THE HOCHSCHILD COHOMOLOGY OF A CLOSED MANIFOLD

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ABSTRACT

Let M be a closed orientable manifold of dimension d and $\mathcal{C}^*(M)$ be the usual cochain algebra on M with coefficients in a field $\mathbb{k}$. The Hochschild cohomology of M, $HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M))$ is a graded commutative and associative algebra. The augmentation map $\varepsilon : \mathcal{C}^*(M) \to \mathbb{k}$ induces a morphism of algebras $I : HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M)) \to HH^*(\mathcal{C}^*(M);\mathbb{k})$. In this paper we produce a chain model for the morphism I. We show that the kernel of I is a nilpotent ideal and that the image of I is contained in the center of $HH^*(\mathcal{C}^*(M);\mathbb{k})$, which is in general quite small. The algebra $HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M))$ is expected to be isomorphic to the loop homology constructed by Chas and Sullivan. Thus our results would be translated in terms of string homology.

1. Introduction

Let M be a simply connected closed oriented d-dimensional (smooth) manifold and $\mathbb{k}$ be a field. We denote by $\mathcal{C}^*(M)$ the cochain algebra of M with coefficients in $\mathbb{k}$ and by $HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M))$ the Hochschild cohomology algebra of $\mathcal{C}^*(M)$ ([11]) with coefficients in itself. The augmentation $\varepsilon : \mathcal{C}^*(M) \to \mathbb{k}$ corresponding to the inclusion of a base point induces a morphism of graded algebras $I : HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M)) \to HH^*(\mathcal{C}^*(M);\mathbb{k})$.

In this paper we give a model for the algebra $HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M))$ and for the morphism I (3.5). From the model we directly deduce:

1. *Theorem (4.1-Theorem 7). — For any field $\mathbb{k}$,*

a) the kernel of I is a nilpotent ideal of nilpotency index less than or equal to $d/2$,

b) the image of I lies in the center of $HH^*(\mathcal{C}^*(M);\mathbb{k})$.

The morphism I connects in fact two well known homotopy invariants of the manifold. First of all, by the Adams Cobar construction ([1], [8]): there is an isomorphism of graded algebras

$$\theta : HH^*(\mathcal{C}^*(M);\mathbb{k}) \xrightarrow{\cong} H_*(\Omega M;\mathbb{k})\,.$$

On the other hand, Jones has established an isomorphism of graded vector spaces ([12])

$$H_*(LM;\mathbb{k}) \xrightarrow{\cong} HH^*(\mathcal{C}^*(M);\mathcal{C}_*(M))\,,$$

where $LM = M^{S^1}$ denotes the free loop space on M. Finally, the Poincaré duality of the manifold yields an isomorphism of graded vectors spaces $D : HH^*(\mathcal{C}^*(M);\mathcal{C}_*(M)) \stackrel{\cong}{\to} HH^{*-d}(\mathcal{C}^*(M);\mathcal{C}^*(M))$ (7.2-Theorem 13), and by composition an isomorphism of graded vector spaces

$$H_*(LM;\Bbbk) \stackrel{\cong}{\to} HH^{*-d}(\mathcal{C}^*(M);\mathcal{C}^*(M))\,.$$

Using those isomorphisms, we can replace I by

$$\theta \circ I : HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M)) \to H_*(\Omega M)\,.$$

Theorem 1 shows that the image of I is in general very small comparatively to the expected growth of $HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M)) \cong H_*(M^{S^1};\Bbbk)$.

When $\Bbbk$ is a field of characteristic zero, Theorem 1 becomes more precise. Let us recall that an element $x \in \pi_q(M)$ is called a *Gottlieb element* ([10]-p.377), if the map $x \vee id_M : S^q \vee M \to M$ extends to the product $S^q \times M$. These elements generate a subgroup $G_*(M)$ of $\pi_*(\Omega M)$ via the isomorphism $\pi_*(\Omega M) \cong \pi_{*+1}(M)$. Finally, we denote by cat M the Lusternik-Schnirelmann category of M normalized so that cat S^n=1.

2. *Theorem (5.2-Theorem 9). — If $\Bbbk$ is a field of characteristic zero then*

a) the kernel of I is a nilpotent ideal of nilpotency index less than or equal to cat M.

b) (Im $\theta \circ I$) $\cap (\pi_*(\Omega M) \otimes \Bbbk) = G_*(M) \otimes \Bbbk$.

c) $\sum_{i=0}^{n}$ dim (Im $\theta \circ I \cap H_i(\Omega M;\Bbbk)) \leq Cn^k$, some constant $C > 0$ and $k \leq$ cat M.

With our model we characterize when I is a surjective morphism:

3. *Theorem (6.3-Theorem 10). — The morphism I is surjective if and only if M has the rational homotopy type of a product of odd dimensional spheres.*

In ([3]), Chas and Sullivan construct a product on the desuspension,

$$\mathbf{H}_*(LM;\Bbbk) = H_{*+d}(LM;\Bbbk)\,,$$

of the free loop space homology of M. This product, called the *loop product*, is defined at the chain level using both intersection product on the chains on M and loop composition. The homology $\mathbf{H}_*(LM;\Bbbk)$ is a graded commutative and associative graded algebra. They refer to $\mathbf{H}_*(LM;\Bbbk)$ endowed with the loop product as the *loop homology* of M.

For an open set $N \hookrightarrow M$ containing the base point we denote by $L_N M$ the space of loops that originate in N. By restriction, the loop product induces a product on $\mathbf{H}_*(L_N M; I\!k)$ so that the induced map $\mathbf{H}_*(L_N M; I\!k) \to \mathbf{H}_*(LM; I\!k)$ becomes a multiplicative morphism. Now the transversal intersection with ΩM defines a morphism $I_N : \mathbf{H}_*(L_N M; I\!k) \to H_*(\Omega M; I\!k)$. The Chas-Sullivan loop homology and the Hochschild cohomology of M are related by a conjecture that extends the previous works of Adams and Jones:

1. *Conjecture. — There exist isomorphisms Φ and Ψ_N of graded algebras making commutative the following diagram*

$$\begin{array}{ccc} \mathbf{H}_*(L_N M; I\!k) & \xrightarrow{\Psi_N} & HH^*(\mathcal{C}^*(M); \mathcal{C}^*(N)) \\ \downarrow {\scriptstyle I_N} & & \downarrow {\scriptstyle HH^*(\mathcal{C}^*(M);\varepsilon)} \\ H_*(\Omega M; I\!k) & \xrightarrow{\Phi} & HH^*(\mathcal{C}^*(M); I\!k) \end{array} .$$

where ϵ denotes the augmentation associated to the base point of N.

There is no complete written proof of this conjecture in the litterature; however Tradler, Cohen, Jones have already understood the situation and have given substantial parts of a proof ([4], [6]). If we assume this result, our computations give a model for the loop product on $\mathbf{H}_*(LM)$ and for the homomorphism $I_M : \mathbf{H}_*(LM; I\!k) \to H_*(\Omega M; I\!k)$.

To prove Theorem 1 we also use the following algebraic result concerning the center of the enveloping algebra of a graded Lie algebra.

4. *Theorem (5.1-Theorem 8). — Let L be a finite type graded Lie algebra defined on a field of characteristic zero, then the center of UL is contained in the enveloping algebra on the radical of L.*

The paper is organized as follows:
2 - Hochschild cohomology and Gerstenhaber product.
3 - A chain model for $I : HH^*(\mathcal{C}^*(M); \mathcal{C}^*(M)) \to HH^*(\mathcal{C}^*(M); