \to \Omega^\bullet_{dR}(BG)$ (see [BS]) where the de Rham forms on the right are the total complex of the simplicial cochain algebra $n \mapsto \Omega^\bullet_{dR}(G^n)$ (which agrees with the de Rham forms of the stack $[*/G]$). Indeed, mapping $\xi \in \mathfrak{g}^*$ to its left invariant 1-form ξ^L yields the canonical map $(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow \Omega^1_{dR}(G) \hookrightarrow \Omega^2_{dR}(BG)$. Denoting $p_1 : G \times H \times H \to G$ the projection, we also get

$$(\mathfrak{g}^*)^{\mathfrak{g}} \hookrightarrow \Omega^1_{dR}(G) \xrightarrow{p_1^*} \Omega^1_{dR}(G \times H \times H) \hookrightarrow \Omega^3_{dR}([G \xrightarrow{i} H]). \qquad (4.24)$$

The action of H on G induces an action of H on $\mathfrak{g}$, and therefore an action on $\mathfrak{g}^*$. Further, the map (4.24) restricts to the invariant subspace $(\mathfrak{g}^*)^{\mathfrak{g},H}$. We thus get

Proposition 4.52 ([3]) *The above map* $(\mathfrak{g}^*)^{\mathfrak{g},H}[3] \to \Omega^\bullet_{dR}([G \xrightarrow{i} H])$ *is a map of cochain complexes and extends uniquely to a map of algebras* $I : S^\bullet\big((\mathfrak{g}^*)^{\mathfrak{g},H}[3]\big) \to H^\bullet([G \xrightarrow{i} H])$.

From Proposition 4.52 and the Leray spectral sequence, we easily deduced in [3]

Proposition 4.53 *Let* A *be an abelian compact Lie group with Lie algebra* $\mathfrak{a}$. *The map* $I : S^\bullet(\mathfrak{a}^*[3]) \to H^\bullet([A \to 1])$ *is an isomorphism of graded algebras.*

Let us recall the (homotopical) construction of *characteristic classes* for Lie (1-)groups. Since the set of isomorphism classes of G-principal bundles over a manifold M is in bijection with the set of homotopy classes of maps $M \to BG$, a principal G-bundle P over M determines a (unique up to homotopy) map $M \xrightarrow{f} BG$ from M to the classifying space of G. Pulling back the generators of $H^*(BG)$ (the universal classes) through f, one obtains characteristic classes of the principal bundle P over M. These characteristic classes coincide with those obtained from a connection by applying the Chern-Weil construction [MiSt].

Lemma and Definition 4.54 ([2]) Let $[G \to H]$ be a Lie 2-group and let Υ be a principal $[G \to H]$-bundle over $\mathbf{\Gamma}$, i.e, a generalized morphism $\mathbf{\Gamma} \overset{\Upsilon}{\rightsquigarrow} [G \to H]$. Passing to cohomology, we obtain the homomorphism

$$\mathbf{CC}_\Upsilon \colon H^\bullet([G \to H]) \xrightarrow{\mathcal{B}^*} H^\bullet(\Gamma) \tag{4.25}$$

which we call *the characteristic homomorphism* of the $[G \to H]$-bundle Υ. It depends only on the isomorphism class of the 2-group bundle.

Now *assume* Γ *is a central* G*-gerbe.* In the case where $G = S^1$, characteristic classes for central gerbes, called *Dixmier-Douady class* have been constructed [BX]. The construction relies on *connection and curvature* arguments much like the construction of Chern-Weil classes. In [2], we generalized the construction of the Dixmier-Douady class to all G-bound gerbes with G a connected and reductive Lie group as follows. Since G is reductive, its Lie algebra $\mathfrak{g}$ decomposes as a direct sum $\mathfrak{g} \cong Z(\mathfrak{g}) \oplus \mathfrak{m}$ of ideals, where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$. Since G is connected, this direct sum decomposition is not only $ad_{\mathfrak{g}}$-invariant but also Ad_G-invariant. Let $pr \colon \mathfrak{g} \to Z(\mathfrak{g})$ be the canonical projection and $d_{tot} \colon \Omega^\bullet_{dR}(\Gamma) \to \Omega^{\bullet+1}_{dR}(\Gamma)$ be the *total* differential [10] of the de Rham complex of a Lie groupoid Γ.

Proposition 4.55 ([2], Proposition 4.13) *Let* $\widetilde{\Gamma} \xrightarrow{\phi} \Gamma \rightrightarrows M$ *be a central* G*-extension with* G *connected and reductive. Let* $\alpha \in \Omega^1(\widetilde{\Gamma}, \mathfrak{g})$ *be a connection 1-form for the right principal* G*-bundle* $\widetilde{\Gamma} \xrightarrow{\phi} \Gamma$.

1. *There exists a cycle* $\Omega_\alpha \in \Omega^3_{dR}(\Gamma, Z(\mathfrak{g}))$ *such that* $\mathrm{pr} \circ d_{tot}\big(\alpha\big) = \phi^*(\Omega_\alpha)$.

10. *i.e.* the sum of the de Rham and simplicial differential on the cosimplicial cochain algebra $\Omega^\bullet_{dR}(N_\bullet(\Gamma))$

2. *Moreover, if* α_1 *and* α_2 *are two different connection 1-forms as above, then* $\Omega_{\alpha_1} - \Omega_{\alpha_2}$ *is a coboundary in* $\Omega^3_{dR}(\Gamma, Z(\mathfrak{g}))$.

We call $\mathbf{DD}_{(\alpha)} := [\Omega_\alpha] \in H^3(\Gamma) \otimes Z(\mathfrak{g})$ *the* Dixmier-Douady class *of the G-central extension.*

A G-central extension being a principal $[Z(G) \to 1]$-bundle (Proposition 4.49), from Proposition 4.53 we get the characteristic map $Z(\mathfrak{g}) \cong H^3([Z(G) \to 1]) \xrightarrow{Z\Upsilon} H^3(\Gamma)$. Dualizing, one obtains a class $\mathbf{CC}_\phi \in H^3(\Gamma) \otimes Z(\mathfrak{g})$. In [2], we proved

Theorem 4.56 *Let G be a compact connected Lie group. For any G-central extension of Lie groupoids $\widetilde{\Gamma} \to \Gamma \rightrightarrows M$, the universal characteristic class (Definition 4.54) coincides with the Dixmier-Douady class (Proposition 4.55).*

As an immediate consequence, we find that the Dixmier-Douady class of a central G-gerbe is integral.

We can also use the spectral sequence to compute the cohomology of a 2-group $[G \xrightarrow{i} H]$ with connected and simply connected compact cokernel $C := \mathrm{Coker}(i) \cong H/i(G))$ and compact kernel $A := \ker(i)$. Indeed, in that case, we showed (using Proposition 4.53) in [3] that the fourth page of the spectral sequence (4.23) is concentrated in bidegree $(p, 3q)$ $(p, q \in \mathbb{N})$ and is given by

$$L_4^{p,3q} \; = \; H^p(BC) \otimes S^q(\mathfrak{a}^*[3]) \tag{4.26}$$

where $\mathfrak{a}$ is the Lie algebra of A. In particular, The (higher) differential $d_4 : L_4^{i,j} \to L_4^{i+4,j-3}$ induces a *transgression homomorphism*

$$T : \mathfrak{a}^* \cong L_4^{0,3} \xrightarrow{d_4} L_4^{4,0} \cong H^4(BC). \tag{4.27}$$

The knowledge of the transgression homomorphism determines all the cohomology of $[G \xrightarrow{i} H]$. Indeed we proved in [3] the following

Proposition 4.57 *There is a linear isomorphism*

$$H^\bullet([G \xrightarrow{i} H]) \cong \big(H^\bullet\big(BC\big)/(\mathrm{im}(T)\big) \otimes S^\bullet\big(\ker(T)[3]\big)$$

which is further an algebra isomorphism if $C = H/i(G)$ is simply connected.

Proposition 4.57 also holds for Fréchet 2-groups.

Example 4.58 The main application in [3] is to compute the cohomology of the *string 2-group* [BCSS] *String*(G), where G is a connected and simply connected compact simple Lie group. We recall that there is an unique left invariant closed 3-form ν on G, which generates $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$ and determines the basic central extension

$$1 \to S^1 \to \widetilde{\Omega G} \xrightarrow{\widetilde{p}} \Omega G \to 1$$

of the based (at identity) loop group ΩG of G. Let also PG be the space of paths $f : [0, 1] \to G$ starting at the identity. The conjugation action of PG on ΩG lifts to

$\widetilde{\Omega G}$. The string 2-group (see [BCSS]) is the Fréchet 2-group corresponding to the crossed module

$$String(G) := [\widetilde{\Omega G} \stackrel{p}{\to} PG],$$

where p is the composition $p : \widetilde{\Omega G} \stackrel{\widetilde{p}}{\to} \Omega G \hookrightarrow PG$. Using the naturality of Leray-Serre spectral sequences, we found in [3] that

$$H^\bullet(String(G)) \cong H^\bullet(BG)/([\omega])$$

where $[\omega] \in H^4(BG)$ is obtained from ν by transgression. These results was also independently found by Baez-Stevenson. In particular, the characteristic classes of a $String(G)$-principal bundle P are the characteristic classes of the associated G-principal bundles $G \times_{String(G)} P$ (induced by the 2-group map $String(G) := [\widetilde{\Omega G} \stackrel{p}{\to} PG] \xrightarrow{f \mapsto f(1)} [1 \to G]$) modulo the first Pontrjagin class $p_1(G \times_{String(G)} P)$.

When one wants to study universal characteristic classes for non-central gerbes, the situation becomes much more complicated. Indeed, the cohomology of $[G \stackrel{i}{\to} \mathrm{Aut}(G)]$ is related to the homology of $GL(n, \mathbb{Z})$ via the spectral sequence (4.23).

Theorem 4.59 ([3]) *If G is a compact Lie group, there are converging spectral sequences (concentrated in bidegree $(p, 3q)$, $p, q \in \mathbb{N}$) of graded commutative algebras*

$$\begin{aligned} E_2^{+\,p,3q} &= H^p\big(SL(n,\mathbb{Z}), S^q((\mathfrak{g}^*)^{\mathfrak{g}}[3])\big) \Longrightarrow H^{p+q}\big([G \xrightarrow{i^+} \mathrm{Aut}^+(G)]\big) \\ E_2^{p,3q} &= H^p\big(GL(n,\mathbb{Z}), S^q((\mathfrak{g}^*)^{\mathfrak{g}}[3])\big) \Longrightarrow H^{p+q}\big([G \stackrel{i}{\to} \mathrm{Aut}(G)]\big) \end{aligned}$$

where $n = \dim((\mathfrak{g}^*)^{\mathfrak{g}})$ is the dimension of $(\mathfrak{g}^*)^{\mathfrak{g}}$, and the $SL(n,\mathbb{Z})$-action (or $GL(n,\mathbb{Z})$-action) on $S^q((\mathfrak{g}^*)^{\mathfrak{g}}[3])$ is induced by the natural action on $(\mathfrak{g}^*)^{\mathfrak{g}} \cong \mathbb{R}^n$.

Example 4.60

– Assume G is a semi-simple Lie group. Then $\mathrm{Out}\,(G)$, $\mathrm{Out}^+(G)$, $\ker(i)$ and $\ker(i^+)$ are finite groups. Thus, $H^n([G \stackrel{i}{\to} \mathrm{Aut}(G)]) \cong H^n([G \xrightarrow{i^+} \mathrm{Aut}^+(G)]) \cong \begin{cases} 0 & \text{if } n > 0, \\ \mathbb{R} & \text{if } n = 0. \end{cases}$

– Using result of Soulé [So], one can compute the spectral sequence given in Theorem 4.59 for $n = \dim(\mathfrak{g}^{\mathfrak{g}}) \leq 3$, see [3]:

$$H^p([G \to \mathrm{Aut}^+(G)]) \cong \begin{cases} \mathbb{R} & \text{if } p = 0,\, 3n \\ 0 & \text{otherwise.} \end{cases}$$

5 Algebraic models for mapping spaces and Hochschild (co)homology

Notations and Conventions 5.1 All along Section 5, we assume that we work over a ground *field k*.

We will consider later various "derived categories" of algebras, modules or chain complexes, which can be described as ∞-categories. Following [R, L-TFT], by an

∞-*category* we mean a *complete Segal space* (though our results do not really depend on the choice of a specific model). The ∞-categories we are interested in arise by *Dwyer-Kan localizations* from closed model categories; they should be thought of as nice *derived categories* in which (weak-)equivalences have been inverted (in a non-naive way). We briefly recall below the construction, but what really matters in this "mémoire" are the following examples.

Example 5.2 sSet **and** Top **:** the model category of simplicial sets sSet yields the ∞-category sSet_∞. The cartesian product of simplicial sets gives sSet a structure of monoidal model category (see [Ho] for example). Thus sSet_∞ inherits the structure of symmetric monoidal ∞-category in the sense of [R, L-TFT]. The model category of topological spaces yields the ∞-category Top_∞. Since sSet and Top are Quillen equivalent, their associated ∞-categories are equivalent (as ∞-categories): $\mathsf{sSet}_\infty \underset{\sim}{\overset{\sim}{\leftrightarrows}} \mathsf{Top}_\infty$, where the left and right equivalences are induced by the singular set and geometric realization functors.

There are also *pointed versions* $\mathsf{sSet}_{*\infty}$ and $\mathsf{Top}_{*\infty}$ of the above ∞-categories.

CDGAs: the model category CDGA of commutative differential graded algebras (CDGA for short) yields the ∞-category CDGA_∞ (in which quasi-isomorphism of CDGAs have been "inverted"). It has a (∞-)monoidal structure induced by tensor products of CDGAs. There are model categories A-Mod and A-CDGA of (differential graded) modules and (differential graded) commutative algebras over a CDGA A from which we get ∞-categories A-Mod_∞ and A-CDGA_∞. The base change functor (for a map $f\colon A \to B$) lifts to an ∞-functor $f_*\colon B\text{-}\mathsf{Mod}_\infty \to A\text{-}\mathsf{Mod}_\infty$.

In particular, k-Mod_∞ is the ∞-category of chain complexes.

E_n**-algebras:** there are similarly ∞-categories E_n-Alg of E_n-algebras where E_n is an (∞)-operad equivalent to the little n-dimensional cubes operad. We denote $Mod_A^{E_k}$ the symmetric monoidal ∞-category of (E_k-)modules over an E_k-algebra A (see [L-HA, F]). Recall that, for instance, $Mod_A^{E_k}$ is equivalent to the category of A-bimodules, while, if A is a CDGA, $Mod_A^{E_\infty}$ is equivalent to the (∞-)category of left A-modules.

Let us recall briefly how to get an ∞-category out of a model category. There is a simplicial structure, denoted $\mathcal{S}e\mathcal{S}p$ on the category of simplicial spaces such that a fibrant object in the $\mathcal{S}e\mathcal{S}p$ is precisely a Segal space. Rezk has shown that the category of simplicial spaces has another simplicial closed model structure, denoted $\mathcal{CS}e\mathcal{S}p$, whose fibrant objects are precisely complete Segal spaces [R, Theorem 7.2]. Let $\mathbb{R} : \mathcal{S}e\mathcal{S}p \to \mathcal{S}e\mathcal{S}p$ be a fibrant replacement functor. Let $\widehat{\cdot} : SeSp \to \mathcal{CS}e\mathcal{S}p$, $X_\bullet \to \widehat{X_\bullet}$, be the completion functor that assigns to a Segal space an equivalent complete Segal space. The composition $X_\bullet \mapsto \widehat{\mathbb{R}(X_\bullet)}$ gives a fibrant replacement functor $L_{\mathcal{CS}e\mathcal{S}p}$ from simplicial spaces to complete Segal spaces. Now, a standard idea to go from a model category to a simplicial space is to use Dwyer-Kan localization. Let $\mathcal{M}$ be a model category and $\mathcal{W}$ be its subcategory of weak-equivalences. We denote $L^H(\mathcal{M}, \mathcal{W})$ its *hammock localization*. It is a *simplicial category* such that the category $\pi_0(L^H(\mathcal{M},\mathcal{W}))$ is the *homotopy category* of $\mathcal{M}$. Any weak equivalence has (weak) inverse in $L^H(\mathcal{M},\mathcal{W})$.

Thus, a model category $\mathcal{M}$ gives rise functorialy to the simplicial category $L^H(\mathcal{M},\mathcal{W})$ hence a simplicial space $N_\bullet(L^H(\mathcal{M},\mathcal{W}))$ by taking its nerve. Composing with the complete Segal Space replacement functor we get a functor $\mathcal{M} \to L_\infty(\mathcal{M}) := L_{\mathcal{CS}e\mathcal{S}p}(N_\bullet(L^H(\mathcal{M},\mathcal{W})))$ from model categories to ∞-categories.

5.1 Brief review on Hochschild cohomology and string topology

Hochschild homology groups of an *associative* algebra A with value in a A-bimodule M are defined as

$$HH_n(A,M) \cong H_n\big(A \otimes^{\mathbb{L}}_{A\otimes A^{op}} A\big) \cong Tor_n^{A\otimes A^{op}}(A,M).$$

while Hochschild cohomology groups are defined as

$$HH^n(A,M) \cong H^n\big(\mathbb{R}Hom_{A\otimes A^{op}}(A,M)\big) \cong Ext^n_{A\otimes A^{op}}(A,M).$$

There is a *standard* chain complex $CH_\bullet^{std}(A,M)$ (resp. $CH^\bullet_{std}(A,M)$) that computes Hochschild homology (resp. cohomology) [Ge, Lo2]. One can extend these definitions to sheaves, differential graded algebras and algebras of smooth functions.

In algebraic topology, Hochschild (co)chains are a model for cochains on the free loop space and string topology. Indeed, there is an isomorphism [CJ, FTV]

$$H_*(LX) \cong HH^*(C^*(X), C_*(X)) \cong HH^*(C^*(X), C^*(X))[d] \qquad (5.1)$$

if X is an oriented and simply connected manifold of dimension d [CJ, FTV]. The isomorphism (5.1) is an isomorphism of (Gerstenhaber) algebras. When X is a triangulated oriented Poincaré duality space, applying Sullivan's techniques, Tradler and Zeinalian proved that the Hochschild cohomology $HH^*(C^*(X), C^*(X))$ is a **BV**-algebra (whose underlying Gerstenhaber algebra is the usual one) [TZ]. The intrinsic reason for the existence of this **BV**structure is that a Poincaré duality is a up to homotopy version of a Frobenius structure and that for Frobenius algebras, the Gerstenhaber structure in Hochschild cohomology is always **BV** [Tr]. It shall be noted that, the fact that $H_*(LM)$ is **BV**does *not* require M to be a *closed* manifold (see Theorem 4.31). However, the aforementioned fact that the Hochschild cohomology $HH^*(C^*(M), C^*(M))$ is **BV** and isomorphic to $H_*(LM)$ *seems* to require M to be closed. Actually, similar results can be obtained using only bimodules maps $C^*(M) \to C_*(M)[d]$ with some properties (as is explained in [GTZ3]). Further, for Calabi-Yau (E_1-)algebras, (extended) topological conformal field theories structure on Hochschild *chains* have been obtained [Co2, L-TFT].

Commutative algebras and Adams operations : When A is further *commutative* and M is a *symmetric* bimodule, the Hochschild (co)chains have additional structure. In fact, Gerstenhaber-Schack [GS] and Loday [Lo1, Lo2] have shown that there are *Adams operations* $(\lambda^k)_{k\geq 1}$ inducing γ-rings (with trivial multiplication) structures on Hochschild cohomology groups $HH^*(A,M)$ and homology groups

$HH_*(A, M)$. In characteristic zero, these operations yield a weight-decomposition called the *Hodge decomposition* whose pieces are precisely (higher) Harisson (or André-Quillen if $\mathrm{char}(k) = 0$) (co)homology. These operations have been widely studied for their use in algebra, geometry and their intrinsic combinatorial meaning. For instance, it is known that Hochschild Hodge decomposition induces the classical Hodge decomposition when applied the structure sheaf $\mathcal{O}_X$ of a smooth scheme X; note that Hochschild homology of $\mathcal{O}_X$ is isomorphic to de Rham forms on X and (periodic) cyclic homology is a model for de Rham cohomology.

We recall that a *γ-ring with trivial multiplication* $(A, (\lambda^k))$ is a k-module A equipped with linear maps $\lambda^n : A \to A$ $(n \geq 1)$ such that λ^1 is the identity map and

$$\lambda^p \circ \lambda^q = \lambda^{pq}.$$

There is a canonical decreasing filtration $F_\bullet^\gamma A$ (see [AT, Lo1]) such that λ^k acts as multiplication by k^n on each associated graded module $\mathrm{Gr}^{(n)} A = F_n^\gamma A / F_{n+1}^\gamma A$. When the filtration splits, we get the *Hodge decomposition* whose pieces are the n^i-eigenspaces of the maps λ^n, see [Lo1, Lo2].

5.2 Hochschild (co)homology of homotopy commutative algebras

The cochain complex $C^*(X)$ of a space is close enough of being commutative: it is naturally an E_∞-algebra. Further, in characteristic zero, one can symmetrize the cochain level cup-product to make $C^\bullet(X)$ a "strictly commutative" and *homotopy associative* algebra, *i.e.*, a C_∞-algebra. These algebras also arise naturally in the work of Tamarkin on Deformation Quantization. This motivated our work [5], in which we study Hochschild (co)homology of commutative and C_∞-algebras, notably to add Adams operations to the string topology picture.

5.2.1 Explicit models for homotopical algebras

We first recall briefly the notions of A_∞ and C_∞-algebras and their (bi)-modules. These are *strong* homotopy versions of algebras over operads (see [5] for details).

Definition 5.3 Let R be a graded k-module.

- An A_∞-algebra structure on R is a coderivation D of degree 1 on the shifted *tensor coalgebra* $T^{\geq 1}(R[1]) = \bigoplus_{n \geq 1} (R[1])^{\otimes n}$. A map between two A_∞-algebras R, S is a map of underlying graded differential coalgebras $T^{\geq 1}(R[1]) \to T^{\geq 1}(S[1])$.
- An A_∞-bimodule structure on M over R is an A_∞-structure on the square zero extension $R \oplus M$, meaning an A_∞ structure on $R \oplus M$ such that the trivial projection $R \oplus M \to R$ induces a map of A_∞-algebras which makes $R \oplus M$ an abelian group object over R, where the group structure is given by addition in M. The latter property can be easily described in terms of square zero and degree 1 coderivation on the tensor bicomodule $T(R[1]) \otimes M \otimes T(R[1])$ compatible with the A_∞-structure of R see [5] for example.

– A C_∞-algebra is an A_∞-algebra (R, D) such that the tensor coalgebra $T(R[1])$, endowed with the shuffle product and the differential D, is a (DG-)bialgebra.
– A (strong) C_∞-bimodule structure on M over (R, D) is given by a C_∞-algebra structure on the square zero extension $R \oplus M$ (wich induces an A_∞-bimodule structure on M over R).

The last definition was first introduced in [5] as an homotopy analogue of the notion of a symmetric bimodule. Note that weaker (*non* symmetric on the nose) and more combinatorial definition of C_∞-modules have also been studied in [5] and the results and definition in this Section 5.2 can be extended to those weaker versions, see [5].

Definition 5.4 Let (R, D) be an A_∞-algebra and M an R-bimodule.
– The Hochschild cochain complex $CH^\bullet_{std}(A, M)$ of A with value in M is the space $s^{-1}\mathrm{CoDer}(R, M)$ of coderivations of (the tensor coalgebra of) R into M equipped with differential b given by

$$b(f) = D_M \circ f - (-1)^{|f|} f \circ D.$$

– The Hochschild chains $CH^{std}_\bullet(A, M)$ of A with value in M is the space $(M \otimes T(sR), b)$, where b is the differential

$$\begin{aligned} b(m, a_0, \dots, a_n) &= \sum_{p+q \leq n} \pm D^M_{p,q}(a_{n-p+1}, \dots, a_n, m, a_1 \dots, a_q) \otimes a_{q+1} \cdots a_{n-p} \\ &\quad + \sum_{i+j \leq n} \pm m \otimes a_1 \otimes \cdots D_{j+1}(a_i, \dots, a_{i+j}) \otimes a_{i+j+1} \otimes \cdots a_n. \end{aligned}$$

Using the notion of symmetric bimodules we can also generalize the classical Harrison (co)homology theory to C_∞-algebraic setting.

Recall that a coderivation $f \in \mathrm{CoDer}\big(R, M\big)$ is determined by its projection $f^i : R^{\otimes i \geq 0} \to M$. Denote $\mathrm{BDer}(R, M)$ the subspace of coderivations f such that the f_i vanishes on the module spanned by the shuffles, *i.e.*,

$$f_i\big(\mathrm{sh}(x, y)\big) = 0 \quad \text{for} \quad i \geq 2, x \in R^{k \geq 1}, y \in R^{i-k \geq 1}.$$

Lemma and Definition 5.5 ([5]) Let (R, D) be a strong C_∞-algebra and (M, D^M) be a strong C_∞-bimodule over R.
– The differential $b(f) = D^M \circ f - (-1)^{|f|} f \circ D$ makes $CHar^*(R, M) := \mathrm{BDer}(R, M)$ a cochain complex whose cohomology is called Harrison cohomology $Har^*(R, M)$ of R with values in M.
– The Harrison homology $Har_*(R, M)$ of R with values in M is the homology of the complex $(CHar_*(R, M) := M \otimes T(R[1])/sh, b)$.

5.2.2 Hodge Decomposition of Hochschild cohomology of homotopy commutative algebras

Recall (see [5]) that a coderivation $f \in \mathrm{CoDer}(R, M)$ is uniquely defined by its components $f_i : R^{\otimes i \geq 0} \to M$. Thus, for $n \geq 1$, we obtain the coderivation

$$\lambda^n(f) := \big(f_i \circ \psi^n /_{R^{\otimes i}}\big)_{i \geq 0}$$

defined by the maps $f_i \circ \psi^n : R^{\otimes i} \to M$. In [5], we proved

Theorem 5.6 *Let (R, D) be a C_∞-algebra and (M, D^M) be a (strong) C_∞-bimodule over R. Then*

1. *The $(\lambda^i)_{i\geq 0}$ give a γ-ring with trivial multiplication structure to $CH^\bullet_{std}(R, M)$ as well as to the Hochschild cohomology $HH^\bullet(R, M)$.*
2. *If k contains $\mathbb{Q}$, there is a natural Hodge decomposition*

$$HH^*(R, M) = \prod_{n\geq 0} HH^*_{(n)}(R, M)$$

into eigenspaces for the maps λ^n. Moreover $HH^*_{(1)}(R, M) \cong Har^*(R, M)$ and $HH^*_{(0)}(R, M) \cong H^*(M)$.

3. *If k is a $\mathbb{Z}/p\mathbb{Z}$-algebra, there is a natural Hodge decomposition*

$$HH^*(R, M) = \bigoplus_{0\leq n\leq p-1} HH^*_{(n)}(R, M)$$

with each λ^n acts by multiplication by n^i on $HH^*_{(i)}(R, M)$.

We also proved [5, Theorem 3.18] a dual result for Hochschild homology. The Hodge decomposition given in Theorem 5.6 agrees with the classical one for strictly associative and commutative algebras and bimodules ([5, Proposition 3.5]) and are natural with respect to maps of C_∞-algebras and C_∞-bimodules by [5, Proposition 3.10]. They are also compatible with the Eilenberg-Moore spectral sequences relating $HH^\bullet(H^\bullet(R), H^\bullet(M))$ to $HH^\bullet(R, M)$ see [5, Proposition 3.7].

In [5], we proved that Hochschild cohomology of A_∞-algebras also has a Gerstenhaber algebra structure. Moreover we proved, for C_∞-algebras:

Proposition 5.7 ([5], Theorem 3.31) *Let R be a C_∞-algebra.*

- *The Harrison cohomology $Har^*(R, R) = HH^*_{(1)}(R, R)$ is stable by the Gerstenhaber bracket.*
- *If $k \supset \mathbb{Q}$, the cup-product and Gerstenhaber bracket are filtered for the Hodge filtration $\mathcal{F}_p HH^*(R, R) = \bigoplus_{n\leq q} HH^*_{(n)}(R, R)$, in the sense that*

$$\mathcal{F}_p HH^*(R, R) \cup \mathcal{F}_q HH^*(R, R) \subset \mathcal{F}_{p+q} HH^*(R, R) \text{ and}$$

$$[\mathcal{F}_p HH^*(R, R), \mathcal{F}_q HH^*(R, R)] \subset \mathcal{F}_{p+q-1} HH^*(R, R)$$

5.2.3 Applications to String Topology

Let X be a triangulated oriented closed space with Poincaré duality, such that the closure of every simplex has the homology of a point. Using Homological Perturbation theory (following the ideas of [TZ, Tr, W]) yields

Lemma 5.8 ([5], Lemma 5.4) *The singular cochains $C_*(X)$ can be endowed with a counital C_∞-coalgebra structure (with structure maps $\delta^i : C_*(X) \to C_*(X)^{\otimes i}$) such that*

– *$(C_*(X), \delta)$ is quasi-isomophic (as an A_∞-coalgebra) to the usual DG-coalgebra structure $(C_*(X), d + \Delta)$;*
– *the cochain $C^*(X)$ inherits an unital C_∞-structure by duality which is quasi-isomorphic (as an A_∞-algebra) to the usual cochain algebra $(C^*(X), d + \cup)$;*
– *there is a Poincaré duality $C_*(X) \stackrel{\Xi}{\to} C^*(X)[\dim(X)]$ of A_∞-modules inducing the Poincaré duality isomorphism in (co)homology.*

We deduce from Section 5.2.2, Lemma 5.8 and the isomorphisms (5.1) the following

Theorem 5.9 ([5]) – *There is a* **BV**-*structure on $HH^*(C^*(X), C^*(X))$ and a compatible γ-ring structure.*
– *If X is simply connected, there is a* **BV**-*algebra structure on $\mathbb{H}_*(LX) := H_{*+d}(LX)$ and a compatible γ-ring structure. When X is a manifold the underlying Gerstenhaber structure of the* **BV**-*structure is the Chas-Sullivan one [CS].*

By a BV-structure on a graded space H^* and compatible γ-ring structure we mean the following:

1. H^* is both a BV-algebra and a γ-ring.
2. The BV-operator Δ and the γ-ring maps λ^k satisfy $\lambda^k(\Delta) = k\Delta(\lambda^k)$.
3. There is an "ideal augmentation" spectral sequence $J_1^{pq} \Rightarrow H^{p+q}$ of BV algebras.
4. On the induced filtration J_∞^{p*} of the abutment H^*, one has, for any $x \in J_\infty^{p*}$ and $k \geq 1$,
$$\lambda^k(x) = k^p x \bmod J_\infty^{p+1*}.$$
5. If $k \supset \mathbb{Q}$, there is a Hodge decomposition $H^* = \prod_{i \geq 0} H^*_{(i)}$ (given by the associated graded of the filtration J_∞^{**}) such that the filtered space $\mathcal{F}_p H^* := \bigoplus H^*_{(n \leq p)}$ is a filtered BV-algebra.

As a consequence of Theorem 5.9, $Har^*(C^*(X), C^*(X))$ has an induced Lie algebra structure. Moreover $J_\infty^{0*}/J_\infty^{1*} \cong H_*(X)$ always splits.

Remark 5.10 The Poincaré duality quasi-isomorphism needed in the proof of Theorem 5.9 depends on choices and, consequently, the **BV**-operator too. It is known, that, in general, there exists different reasonable choices of **BV**-operator yielding to Chas-Sullivan Gerstenhaber algebra structure, see [Me].

Remark 5.11 The techniques of higher Hochschild chains we developed later allows to lift (and study in more depth) the results of this Section to the case of E_∞-algebras. We will illustrate this fact in [GTZ3] and future work. Note however, that the C_∞-algebras models sometimes carries combinatorial meaningful informations [ChGe]).

5.3 Higher Hochschild (co)homology as Mapping spaces

In this section we will explain how to generalize the relationship between Hochschild theory and loop spaces to general mapping spaces (as mentioned in the introduction) and gave an application to the surface product.

5.3.1 Hochschild (co)chains over spaces

Following [P, 6, 7], the higher Hochschild complex over spaces is a functor $CH : \mathsf{sSet} \times \mathsf{CDGA} \to \mathsf{CDGA}$. This functor is defined as follows.

Let $(A = \bigoplus_{i\in\mathbb{Z}} A^i, d, ^\bullet)$ be a differential graded, associative, commutative algebra and let n_+ be the set $n_+ := \{0, \dots, n\}$. We define $CH_{n_+}(A) := A^{\otimes n+1} \cong A^{\otimes n_+}$. Let $f : k_+ \to \ell_+$ be any function, we denote by $f_* : A^{\otimes k_+} \to A^{\otimes \ell_+}$, the linear map given by

$$f_*(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = (-1)^\epsilon \cdot b_0 \otimes b_1 \otimes \cdots \otimes b_\ell, \tag{5.2}$$

where $b_j = \prod_{i\in f^{-1}(j)} a_i$ (or $b_j = 1$ if $f^{-1}(j) = \emptyset$) for $j = 0, \dots, \ell$. The sign ϵ in equation (5.2) is determined by the usual Koszul sign rule of $(-1)^{|x|\cdot|y|}$ whenever x moves across y. In particular, $n_+ \mapsto CH_{n_+}(A)$ is functorial. Taking tensor products indexed by any finite sets, and extending the construction by colimit we obtain a well-defined functor

$$Y \mapsto CH_Y(A) := \varinjlim_{Fin \ni K \to Y} CH_K(A)$$

from pointed sets to (DG) k-modules. Since the tensor products of CDGAs is a CDGA, this functor actually takes values in CDGA. Now, if $Y_\bullet$ is a simplicial set, we get a simplicial CDGA $CH_{Y_\bullet}(A)$ and by the Dold-Kan construction a CDGA (whose product is induced by the *shuffle* product, we refer to our paper[7] for details).

Definition 5.12 Let $Y_\bullet$ be a simplicial set. The Hochschild chains over $Y_\bullet$ of A is the commutative differential graded algebra $CH_{Y_\bullet}(A)$, whose homology, denoted $HH_{Y_\bullet}(A)$, is called higher Hochschild homology of A over $Y_\bullet$.

Note that if $Y_\bullet$ is a *pointed* simplicial set, there is a canonical CDGA map $A \xrightarrow{\sim} CH_{pt_\bullet}(A) \to CH_{Y_\bullet}(A)$. This allows to add a module structure to the previous construction as well as to define higher Hochschild cochains as we did in [6] (for Hochschild cochains) and [7].

Definition 5.13 The *Hochschild chains* of a CDGA A with value in a A-module M over a pointed simplicial set $Y_\bullet$ is defined as

$$CH_{X_\bullet}(A, M) = M \underset{A}{\otimes} CH_{X_\bullet}(A)$$

The *Hochschild cochains* of a CDGA A with value in M over the (pointed) simplicial set $Y_\bullet$ is given by

$$CH^{X_\bullet}(A, M) = Hom_A(CH_{X_\bullet}(A), M).$$

Let us denote Mod^{CDGA} the ∞-category of DG-modules over (some) CDGA, which informally can be thought of as the category of pairs (A, M) of a CDGA A and a A-module M (see[L-HA, F, Fr1] for details on ∞-categories of modules). The *Hochschild (co)chains functors are homotopy invariant* with respect to both arguments (see [P, 8]); it allows to lift these functors to ∞-categories.

Proposition 5.14 ([8, GTZ3]) *The derived Hochschild chain* $(X_\bullet, A) \mapsto CH_{X_\bullet}(A)$ *lifts as a functor of* $(\infty,1)$*-categories*

$$CH : \mathsf{sSet}_\infty \times \mathsf{CDGA}_\infty \to \mathsf{CDGA}_\infty.$$

The derived Hochschild chain $CH_{X_\bullet}(A, M)$ *given by Definition 5.13 lifts as a functor of* $(\infty,1)$*-categories* $CH : (X_\bullet, M) \mapsto CH_{X_\bullet}(\iota(M), M)$ *from* $\mathsf{sSet}_{*\infty} \times Mod^{\mathsf{CDGA}}$ *to* Mod^{CDGA}.

The derived Hochschild cochain $CH^{X_\bullet}(A, M)$ *lifts as a functor of* $(\infty,1)$*-categories* $\mathsf{sSet}_{*\infty} \times Mod^{\mathsf{CDGA}}$ *to* $\mathsf{k}\text{-}\mathsf{Mod}_\infty$.

Since the ∞-category of simplicial sets is equivalent to the ∞-category of spaces, all the above functors can be defined with Top_∞ (or its pointed version) instead of sSet_∞. In particular, for any space X, there is a canonical equivalence (in CDGA_∞) $CH_X(A) \cong CH_{S_\bullet(X)}(A)$ where $S_\bullet(X) := \mathrm{Map}(\Delta^\bullet, X)$ is the singular set of X.

Remark 5.15 The above definitions make sense in a much broader context. In fact, they have analogue for any symmetric monoidal ∞-category (in the sense of [L-HA]) in place of CDGA, since they can be *tensored* over spaces. In the case of CDGAs, the monoidal structure agrees with the coproduct and is fairly easy to describe and work with *explicitly* as we did above and in [6, 7, 8].

In particular, the above definitions 5.12 and 5.13 extend in a completely analogous way to E_∞-algebras as well as Proposition 5.14

5.3.2 Models for mapping spaces and Iterated Chen integrals

The Hochschild chain functor models mapping spaces in two different sense. First, there is generalization of Chen Iterated integrals that we studied in [7]. Let M be a compact, oriented manifold, and denote by $\Omega_{dR} = \Omega^\bullet_{dR}(M)$ the space of differential forms on M and let $Y_\bullet$ be a simplicial set with geometric realization $Y := |Y_\bullet|$. Denote by $M^Y := \mathrm{Map}_{sm}(Y, M)$ the space of continuous maps from Y to M, which are smooth on the interior of each simplex $Image(\eta(i)) \subset Y$. Recall from Chen [Ch, Definition 1.2.1], that a differentiable structure on M^Y is specified by the set of plots $\phi : U \to M^Y$, where $U \subset \mathbb{R}^n$ for some n, which are those maps whose adjoint $\phi_\sharp : U \times Y \to M$ is continuous on $U \times Y$, and smooth on the restriction to the interior of each simplex of Y, i.e. $\phi_\sharp|_{U\times(\text{simplex of }Y)^\circ}$ is smooth. Following [Ch, Definition 1.2.2], a p-form $\omega \in \Omega^p_{dR}(M^Y)$ on M^Y is given by a p-form $\omega_\phi \in \Omega^p_{dR}(U)$ for each plot $\phi : U \to M^Y$, which is invariant with respect to smooth transformations of the domain.

We now define the space of Chen (generalized) *iterated integrals* $\mathcal{C}hen(M^Y)$ of the mapping space M^Y. Let $\eta\colon Y_\bullet \to S_\bullet|Y_\bullet|$ be the canonical simplicial map (induced by adjunction) which is given for $i \in Y_k$ by maps $\eta(i) : \Delta^k \to Y$ in the following way,

$$\eta(i)(t_1 \leq \cdots \leq t_k) := [(t_1 \leq \cdots \leq t_k) \times \{i\}] \in \left(\coprod \Delta^\bullet \times Y_\bullet / \sim\right) = Y.$$

The map η allows to define, for any plot $\phi : U \to M^Y$, a map $\rho_\phi := ev \circ (\phi \times id)$,

$$\rho_\phi : U \times \Delta^k \xrightarrow{\phi \times id} M^Y \times \Delta^k \xrightarrow{ev} M^{Y_k}, \tag{5.3}$$

where ev is defined as the evaluation map,

$$ev(\gamma : Y \to M, t_1 \leq \cdots \leq t_k) = \big(\ldots, \big(\gamma \circ \eta(i)\big)(t_1 \leq \cdots \leq t_k), \ldots, \big)_{i \in Y_k}. \tag{5.4}$$

Now, if we are given $y_k := \#Y_k$ many forms on M, $a_0, \ldots, a_{y_k} \in \Omega = \Omega^\bullet_{dR}(M)$, or more precisely a form $a_0 \ldots a_{y_k} \in \big(\Omega_{dR}(M)\big)^{\otimes Y_k}$, the pullback $(\rho_\phi)^*(a_0 \otimes \cdots \otimes a_{y_k}) \in \Omega^\bullet(U \times \Delta^k)$, may be integrated along the fiber Δ^k, and is denoted by

$$\left(\int_{\mathcal{C}} a_0 \ldots a_{y_k}\right)_\phi := \int_{\Delta^k} (\rho_\phi)^*(a_0 \otimes \cdots \otimes a_{y_k}) \quad \in \Omega^\bullet_{dR}(U).$$

The resulting $p = (\sum_i deg(a_i) - k)$-form $\int_{\mathcal{C}} a_0 \ldots a_{y_k} \in \Omega^p_{dR}(M^Y)$ is called the (generalized)*iterated integral* of $a_0, \ldots, a_{y_k}$. The subspace of the space of De Rham forms $\Omega^\bullet(M^Y)$ generated by all iterated integrals is denoted by $\mathcal{C}hen(M^Y)$. In short, we may picture an iterated integral as the pullback composed with the integration along the fiber Δ^k of a form in M^{Y_k},

$$M^Y \xleftarrow{\int_{\Delta^k}} M^Y \times \Delta^k \xrightarrow{ev} M^{Y_k}$$

Definition 5.16 We define $\mathcal{I}t^{Y_\bullet} : CH_{Y_\bullet}(\Omega, \Omega) \cong \Omega^{\otimes Y_\bullet} \to \mathcal{C}hen(M^Y)$ by

$$\mathcal{I}t^{Y_\bullet}(a_0 \otimes \cdots \otimes a_{y_k}) := \int_{\mathcal{C}} a_0 \ldots a_{y_k}. \tag{5.5}$$

Theorem 5.17 ([7]) *The iterated integral map $\mathcal{I}t^{Y_\bullet} : CH_{Y_\bullet}(\Omega, \Omega) \to \Omega^\bullet_{dR}(M^Y)$ is a (natural) map of CDGAs.*

Further, assume that $Y = |Y_\bullet|$ is n-dimensional, i.e. the highest degree of any non-degenerate simplex is n, and assume that M is n-connected. Then, $\mathcal{I}t^{Y_\bullet}$ is a quasi-isomorphism.

Dualizing the construction of iterated integrals, we obtained [7, Corollary 2.5.5],

Corollary 5.18 *Under the assumptions of Theorem 5.17, we have a quasi-isomorphism $(\mathcal{I}t^{Y_\bullet})^* : C_\bullet(\mathrm{Map}(Y, M)) \to CH^{Y_\bullet}(\Omega, \Omega^*)$.*

Explicit examples of iterated integrals are described carefully in [7].

Remark 5.19 Theorem 5.17 and Corollary 5.18 have analogs within the E_∞-algebra context as we prove in [GTZ3].

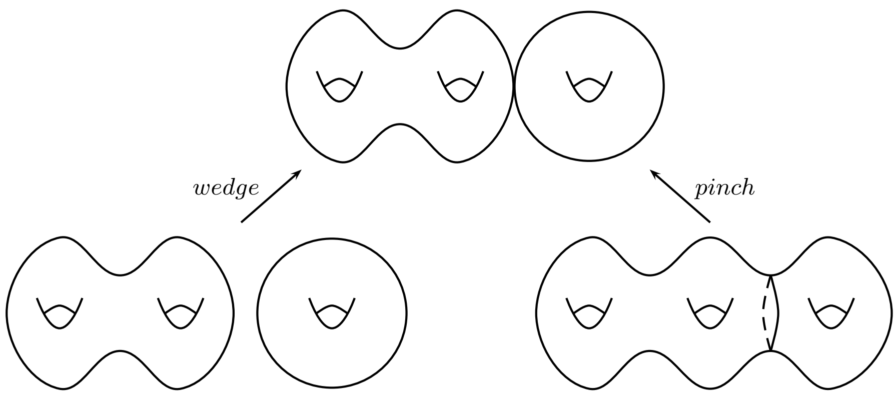
There is also a (derived) algebraic geometry interpretation of Higher Hochschild chains in terms of *derived mapping spaces*. Let $\mathbf{dSt}_k$ be the (model) category of derived stacks over k described in details in [TV, Section 2.2] (which is a derived enhancement of the category of stacks over k). This category admits internal Hom's that we denote by $\mathbb{R}\,Map(F,G)$ following [TV]. To any simplicial set $X_\bullet$, we associate the constant simplicial presheaf $k-Alg \to \mathsf{sSet}$ defined by $R \mapsto X_\bullet$ and we denote $\mathfrak{X}$ the associated stack. For a (derived) stack $\mathfrak{Y}$, we denote $\mathcal{O}_{\mathfrak{Y}}$ its functions [TV] (*i.e.*, $\mathcal{O}_{\mathfrak{Y}} := \mathbb{R}\underline{Hom}(\mathfrak{Y}, \mathbb{A}^1)$).

Proposition 5.20 ([8]) *Let $\mathfrak{R} = \mathbb{R}\,Spec(R)$ be an affine derived stack (for instance an affine stack) [TV]. Then the Hochschild chains over $X_\bullet$ with coefficient in R represent the mapping stack $\mathbb{R}\,Map(\mathfrak{X}, \mathfrak{R})$. That is,*

$$\mathcal{O}_{\mathbb{R}\,Map(\mathfrak{X},\mathfrak{R})} \;\cong\; CH_{X_\bullet}(R).$$

5.3.3 Application: the surface product

The collection of compact surfaces of any genus is naturally equipped with a product similar to the loop product of string topology [CS]. The idea behind this product, that we call the surface product, is shown in the following picture.



(5.6)

In our work [7, Section 3.1], we gave an explicit simplicial set model, denoted $\Sigma^g_\bullet$ of a genus g-surface, together with maps of simplicial sets Pinch$g,h\colon \Sigma^{g+h}_\bullet \to \Sigma^g_\bullet \vee \Sigma^h_\bullet$ (for $g,h \geq 1$) describing the pinching maps (after passing to geometric realization) as in figure (5.6). These maps further satisfy an associativity relation [7, Lemma 3.1.4]. Roughly, the simplicial set $\Sigma^g_\bullet$ is obtained by representing a genus $g \geq 1$ surface as a $4g$-gon, where the boundary is identified via a word $a_1b_1a_2b_2\dots a_gb_ga_g^{-1}b_g^{-1}\dots a_2^{-1}b_2^{-1}a_1^{-1}b_1^{-1}$. The polygon is subdivided into g^2 standard simplicial models of the squares, and further e the off diagonal squares are subdivided further into two standard simplicial models of triangles. The pinching map is obtained by collapsing some of the off diagonal squares to the (simplicial model of the) point. We refer to [7, Section 3.1] for details and figures. There is also a simplicial set model for the pinching map $\mathrm{Pinch}_{g,0}\colon \Sigma^g \to \Sigma^g \vee \Sigma^0$ induced by a simplicial set map $\widetilde{\Sigma}^g_\bullet \to \Sigma^g_\bullet \vee \Sigma^0_\bullet$ where $\widetilde{\Sigma}^g$ is another simplicial model for Σ^g, see [7, Section 3.2].

Denote by $\mathrm{Map}(\Sigma^g, M)$ the space of (continuous, non pointed) maps from a surface Σ^g to a closed oriented manifold M. Then there are induced maps

$$\mathrm{Map}(\Sigma^g, M) \times \mathrm{Map}(\Sigma^h, M) \overset{\rho_{in}}{\leftarrow} \mathrm{Map}(\Sigma^g \vee \Sigma^h, M) \overset{\rho_{out}}{\rightarrow} \mathrm{Map}(\Sigma^{g+h}, M),$$

where ρ_{in} is given by including to the first and second component in $\Sigma^g \vee \Sigma^h$ and ρ_{out} is induced by the realization of the pinching map $\mathrm{Pinch}g,h\colon |\Sigma^{g+h}_\bullet| \to |\Sigma^g_\bullet| \vee |\Sigma^h_\bullet|$.

Note that the map ρ_{in} is given as a pullback of diagrams

$$\begin{array}{ccc} \mathrm{Map}(\Sigma^g \vee \Sigma^h, M) & \xrightarrow{\rho_{in}} & \mathrm{Map}(\Sigma^g, M) \times \mathrm{Map}(\Sigma^h, M) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\text{diagonal}} & M \times M \end{array} \tag{5.7}$$

In particular, ρ_{in} is an embedding of infinite dimensional manifolds with finite codimension equal to the dimension of M, $\mathrm{codim}(\rho_{in}) = \dim(M)$. Using either Thom space (as in [7, Section 3.2]) or our bivariant theory (Section 4.2 or [4]) we obtain a Gysin homomorphism,

$$\begin{aligned} (\rho_{in})_! : H_\bullet(\mathrm{Map}(\Sigma^g, M)) \otimes H_\bullet(\mathrm{Map}(\Sigma^h, M)) \\ \cong H_\bullet(\mathrm{Map}(\Sigma^g, M) \times \mathrm{Map}(\Sigma^h, M)) \\ \longrightarrow H_{\bullet - \dim(M)}(\mathrm{Map}(\Sigma^g \vee \Sigma^h, M)). \end{aligned} \tag{5.8}$$

Definition 5.21 The *surface product* is the composition $\uplus := (\rho_{out})_* \circ (\rho_{in})_!$:

$$\begin{aligned} \uplus : H_\bullet(\mathrm{Map}(\Sigma^g, M)) \otimes H_\bullet(\mathrm{Map}(\Sigma^h, M)) \\ \overset{(\rho_{in})_!}{\longrightarrow} H_\bullet(\mathrm{Map}(\Sigma^g \vee \Sigma^h, M)) \overset{(\rho_{out})_*}{\longrightarrow} H_\bullet(\mathrm{Map}(\Sigma^{g+h}, M)) \end{aligned}$$

Let $i_g : M \to \mathrm{Map}(\Sigma^g, M)$ be the obvious embedding of M (as constant functions) and $[M]_g := i_{g*}([M]) \in H_{\dim(M)}(\mathrm{Map}(\Sigma^g, M))$ be the image of the fundamental class.

Theorem 5.22 ([7], Section 3.2) *Let M be a closed oriented manifold.*

1. *The surface product $\uplus$ (Definition 5.21) is associative;*
2. *for $g = 0$, the surface product is Sullivan-Voronov Brane product [CV].*
3. *$H_{\bullet+\dim(M)}(\mathrm{Map}(\Sigma^g, M))$ is a symmetric $H_{\bullet+\dim(M)}(\mathrm{Map}(\Sigma^0, M))$-bimodule,* i.e., *one has $[M]_0 \uplus x = x$ and $x \uplus y = (-1)^{|y|\cdot|x|} y \uplus x$. In particular the surface product is unital, with unit $[M]_0$.*

Remark 5.23 The assumption that M is closed is not essential at that point if we forget about the unitality assumption. Indeed, using the framework we build in Section 4.2, one can show that if $\mathfrak{X}$ is any oriented Hurewicz stack, the surface product $\uplus : H_{\bullet+\dim(\mathfrak{X})}(\mathrm{Map}(\Sigma^g,\mathfrak{X})) \otimes H_{\bullet+\dim(\mathfrak{X})}(\mathrm{Map}(\Sigma^h,\mathfrak{X})) \to H_{\bullet+\dim(\mathfrak{X})}(\mathrm{Map}(\Sigma^{g+h},\mathfrak{X}))$ is well defined and Theorem 5.22.1 and 2 holds.

There is an algebraic Hochschild model for the surface product originally introduced in our work [7]. Here we briefly described it, using the techniques of [8] (Proposition 5.14 in particular) to simplify the exposition.

The Hochschild wedge-product described below (5.10) allows to define

Definition 5.24 Let B be an A-algebra. The *cup-product* is the composition

$$\cup : CH^{\Sigma^g}(A,B)\otimes CH^{\Sigma^h}(A,B) \xrightarrow{\vee} CH^{\Sigma^g\vee\Sigma^h}(A,B) \overset{\mathrm{Pinch}_{g,h}^*}{\longrightarrow} CH^{\Sigma^{g+h}}(A,B)$$

Putting together various simplicial models for pinching maps and surfaces, using naturality and homotopy invariance of Hochschild cochains, we proved

Theorem 5.25 ([7], Section 3.3) *Let A be a CDGA and B an A-algebra.*

1. *The cup product (Definition 5.24) makes $\bigoplus_{g\geq 0} HH^{\Sigma^g}(A,B)$ into an associative (bi)graded algebra. Furthermore, $\bigoplus_{g\geq 0} HH^{\Sigma^g}(A,B)$ is unital with unit being the class $1_B \in H^0(B,d_B)\cong \big(HH^{\Sigma^0}(A,B)\big)^0$.*
2. *The cup-product is functorial with respects to maps of CDGAs in both arguments and preserves (weak) equivalences.*
3. *$HH^{\Sigma^0}(A,A)$ lies in the center of $\bigoplus_{g\geq 0} HH^{\Sigma^g}(A,B)$; in particular it is a commutative (sub-)algebra.*

Remark 5.26 The techniques we introduced in [7, Section 3.3] together with section 5.6 below actually show that Theorem 5.25 can be lifted at the level of the (∞-)category of differential graded algebras by replacing Hochschild cohomology by Hochschild cochains. In that case, the point 3, should be read as $CH^{\Sigma^0}(A,A)$ lies in the *derived center* of $\bigoplus_{g\geq 0} CH^{\Sigma^g}(A,B)$; in particular it is an E_2-algebra.

Now we relate the algebraic and topological construction. Let M be a simply connected compact manifold and denote again $\Omega = \Omega^\bullet_{dR}M$ its de Rham cochain algebra and $\Omega^* = Hom(\Omega,k)$ its linear dual.

Lemma 5.27 *There are natural "Poincaré duality" isomorphisms*

$$\mathcal{P} : HH_{\Sigma^g}(\Omega,\Omega) \overset{\cong}{\to} HH_{\Sigma^g}(\Omega,\Omega^*)[\dim(M)],\ \ \mathcal{P} : HH^{\Sigma^g}(\Omega,\Omega) \overset{\cong}{\to} HH^{\Sigma^g}(\Omega,\Omega^*)[\dim(M)]$$

which are functorial with respect to smooth oriented maps between manifolds of the same dimension.

Composing with the Chen iterated integral morphism we get a linear map

$$\mathcal{I}t_{\mathcal{P}}^{\Sigma^\bullet}: \bigoplus_{g\geq 0} H_\bullet(\mathrm{Map}(\Sigma^g, M))[\dim(M)]$$

$$\stackrel{\oplus(\mathcal{I}t^{\Sigma^g_\bullet})^*}{\longrightarrow} \bigoplus_{g\geq 0} HH^{\Sigma^g}(\Omega,\Omega^*)[\dim(M)] \stackrel{\oplus\mathcal{P}}{\longrightarrow} \bigoplus_{g\geq 0} HH^{\Sigma^g}(\Omega,\Omega). \quad (5.9)$$

that we call the *Poincaré dual of Chen iterated integral.*

Theorem 5.28 ([7], Section 3.4) *Let M be a 2-connected compact manifold. The linear map* (5.9)

$$\mathcal{I}t_{\mathcal{P}}^{\Sigma^\bullet}: (\bigoplus_{g\geq 0} H_\bullet(\mathrm{Map}(\Sigma^g, M))[\dim(M)] \to \bigoplus_{g\geq 0} HH^{\Sigma^g}(\Omega,\Omega)$$

is an isomorphism of algebras (with respect to the surface product and cup-product).

Corollary 5.29 *Let M, N be 2-connected compact manifolds with equal dimensions, and let $i: M \to N$ be an homotopy equivalence. Then*

$$i_*: \big(\bigoplus_{g\geq 0} H_{\bullet+\dim(M)}(\mathrm{Map}(\Sigma^g, M)), \uplus\big) \to \big(\bigoplus_{g\geq 0} H_{\bullet+\dim(M)}(\mathrm{Map}(\Sigma^g, N)), \uplus\big)$$

is an isomorphism of algebras.

In particular, the surface product is homotopy invariant for 2-connected manifolds.

Ine [7, Section 4.4], we proved an analogue of Hochschild-Kostant-Rosenberg Theorem, and, using the Hochschild model we compute explicitly the surface product when M is a Lie group.

5.4 Higher Hochschild homology and Field Theories

To fully appreciate the higher Hochschild functors of Section 5.3, one *needs* to go beyond mere homology and work with the derived lifts from Proposition 5.14. Indeed, the lift allows to formulate a *locality axiom*, reminiscent of the locality axioms of topological field theories. This locality axiom is the analogue of the *excision* axiom of Eilenberg-Steenrod for usual (co)homology theories. This gluing property together with the homotopy invariance allow to study many examples of Hochschild chain complexes and to do computations as was demonstrated in [7, 8]. Further, such an enhancement is needed in order to correctly compare higher Hochschild with more sophisticated concepts, such as topological chiral homology, which naturally lies in a homotopical setting.

5.4.1 Axiomatic characterization of higher Hochschild homology à la "Eilenberg-Steenrod"

Putting together Theorem 4.2.2 and Theorem 4.3.1 in our paper [8], we get

Theorem 5.30 *The (derived) Hochschild chains functor* $CH : \mathsf{Top}_\infty \times \mathsf{CDGA}_\infty \to \mathsf{CDGA}_\infty$ *satisfies the following axioms*

1. **value on a point:** *there is a natural equivalence* $CH^\bullet_{pt}(A) \cong A$ *in* CDGA_∞.
2. **monoidal:** *there are natural equivalences (in* CDGA_∞*)*

$$CH_{\coprod\limits_{i\in I} X_i}(A) \;\cong\; \bigotimes_{i\in I} CH_{X_i}(A)$$

3. **homotopy glueing/pushout:** CH *sends homotopy pushout in* Top_∞ *to homotopy pushout in* CDGA_∞. *More precisely, given maps* $Z \xrightarrow{f} X$ *and* $Z \xrightarrow{g} Y$ *in* Top_∞, *and* $W \cong X \bigcup^h_Z Y$ *a homotopy pushout, there is a natural equivalence*

$$CH_W(A) \cong CH_X(A) \otimes^{\mathbb{L}}_{CH_Z(A)} CH_Y(A).$$

4. *Similarly, the (derived) Hochschild chains with coefficients functor* $CH : \mathsf{Top}_{*\infty} \times Mod^{\mathsf{CDGA}} \to Mod^{\mathsf{CDGA}}$ *has the property that* $CH_X(A, M)$ *is a* $CH_X(A)$*-module which further satifies the following axioms*
   (a) $CH_{pt}(A, M) \cong M$ *in* $A - Mod_\infty$.
   (b) $CH_{X\coprod Y}(A, M) \cong CH_X(A, M) \otimes CH_Y(A)$ *in* $CH_{X\coprod Y}(A) - Mod_\infty$
   (c) $CH_{X\bigcup^h_Z Y}(A, M) \cong CH_X(A, M) \otimes^{\mathbb{L}}_{CH_Z(A)} CH_Y(A)$ *in* $CH_{X\bigcup^h_Z Y}(A) - Mod_\infty$.

The above axioms actually fully characterized the Hochschild chain functor.

Theorem 5.31 *The Hochschild chains is the* unique *(up to natural equivalence) (*∞*-)functor* $\mathsf{Top}_\infty \times \mathsf{CDGA}_\infty \to \mathsf{CDGA}_\infty$ *satisfying the axioms (1), (2), (3) in Theorem 5.30.*

Sketch of Proof. The Theorem follows from Theorem 4.2.7 Proposition 4.3.2 in our paper [8]. The argument is essentially that simplicial sets can be reconstructed by taking disjoint union and pushout from a point. □

Let us mention a few easy but nevertheless useful corollaries, see [8].

– The derived Hochschild chains functors commute with homotopy colimits in Top_∞ and in CDGA_∞.
– *Exponential law:* . There is a natural equivalence (in CDGA_∞)

$$CH_{X\times Y}(A) \xrightarrow{\sim} CH_X\left(CH_Y(A)\right)).$$

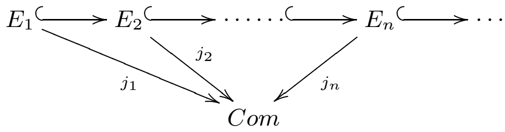
– there are converging Eilenberg-Moore spectral sequences of differential $HH_Z(A)$-modules

$$E_2^{p,q} := Tor^{HH_Z(A)}_{p,q}\left(HH_X(A, M), HH_Y(A, N)\right) \Longrightarrow HH_{X\cup^h_Z Y}\Big(A, M \overset{\mathbb{L}}{\underset{A}{\otimes}} N\Big)$$

where q is the *internal* grading.

5.4.2 Higher Hochschild homology is chiral homology of commutative algebras

We now establish a relationship between the higher Hochschild functor and the topological chiral homology [11] functor defined by Lurie [L-TFT, L-VI, L-HA]. To obtain a comparison between these functors, it is important to note that they are defined in two different setting with a common intersection. *Topological chiral homology*, denoted $\int_M A$, is defined for any E_n-algebra A and an m-dimensional manifold M, $m \leq n$, such that $M \times D^{n-m}$ is framed (we say M is n-framed). For any framed manifold N and E_∞-algebra, $\int_N A$ is functorial with respect to maps of E_∞-algebras and framed embeddings of manifolds. Further $\int_M A$ is an E_{n-m}-algebra which is also a module over the E_{n-m+1}-algebra $\int_{\partial M} A$. We refer to [L-VI, L-HA, F] for details and definitions. Topological chiral homology can be interpreted as an *invariant of framed manifolds* produced by an extended (∞, n)-Topological Field Theory in the sense of [L-TFT]; the theory in question taking value in an (∞, n)-category of E_n-algebras whose n-morphisms are (homotopy types) of chain complexes. Note that topological chiral homology depends on and comes with a choice of commutative diagram of (∞-)operads



which allows one to interpret a CDGA as an E_n-algebra for any n.

In fact [L-VI, Section 3.2] (also see[L-HA, F, L-TFT]) Lurie also defined topological chiral homology for a *(locally constant) family of E_n-algebras* parametrized by the points in $M \times D^{m-n}$ even if M is *not* n-framed. Such families objects are (locally constant) algebras over an operad $\mathbb{E}_n[M] := \mathbb{E}[M \times D^{n-m}]$, the operad of little n-cubes in $M \times D^{m-n}$, see [L-VI, Definition 3.2.1] and § 5.5 below. Here M is still of dimension m (and $m \leq n$). He also outlined [L-HA, L-VI] (and [L-TFT]) that the chiral homology also satisfy an excision axiom. A proof was given in our paper [8] as well as independently by Francis [F] too. More precisely

Lemma 5.32 *Let M be a manifold and $\mathcal{A}$ be an $\mathbb{E}[M]$-algebra. Assume that there is a codimension 1 submanifold N of M with a trivialization $N \times D^1$ of its neighborhood such that M is decomposable as $M = X \cup_{N \times D^1} Y$ where X, Y are submanifolds of M glued along $N \times D^1$. Then*

1. *$\int_{N\times D^1} \mathcal{A}$ is an E_1-algebra and $\int_X \mathcal{A}$ and $\int_Y \mathcal{A}$ are right and left modules over $\int_{N\times D^1} \mathcal{A}$.*
2. *The natural map $\int_X \mathcal{A} \otimes^{\mathbb{L}}_{\int_{N\times D^1} \mathcal{A}} \int_Y \mathcal{A} \longrightarrow \int_M \mathcal{A}$ is an equivalence.*

The locality axiom implies that chiral homology of CDGA agrees with higher Hochschild functor:

11. also called factorization homology; despite its name it takes value in $\mathsf{k}\text{-}\mathsf{Mod}_\infty$

Theorem 5.33 ([8], Theorem 6.1) *Let M be a manifold of dimension m endowed with a framing of $M \times D^k$ and A be a CDGA viewed as an E_{m+k}-algebra. Then, the topological chiral homology of M with coefficients in A, denoted by $\int_M A$ is equivalent to $CH^{\bullet}_M(A)$ (viewed as an E_k-algebra).*

Sketch of Proof. By the value on a point axiom (Theorem 5.30), both topological chiral homology and higher Hochschild chains agree on a point and further on any disk D^k. Further, by Theorem 5.30.2 and Lemma 5.32, they both satisfy the excision axiom. Now the result essentially follows using handle decompositions, since one can chop manifolds on disks which are glued along their boundaries. □

Corollary 5.34 ([8]) *Chiral homology of dimension n-framed manifold with value in CDGAs is an homotopy invariant.*

In particular, $\int_M A$ is *independent* of the n-framing for a CDGA A.

Let us outline the philosophy intertwining the different concepts studied here. Given an n-framed manifold M of dimension m, and an E_n-algebra A, we can form the topological chiral homology $\int_M A$, which can be thought of as a colimit of tensor products of A indexed by balls in the manifold. Now, if we embed $M \times \mathbb{R}^{n-m}$ in $M \times \mathbb{R}^{n-m+1}$ equipped with the induced framing, one can form $\int_M B$ for an E_{n+1}-algebra. But two different framings of $M \times \mathbb{R}^{n-m}$ may become equivalent after the embedding. Since a CDGA C is an E_k-algebra for all k, $\int_M C$ should not be able to distinguish different framings. Further, since any CW-complex is homotopy equivalent to a framed manifold, we see that, for a CDGA C, $\int_M C$ should makes sense for any manifold. This suggests that for CDGAs topological chiral homology may be extended to any CW-complex and is a homotopy invariant, which is precisely realized by the derived higher Hochschild functor.

5.5 Chiral homology and factorization algebras

Here, we explain how to see topological chiral homology in terms of factorization algebras, a structure arising naturally when studying quantum field theory. The point is that the data of an E_n-algebra are equivalent to those of a locally constant factorization algebra in $\mathbb{R}^n$ [CoGw, L-VI]; the latter together with chiral homology theory give are powerful tool to study derived algebraic geometry of E_n-algebras [L-HA, L-TFT, F].

5.5.1 Factorization algebras and E_n-algebras

Let us briefly outlined the definition of a factorization algebra[12]. Given a topological manifold M of dimension n, one can define a colored operad whose objects are open subsets of M that are homeomorphic to $\mathbb{R}^n$ and whose morphisms from $\{U_1, \cdots, U_n\}$ to V are empty unless except when U_i's are mutually disjoint

12. by factorization algebra we mean homotopy factorization algebra in the sense of [CoGw]; our terminology is closer to the one of [F]

subsets of U, in which case they are singletons. The ∞-operad associated to this colored operad is denoted by $N(Disk(M))$ (see [L-HA], Remark 5.2.4.7).

An algebra over this ∞-operad [L-VI, L-HA], with value in chain complexes, is a rule that assigns to any open disk [13] U a chain complex $\mathcal{F}(U)$ and, to any finite family of disjoint open disks $U_1, \dots, U_n \subset V$ included in a disk V, a natural map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$. It is called **locally constant** if for any inclusion of open disks $U \hookrightarrow V$ in X, the structure map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a quasi-isomorphism.

A **locally constant factorization algebra** on a manifold X (in the sense of Costello [CoGw]) is a similar construction where the U_i are allowed to be any open subsets, satisfying a kind of (homotopy) cosheaf condition (see [CoGw, 8]) and still being locally constant in the above sense. Key properties of factorization algebras are the fact that they are completely determined by their restriction on a (factorizing) basis of opens and further they are endowed with a *pushforward functor*. Indeed, a continuous map $f : X \to Y$ induces a functor $f_* : Fac_X \to Fac_Y$ given, for V open in Y and $\mathcal{F} \in Fac_X$ by $f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$. The **homology** (or global derived sections) $HF(\mathcal{F}, X)$ of a factorization algebra $\mathcal{F}$ on X is the pushforward $HF(\mathcal{F}, X) := p_*(\mathcal{F}) \in \mathsf{k\text{-}Mod}_\infty$ where $p : X \to pt$ is the unique map.

One can define similarly factorization algebras with values in any *symmetric monoidal ∞-category* $\mathcal{C}$. We denote $Fac_X^{lc}(\mathcal{C})$ the ∞-category of locally constant factorization algebras on X with value in $\mathcal{C}$. Theorem 5.2.4.9 of [L-HA, L-VI] provides an equivalence of ∞-categories between E_n-Alg and the ∞-category of *locally constant factorization algebra* on the open disk D^n. In fact Lurie has proved the following

Proposition 5.35 *The ∞-category of E_n-algebras is naturally equivalent to the ∞-category of locally constant algebras over the operads (in sets) $N(Disk(\mathbb{R}^n))$. Further, the latter ∞-category is equivalent to the ∞-category $Fac_{\mathbb{R}^n}^{lc}$ of locally constant factorization algebras on $\mathbb{R}^n$.*

From Proposition 5.35 and operations for factorization algebras, in[8] we proved

Lemma 5.36 *Let M be a manifold and $\pi_1 : M \times \mathbb{R}^d \to M$ the canonical projection. The pushforward by π_1 induces an equivalence of ∞-categories*

$$\pi_{1*} : Fac_{M\times\mathbb{R}^d}^{lc}(\mathsf{k\text{-}Mod}_\infty) \xrightarrow{\simeq} Fac_M^{lc}(E_d\text{-}Alg)$$

In particular, if $\mathcal{F} \in Fac_{M\times\mathbb{R}^d}^{lc}(\mathsf{k\text{-}Mod}_\infty)$, then its factorization homology

$$HF(\mathcal{F}, M \times \mathbb{R}^d) = p_*(\mathcal{F})(pt) \cong p_* \circ \pi_{1*}(\mathcal{F})(pt) \cong HF(\pi_{1*}(\mathcal{F}), M)$$

is an E_d-algebra.

5.5.2 Chiral homology is factorization homology

Chiral homology precisely computes (derived) sections of a factorization algebra.

13. i.e. an open subset of X homeomorphic to a euclidean ball

Theorem 5.37 ([8]) *Let M be a manifold of dimension n.*

1. *Topological chiral homology defines a functor $\mathcal{TC}_M$, from the category of $\mathbb{E}[M \times \mathbb{R}^d]$-algebras to the category of locally constant factorization algebras on M with value in E_d-algebras, such that $\int_M A \cong HF(\mathcal{TC}_M, M)$.*
2. *Any locally constant factorization algebra on M with values in E_d-algebras is equivalent to $\mathcal{TC}_M(A)$ for a unique (up to equivalences) $\mathbb{E}[M \times \mathbb{R}^d]$-algebra.*

Said otherwise, a locally constant $N(Disk(M))$-algebra can be extended to a full factorization algebra whose section over any open subset U of M is precisely computed by chiral homology over U.

Sketch of Proof. This is [8, Theorem 6.3.6]. The key point is to use an excision axiom for locally constant factorization algebras. Indeed, we proved

Lemma 5.38 ([8], Lemma 6.3.4) *Let $\mathcal{A}$ be a locally constant factorization algebra on a manifold M and assume that there is a codimension 1 submanifold (possibly with corners) N of M with a trivialization $N \times D^1$ of its neighborhood such that M is decomposable as $M = X \cup_{N\times I} Y$ where X, Y are submanifolds (with corners) of M glued along $N \times D^1$. $HF(\mathcal{A}_{|X})$ and $HF(\mathcal{A}_{|Y})$ are right and left $HF(\mathcal{A}_{|N\times D^1})$-modules and further,*

$$HF(\mathcal{A}) \;\cong\; HF(\mathcal{A}_{|X}) \overset{\mathbb{L}}{\underset{HF(\mathcal{A}_{|N\times D^1})}{\otimes}} HF(\mathcal{A}_{|Y}).$$

Recall that according to Lemma 5.36 above $HF(\mathcal{A}_{|N\times D^1})$ is an E_1-algebra. Then one applies many times handles decomposition, the above Lemma and Lemma 5.32 to prove the result. □

The relationship between higher Hochschild and chiral homology for CDGAs over a manifold can be pushed further, using factorization algebras, *over any CW-complex.* Indeed, we proved in [8]

Theorem 5.39 *Let A be a CDGA and X be a topological space with a basis of open sets which is also a factorizing good cover. Then the rule $\mathcal{CH}_X : U \mapsto CH_U(A)$ makes higher Hochschild chains a factorization algebra on X.*

Further, any factorization algebra for which $\mathcal{F}(U)$ (for contractible U) is naturally equivalent to a CDGA A is canonically equivalent to $\mathcal{CH}_X(A)$.

5.5.3 Hochschild cohomology over spheres and deformations in E_n-algebras

We now explain the relationship between E_n-algebras and higher Hochschild cochains over the n-sphere S^n.

Deformations of an E_n-algebra A are closely related (see [KS, K, F]) to $RHom_A^{E_n}(A, A)$. Here $RHom_A^{E_n}$ denotes the hom space in the (∞-)category $Mod_A^{E_n}$ of E_n-A-modules. Note that in the case $n = 1$, the latter complex is precisely the standard Hochschild cochains of A since E_1-A-modules are bimodules (see Example 5.2).

Proposition 5.40 ([GTZ3]) *If A is a CDGA and M an A-module, there is a natural equivalence $RHom_A^{E_n}(A,M) \cong CH^{S^n}(A,M)$, where CH^{S^n} denotes the derived higher Hochschild cochain functor (Proposition 5.14).*

SKETCH OF PROOF. There is an equivalence of ∞-categories $A\text{-}Mod^{E_n} \cong \big(\int_{S^{n-1}} A\big)\text{-}LMod$ where $\int_{S^{n-1}} A$ is the chiral homology of S^{n-1} with value in A (see [F]). Thus we have natural equivalences

$$\begin{aligned} RHom_A^{E_n}(A,M) &\cong RHom^{left}_{\int_{S^{n-1}} A}(A,M) \\ &\cong RHom_A^{left}\Big(CH_{D^n}(A) \otimes^{\mathbb{L}}_{CH_{S^{n-1}}(A)} A, M\Big) \\ &\cong RHom_A^{left}\left(CH_{S^n}(A), M\right) \cong CH^{S^n}(A,M) \end{aligned}$$

where $RHom_B^{left}$ denotes the hom space of morphisms of left B-modules. □

5.6 Operations on Hochschild (co)homology over spheres

Our quest for higher Hochschild theory was motivated by the aim to apply functoriality to get algebraic structures on models for mapping spaces. We have already seen some of applications along this line in Section 5.3 with the surface product and Section 5.2 as well. We now turn to applications to (variants of) higher Deligne conjecture [K] and brane topology operations following [6, 8, GTZ3].

5.6.1 Wedge product in higher Hochschild cohomology

Let $A \xrightarrow{f} B$ be a map of CDGAs. Note that it makes B into an A-algebra as well as an $A\otimes A$-algebra (since the multiplication $A \otimes A \to A$ is an algebra morphism). The excision axiom 5.30.2 implies

Lemma 5.41 *Let M be an A-module and X, Y be pointed topological spaces. There is a natural equivalence*

$$\mu:\ Hom_{A\otimes A}\left(CH_X(A) \otimes CH_Y(A), M\right) \xrightarrow{\simeq} CH^{X\vee Y}(A,M)$$

We use Lemma 5.41 to define the *wedge product* of Hochschild cochains (which we first introduced in [6, Section 3]) as the linear map

$$\begin{aligned} \mu_\vee : CH^X(A,B) \otimes CH^Y(A,B) &\longrightarrow Hom_{A\otimes A}\Big(CH_X(A) \otimes CH_Y(A), B \otimes B\Big) \\ &\xrightarrow{(m_B)_*} Hom_{A\otimes A}\Big(CH_X(A) \otimes CH_Y(A), B\Big) \cong CH^{X\vee Y}(A,B) \qquad (5.10) \end{aligned}$$

where the first map is given by the tensor products $(f,g) \mapsto f \otimes g$ of functions. We give an explicit description of the wedge products in [6]. On can check easily that the wedge-product is associative. It follows that if X is endowed with an (homotopy) co-associative diagonal $\delta\colon X \to X \vee X$, Hochschild cochains $CH^X(A,B)$ over X inherits an (homotopy) associative algebra structure:

$$\cup_X : CH^X(A,B) \otimes CH^X(A,B) \xrightarrow{\mu_\vee} CH^{X\vee X}(A,B) \xrightarrow{\delta^*} CH^X(A,B).$$

Example 5.42 A standard example of space with a diagonal is given by spheres $X = S^n$. Actually, one can check that for $d = 1$, the cup-product $\cup_{S^1}$ is (homotopy) equivalent to the usual cup-product for Hochschild cochains as in [Ge] and for $n = 2$, $\cup_{S^2}$ is (homotopy equivalent to) the Riemann sphere product as defined in [7, 6] and § 5.3.3. Note that the diagonal $S^n \to S^n \vee S^n$ becomes more commutative as n-increases. This can be use to lift the cup-product to E_n-algebra structure as we show in the next section.

5.6.2 The E_n-structure of Hochschild (co)homology over S^n

In [6], we also extended the cup-product for spheres S^n, defined in § 5.6 (Example 5.42), into an E_n-algebra structure (at the level of cochains). This result is actually a version of *higher Deligne conjecture for morphisms* of CDGAs, *i.e.*, an explicit construction of Lurie's notion of (derived) *centralizers* of a map of CDGAs in the category of E_n-algebras. We now sketch the construction of [6].

Let $\mathcal{C}_n$ be the usual n-dimensional little cubes operad (as an operad of topological spaces). Recall that $\mathcal{C}_n(k)$ is the configuration space of k n-dimensional open cubes in I^n. Any element $c \in \mathcal{C}_d(k)$ defines a map $pinch_c : S^d \to \bigvee_{i=1\dots k} S^d$ by collapsing the complement of the interiors of the cubes to the base point. The maps $pinch_c$ assemble together to give a continuous map

$$pinch : \mathcal{C}_d(k) \times S^d \longrightarrow \bigvee_{i=1\dots k} S^d. \tag{5.11}$$

Note that the map *pinch* preserves the base point of S^d hence passes to the pointed category Top_*.

Applying the contravariance of Hochschild cochains and using the wedge product $\mu_\vee$ (*i.e.* the map (5.10)), we get, for all $n \geq 1$, a morphism

$$\begin{aligned} pinch^*_{S^n,r} : C_*\big(\mathcal{C}_n(r)\big) \otimes \Big(CH^{S^n}(A,B)\Big)^{\otimes r} & \\ \stackrel{(\mu_\vee)^{(n-1)}}{\longrightarrow} C_*\big(\mathcal{C}_n(r)\big) \otimes CH^{\bigvee_{i=1}^r S^n}\big(A,B\big) & \stackrel{pinch^*}{\longrightarrow} CH^{S^n}(A,B) \end{aligned} \tag{5.12}$$

in $\mathsf{k}\text{-}\mathsf{Mod}_\infty$ (where $(\mu_\vee)^{(n-1)}$ is the iteration of the wedge product). Using Proposition 5.14 and Theorem 5.30 we get

Theorem 5.43 ([6]) *Let $A \stackrel{f}{\to} B$ be a CDGA map. The collection of maps $(pinch_{S^n,k})_{k\geq 1}$ makes $CH^{S^n}(A,B)$ an E_n-algebra (naturally in A, B).*

In particular, for $n > 1$, the induced cup-product on the cohomology groups $HH^{S^n}_\bullet(A,B)^{\otimes 2} \to HH^{S^n}_\bullet(A,B)$ is commutative.

Let us be more precise about the naturality. Let $A \stackrel{f}{\to} B$ and $B \stackrel{g}{\to} C$ be maps of CDGAs. Using Proposition 5.40, we can define the (derived) *composition of higher Hochschild cochains*:

$$\begin{aligned} CH^{S^n}(A,B) \otimes CH^{S^n}(B,C) \cong RHom_A^{E_n}\big(A,B\big) \otimes RHom_A^{E_n}\big(B,C\big) & \\ \stackrel{\circ}{\longrightarrow} RHom_A^{E_n}\big(A,C\big) \cong CH^{S^n}(A,C). & \end{aligned} \tag{5.13}$$

where the middle arrow is (derived) composition of E_n-modules morphisms.

Lemma 5.44 *The derived composition* $CH^{S^n}(A,B) \otimes CH^{S^n}(B,C) \to CH^{S^n}(A,C)$ *is a map of* E_n*-algebras.*

Sketch of Proof. The E_n-algebra structure on the tensor product $CH^{S^n}(A,B) \otimes CH^{S^n}(B,C)$ is induced by the diagonal maps $\mathcal{C}_n(r) \to \mathcal{C}_n(r) \times \mathcal{C}_n(r)$ in Top_∞. The equivalence provided by Proposition 5.40 factors through an equivalence

$$CH^{S^n}(A,B) \cong RHom^{left}_{CH_{S^{n-1}}(A)}\left(CH_{I^n}(A), CH_{I^n}(B)\right). \tag{5.14}$$

(where S^{n-1} is identified with the boundary of the cube I^n) and the right hand side is the hom space in left modules. The E_n-algebra structure on higher Hochschild cochains is induced by the pinching map, which itself is induced by inclusions of (configurations of) cubes in the right hand side of the equivalence (5.14), *i.e.* the definition of the little n-cubes operadic structure. It is now straightforward to check that the derived composition (5.13) preserves the $\mathcal{C}_n$-action (also see [GTZ3]). $\square$

Remark 5.45 Let $I_\bullet$ be the standard simplicial model of the interval ([6, 7]); its boundary $\partial I^n_\bullet$ is a simplicial model forS^{n-1}. Then the equivalence (5.14) allows to see the derived composition (5.13) as the usual composition (of left dg-modules)

$$Hom^{left}_{CH_{\partial I^n_\bullet}(A)}\left(CH_{I^n_\bullet}(A), CH_{I^n_\bullet}(B)\right) \otimes Hom^{left}_{CH_{\partial I^n_\bullet}(B)}\left(CH_{I^n_\bullet}(B), CH_{I^n_\bullet}(C)\right) \stackrel{\circ}{\longrightarrow} Hom^{left}_{CH_{\partial I^n_\bullet}(A)}\left(CH_{I^n_\bullet}(A), CH_{I^n_\bullet}(C)\right)$$

since $CH_{I^n_\bullet}(A)$ is a (semi-)free $CH_{\partial I^n_\bullet}(A)$-algebra.

Example 5.46 If $A = k$, there is a canonical equivalence of E_n-algebras $CH^{S^n}(k,B) \cong B$ (which actually is the restriction of an equivalence of CDGAs). If $B = k$, then the E_n-algebra structure of $CH^{S^n}(A,k)$ is the dual of the E_n-coalgebra structure given by the n-times iterated Bar construction $Bar^{(n)}(A)$ (as defined in [Fr2, F, L-TFT, GTZ3]).

5.6.3 Derived centralizers of algebras morphisms and higher Deligne conjecture

Following Lurie [L-HA, L-VI], the (derived) **centralizer** of an E_n-algebra map $f : A \to B$ is the *universal* E_n-algebra $\mathfrak{z}(f)$ endowed with a morphism of E_n-algebras $\kappa : A \otimes \mathfrak{z}(f) \to B$ making the following diagram

$$\begin{array}{ccc} & A \otimes \mathfrak{z}(f) & \\ {\scriptstyle id\otimes 1_{\mathfrak{z}(f)}} \nearrow & & \searrow {\scriptstyle \kappa} \\ A & \xrightarrow{\quad f \quad} & B \end{array} \tag{5.15}$$

commutative in E_n-Alg. Its existence is proved in [L-HA].

The following is a special case of a result we prove in the work in progress [GTZ3].

Proposition 5.47 *Let $A \xrightarrow{f} B$ be a map of CDGAs. Then $CH^{S^n}(A,B)$ is the centralizer of f in the category of E_n-algebras.*

Note that we do not use Lurie's existence Theorem but rather reprove it.

Sketch of Proof. By naturality of the E_n-algebra structure (Lemma 5.44) and Example 5.46, there is an natural evaluation map $\mathrm{eval} : A \otimes CH^{S^n}(A,B) \to B$ which is a map of E_n-algebras. It follows that the following diagram

$$\begin{array}{ccccc} & & A \otimes CH^{S^n}(A,B) & & \\ & {}^{id\otimes 1}\nearrow & & \searrow^{\mathrm{eval}} & \\ A & & \xrightarrow{\quad f \quad} & & B \end{array}$$

is commutative in $E_n - Alg$.

Now let $\mathfrak{z}$ be an E_n-algebra, endowed with a E_n-algebra map $\phi : A \otimes \mathfrak{z} \to B$ fitting in a commutative diagram

$$\begin{array}{ccccc} & & A \otimes \mathfrak{z} & & \\ & {}^{id\otimes 1_{\mathfrak{z}}}\nearrow & & \searrow^{\phi} & \\ A & & \xrightarrow{\quad f \quad} & & B\,. \end{array} \tag{5.16}$$

By adjunction (in $\mathsf{k\text{-}Mod}_\infty$), the map ϕ has a (derived) adjoint $\theta_\phi : \mathfrak{z} \to RHom(A,B)$. Since ϕ is a map of E_n-algebras and diagram (5.16) is commutative, one check that θ_ϕ factors through a map

$$\begin{aligned} \widetilde{\theta_\phi} \colon \mathfrak{z} &\cong k \otimes \mathfrak{z} \\ &\overset{1_{RHom_A^{E_n}(A,A)}\otimes id}{\longrightarrow} RHom_A^{E_n}(A,A) \otimes \mathfrak{z} \cong RHom_A^{E_n}(A,A) \otimes RHom_k^{E_n}(k,\mathfrak{z}) \\ &\longrightarrow RHom_A^{E_n}(A, A\otimes \mathfrak{z}) \xrightarrow{\phi_*} RHom_A^{E_n}(A,B) \cong CH^{S^n}(A,B). \end{aligned} \tag{5.17}$$

It follows from naturality of the E_n-algebra structure of Hochschild cochains over S^n that this composition $\widetilde{\theta_\phi} \colon \mathfrak{z} \to CH^{S^n}(A,B)$ is actually a map of E_n-algebras. Further, by definition of θ_ϕ, the identity

$$\mathrm{eval} \circ \big(id_A \otimes \theta_\phi\big) \;=\; \phi$$

holds. Now, the uniqueness of the map $\widetilde{\theta_\phi}$ follows quite easily from the fact that the composition

$$\begin{aligned} RHom_A^{E_n}(A,B) &\cong RHom_k^{E_n}\Big(k, RHom_A^{E_n}(A,B)\Big) \\ &\overset{1_{RHom_A^{E_n}(A,A)}\otimes id}{\longrightarrow} RHom_A^{E_n}(A,A) \otimes \Big(k, RHom_A^{E_n}(A,B)\Big) \\ &\longrightarrow RHom_A^{E_n}\Big(A, A \otimes RHom_A^{E_n}(A,B)\Big) \\ &\xrightarrow{ev_*} RHom_A^{E_n}(A,B) \end{aligned} \tag{5.18}$$

is the identity map. Hence $CH^{S^n}(A,B)$ satisfies the universal property of the derived center $\mathfrak{z}(f)$. □

By Lemma 5.44 above, the derived composition

$$CH^{S^n}(A,A)\otimes CH^{S^n}(A,A) \longrightarrow CH^{S^n}(A,A) \tag{5.19}$$

is a homomorphism of E_n-algebras (with unit given by the identity map 1_A). The composition of morphisms is further (homotopy) associative and unital (with unit 1_A); thus $CH^{S^n}(A,A)$ is actually an E_1-algebra in the ∞-category E_n-Alg.

By the ∞-category version of Dunn Theorem [Du, L-HA, L-VI], there is an equivalence of ∞-categories $E_1-Alg\big(E_n-Alg\big)\cong E_{n+1}-Alg$. Thus the multiplication (5.19) lifts the E_n-algebra structure of $CH^{S^n}(A,A)\cong RHom_A^{E_n}(A,A)$ to an E_{n+1}-algebra structure so that we have proved

Corollary 5.48 (Higher Deligne Conjecture) *Let A be a CDGA. There is a natural E_{n+1}-algebra structure on $CH^{S^n}(A,A)$ whose underlying E_n-algebra structure is the one given by Theorem 5.43. In particular, the underlying E_1-algebra structure is given by the standard cup-product.*

By Proposition 5.47, the E_{n+1}-structure trivially agrees with the one in [L-HA]. We conjecture that this E_{n+1}-structure is also equivalent to the one of [F].

Remark 5.49 The result of this section can be extended from CDGAs to E_n-algebras using factorization algebras and chiral homology instead of Hochschild chains. We refer to our work in progress [GTZ3] for details.

5.6.4 Hodge decomposition of Hochschild (co)chains over spheres

The Hodge decomposition of the usual Hochschild (co)homology has an immediate analogue in higher Hochschild. Let A be a CDGA and M be an A-module. We define ψ^m as the composition

$$CH_{S^d}(A,M)\stackrel{\vee id^*}{\longrightarrow} CH_{S^d\vee\cdots\vee S^d}(A,M)\stackrel{p^*}{\longrightarrow} CH_{S^d}(A,M)$$

where $p: S^d\to S^d\vee\cdots\vee S^d$ (m-factors) is the iterated pinch map and $\vee id: S^d\vee\cdots\vee S^d\to S^d$ is the identity on each factor of the wedges. Similarly, for cochains, we have

$$CH^{S^d}(A,M)\stackrel{p^*}{\longrightarrow} CH_{S^d\vee\cdots\vee S^d}(A,M)\stackrel{\vee id^*}{\longrightarrow} CH^{S^d}(A,M).$$

The following is (a slight rewriting of) a result of our note [5].

Proposition 5.50 *The maps ψ^m satisfy the identity $\psi^p\circ\psi^q=\psi^{pq}$ for any $p,q\geq 1$;*

i) *if k is of characteristic 0, then there is a splitting in cohomology $HH^{S^d}(A,M)=\prod_{j\geq 0} HH^{S^d}_{(j)}(A,M)$ where the vector spaces $HH^{S^d}_{(j)}(A,M)$ are isomorphic to $\ker(\psi^m-m^j.\mathrm{id})$.*

ii) *For $d = 1$, the maps ψ^m coincides with the usual Adams operations in Hochschild (co)homology [Lo1, GS].*

Remark 5.51 It is easy, using the edgewise subdivision functor, to describe *explicitly* the maps ψ^m on the standard simplicial set models of S^d. See [6, Section 4] for details.

The fact that for $d > 1$, the (homotopy) commutativity of the cup-product $\cup_{S^d}$ (of example 5.42) can be induced by a base-point preserving homotopy implies

Proposition 5.52 ([6]) *For $d > 1$, the Adams operations ψ^m acting on Hochschild cochains $CH^{S^d}(A,B)$ commutes with the cup-product. That is one has $\psi^k(f) \cup_{S^d} \psi^k(g) = \psi^k(f \cup_{S^d} g)$ for all $f, g \in CH^{S^d}(A,B)$.*

Recall [BW] that this result is false for $d = 1$.

5.6.5 Applications to Brane topology

We now apply the previous results on higher Hochschild cochains to give an algebraic models fro Brane topology [CV], the analogue of string topology for free spheres spaces. Further, we get *chain level* construction. The n-dimensional free sphere space is denoted $X^{S^n} = \mathrm{Map}(S^n, X)$. Let M be a closed oriented n-connected manifold and $\Omega := \Omega^\bullet_{dR}(M)$ its de Rham algebra. Then (see [6, 8]) there exists an natural equivalence

$$CH^{S^n}(\Omega, \Omega) \xrightarrow{\simeq} CH^{S^n}\big(\Omega, (\Omega)^*\big) \qquad (5.20)$$

and by Corollary 5.18 an equivalence $CH^{S^n}(\Omega, \Omega^*) \xleftarrow{\simeq} C_\bullet(\mathrm{Map}(S^n, X))$. Hence, using the solution of the higher Deligne conjecture (Corollary 5.48) we obtain easily the following result (which expands the results of [6])

Theorem 5.53 *Let M be a closed oriented n-connected manifold. Then the shifted chain complex $C_{\bullet+\dim(M)}(M^{S^n})$ has a natural*[14] *E_{n+1}-algebra structure (which induces the sphere product [CV, Section 5] $H_p\big(M^{S^n}\big) \otimes H_q\big(M^{S^n}\big) \to H_{p+q-\dim(M)}\big(M^{S^n}\big)$ in homology)*

Remark 5.54 In particular the above Theorem 5.53 is a chain level construction of Brane topology operations. However, we only deal with the non-framed version of the E_{n+1}-structure. We conjecture that the structure given above can be lifted to an action of the framed E_{n+1}-operad.

The algebraic model given by higher Hochschild cochains can be computed using rational homotopy techniques. For instance, if A is a CDGA, in [6] (also see[7]), we constructed a canonical map

$$HKR : \mathrm{Hom}_A(S^\bullet(\Omega_A[n]), M) \to HH^{S^n}(A, M) \qquad (5.21)$$

14. with respect to maps of Poincaré duality spaces

where Ω_A is the space of Kähler differentials (recall that $\mathrm{Hom}_A(\Omega A, M) \cong \mathrm{Der}(A, M)$) and $S^\bullet$ is the graded symmetric algebra functor. There are Adams operations ψ^m on the left hand side, defined on $\mathrm{Hom}_A(S^j(\Omega_A[d]), A)$ by the multiplication by m^j. We proved in [6] the following

Corollary 5.55 *Let $n > 1$, A be a Sullivan algebra*[15] *and* $\mathrm{char}(k) = 0$. *The map*

$$HKR : H^\bullet(\mathrm{Hom}(S^\bullet(\Omega_A[n]), A), d_A) \to HH^{S^n}(A, A)$$

is an isomorphism of algebras commuting with the Adams operations.

Corollary 5.55 can be applied to any model Ω of the de Rham forms on M. In particular, it adds the Hodge decomposition to the Brane topology story studied (at the homology level) in [CV].

Remark 5.56 Note that $\mathrm{Hom}_A(S^\bullet(\Omega_A[n]), A)$ is in fact naturally a P_{n+1}-algebra, *i.e.* is equipped with a degree n Lie bracket induced from the Lie bracket of derivations by the the identification $\mathrm{Hom}_A(\Omega_A, A) \cong \mathrm{Der}(A, A)$ and extended to the whole space by the Leibniz rule. When A is smooth, the HKR map (5.21) allows to transfer these structure to $HH^{S^n}(A, A)$. The E_{n+1}-structure on $CH^{S^n}(A, A)$ given by Theorem 5.53 also induces an P_{n+1}-algebra structure on the cohomology $HH^{S^n}(A, A)$. We conjecture (and are working on a proof) that these two structures are the same and actually that *formality* holds, *i.e.* that the HKR map can be lifted to an equivalence $\mathrm{Hom}_A(S^\bullet(\Omega_A[n]), A) \to CH^{S^n}(A, A)$ of E_{n+1}-algebras.

References

[AZ] H. Abbaspour, M. Zeinalian, *String bracket and flat connections*, Algebr. Geom. Topol. **7** (2007), 197–231.

[ACJ] P. Aschieri, L. Cantini, B. Jurco, *Nonabelian bundle gerbes, their differential geometry and gauge theory*, Comm. Math. Phys. **254** (2005), no. 2, 367–400.

[AT] M. Atiyah, D. Tall, *Group representations, λ-rings and the J-homomorphism*, Topology 8 (1969) 253–297.

[BS] J. Baez, U. Schreiber, *Higher Gauge Theory*, Categories in algebra, geometry and mathematical physics, 7–30, Contemp. Math., 431, Amer. Math. Soc., RI, 2007.

[BCSS] J. Baez, A. Crans, U. Schreiber, D. Stevenson, *From loop groups to 2-groups*, Homology, Homotopy Appl. **9** (2007), no. 2, 101–135.

[BD] A. Beilinson, V. Drinfeld, *Chiral algebras*, American Mathematical Society Colloquium Publications, 51. American Mathematical Society, Providence, RI, 2004

[Be] K. Behrend, *Cohomology of stacks*, Intersection theory and moduli, ICTP Lect. Notes, XIX, Trieste (2004), 249–294.

[BX] K. Behrend, P. Xu, *Differentiable stacks and gerbes*, preprint `DG/0605674`, to appear Journal of Symplectic Geometry.

[BF] C. Berger, B. Fresse, *Combinatorial operad actions on cochains*, Math. Proc. Cambridge Philos. Soc. 137 (2004), 135-174.

15. that is a $CDGA$ of the form $(S(V), d)$

[BW] N. Bergeron, L. Wolfgang *The decomposition of Hochschild cohomology and Gerstenhaber operations*, J. Pure Appl. Algebra **79** (1995) 109–129

[BS] R. Bott, *On the Chern-Weil homomorphism and continuous cohomology of Lie groups*, Advances in Math. **11** (1973), 289–303.

[Br] L. Breen, *On the classification of* 2*-gerbes and* 2*-stacks*, Astérisque No. **225** (1994).

[CFP] A. Cattaneo, J. Fröhlich, B. Pedrini, *Topological field theory interpretation of string topology*, Comm. Math. Phys. **240** (2003), no. 3, 397–421.

[CS] M. Chas, D. Sullivan, *String Topology*, arXiv:math/9911159

[CM] D. Chataur, L. Menichi, *String topology of classifying spaces*, preprint AT/0801.0174.

[CR] L. Chen, Y. Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. **248** (2004) no. 1, 1–31.

[Ch] K.-T. Chen, *Iterated integrals of differential forms and loop space homology*, Ann. of Math. (2) 97 (1973), 217–246.

[ChGe] X.Z. Cheng, E. Getzler, *Transferring homotopy commutative algebraic structures* J. Pure Appl. Algebra **212** (2008), no. 11, 2535–2542.

[CG] R. Cohen, V. Godin, *A polarized view of string topology*, Topology, Geometry and Quantum Field Theory, London Math. Soc. Lecture Notes **308** (2004), 127–154.

[CJ] R. Cohen, J. Jones, *A homotopy theoretic realization of string topology*, Math. Ann. **324** (2002), no. 4, 773–798.

[CV] R. Cohen, A. Voronov, *Notes on String topology*, String topology and cyclic homology, 1–95, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2006.

[CoGw] K. Costello, O. Gwilliam, *Factorization algebras in perturbative quantum field theory*,
Online wiki available at http://math.northwestern.edu/∼costello/factorization_public.html

[Co1] K. Costello, *Topological conformal field theories and gauge theories*, Geom. Topol. **11** (2007), 1539–1579.

[Co2] K. Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214.

[Du] G. Dunn, *Tensor product of operads and iterated loop spaces*, J. Pure Appl. Algebra 50 (1988), no. 3, 237–258.

[FTV] Yves Félix, Jean-Claude Thomas, Micheline Vigué, *The Hochschild cohomology of a closed manifold.* Publ. Math. Inst. Hautes Études Sci. No. 99 (2004), 235–252.

[F] J. Francis *The tangent complex and Hochschild cohomology of E_n-rings*, preprint AT/1104.0181

[Fr1] B. Fresse, *Modules over operads and functors*, Lecture Notes in Mathematics, 1967. Springer-Verlag, Berlin, 2009.

[Fr2] B. Fresse, *The bar complex of an E_∞-algebra*, Adv. Math. 223 (2010), pp. 2049–2096.

[FMcP] W. Fulton, R. MacPherson, *Categorical framework for the study of singular spaces*, Mem. Amer. Math. Soc. **31** (1981), no. 243.

[Ge] M. Gerstenhaber, *The Cohomology Structure Of An Associative ring* Ann. Maths. 78(2) (1963).

[GS] M. Gerstenhaber, S. Schack, *A Hodge-type decomposition for commutative algebra cohomology* J. Pure Appl. Algebra 48 (1987), no. 3, 229–247

[Get] E. Getzler, *Batalin-Vilkovisky alegbras and two dimensional topological field theory*, Comm. Math. Phys. **159** (1994), no 2, 265-285.

[GiNo] G. Ginot, B. Noohi, *Group actions on Stacks and Applications to String Topology for stacks*, Preprint in progress

[GTZ3] G. Ginot, T. Tradler, M. Zeinalian, *Higher Hochschild cohomology of E_∞-algebras, Brane topology and centralizers of E_n-algebra maps*, Preprint

[G] J. Giraud, *Cohomologie non abélienne*, Springer-Verlag, 1971.

[Go] V. Godin, *Higher string topology operations*, preprint arXiv:0711.4859.

[GLSU] A. González, E. Lupercio, C. Segovia, B. Uribe, *Chen-Ruan Cohomology of cotangent orbifolds and Chas-Sullivan String Topology*, Math. Res. Lett. **14** (2007), no3., 491–501.

[He] A. Henriques, *Integrating L_∞-algebras*, Compos. Math. **144** (2008), no. 4, 1017–1045.

[HS] M. Hilsum, G. Skandalis, *Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture d'A. Connes).* Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 3, 325–390.

[Ho] M. Hovey, *Model Categories*, Mathematical Surveys and Monographs, 63. American Mathematical Society, Providence, RI, 1999. xii+209 pp.

[JKK] T. Jarvis, R. Kaufmann and T. Kimura, *Stringy K-theory and the Chern character*, Invent. Math. **168** (2007), no. 1, 23–81.

[K] M. Kontsevich, *Operads and motives in deformation quantization*. Lett. Math. Phys. **48** (1999), 35–72.

[KS] M. Kontsevich, Y. Soibelman. *Deformation Theory, Vol. 1.*, Unpublished book draft. Available at www.math.ksu.edu/ soibel/Book-vol1.ps

[LSX] C. Laurent-Gengoux, M. Stiénon, P. Xu, *Non-abelian differentiable gerbes*, Adv. Math. **220** (2009), no. 5, 1357–1427.

[Lo1] J.-L. Loday, *Opérations sur l'homologie cyclique des algèbres commutatives*, Invent. Math. 96 (1989), No. 1, 205–230

[Lo2] J.-L. Loday, *Cyclic Homology*, Grundlehren der mathematischen Wissenschaften 301 (1992), Springer Verlag

[LUX] E. Lupercio, B. Uribe, M. Xicoténcatl, *Orbifold String Topology*, Geom. Topol. **12** (2008), no. 4, 2203–2247.

[L-HTT] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, 170. Princeton University Press, Princeton, NJ, 2009. xviii+925 pp.

[L-VI] J. Lurie, *Derived Algebraic Geometry VI: E_k Algebras*, arXiv:0911.0018v1.

[L-TFT] J. Lurie, *On the Classification of Topological Field Theories*, preprint, arXiv:0905.0465v1.

[L-HA] J. Lurie, *Higher Algebra*, book available at http://www.math.harvard.edu/∼lurie/

[MP] M. Mackaay, R. Picken, *Holonomy and parallel transport for abelian gerbes*, Adv. Math. **170** (2002), no. 2, 287–339.

[Me] L. Menichi, *String topology for spheres*, Comment. Math. Helv. 84 (2009), no. 1, 135–157.

[MiSt] , J. Milnor, J. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, No. 76. Princeton University Press.

[MM] I. Moerdijk, J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, 91, 2003

[MS] I. Moerdijk, J. Svensson, *Algebraic classification of equivariant homotopy 2-types*, J. Pure Appl. Algebra **89** (1993), 187–216.

[No1] B. Noohi, *Mapping stacks of topological stacks*, J. Reine Angew. Math. **646** (2010), 117-133.

[No2] B. Noohi, *Foundations of topological stacks, I*, preprint, `AG/0503247`.

[No3] B. Noohi, *Notes on 2-groupoids, 2-groups and crossed modules*, Homology, Homotopy Appl. **9** (2007), no. 1, 75–106.

[P] T. Pirashvili, *Hodge Decomposition for higher order Hochschild Homology*, Ann. Sci. École Norm. Sup. (4) 33 (2000), no. 2, 151–179.

[R] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. 353 (3) (2001), 937–1007.

[So] C. Soulé, *The cohomology of* $SL_3(\mathfrak{Z})$, Topology, **17** (1978), 1–22.

[St] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, 229 pp.

[S] D. Sullivan, *String Topology: Background and Present State*, `AT/0710.4141`

[TV] B. Toën, G. Vezzosi, *Homotopical Algebraic Geometry II: geometric stacks and applications*, Mem. Amer. Math. Soc. 193 (2008), no. 902.

[TV2] B. Toën, G. Vezzosi, *A note on Chern character, loop spaces and derived algebraic geometry*, Abel Symposium, Oslo (2007), Volume 4, 331-354.

[TZ] T. Tradler, M. Zeinalian *Infinity structure of Poincaré duality spaces*, Algebr. Geom. Topol. 7 (2007), 233–260.

[Tr] T. Tradler, *The Batalin-Vilkovisky Algebra on Hochschild Cohomology Induced by Infinity Inner Products*, Ann. Inst. Fourier **58** (2008), no. 7, 2351–2379.

[T] V. Turaev, *Skein quantization of Poisson algebras of loops on surfaces* Ann. Sci. École Norm. Sup. (4) **24** (1991), no. 6, 635–704.

[W] S. Wilson, *On the Algebra and Geometry of a Manifold's Chains and Cochains*, PhD dissertation, Stony Brook (2005).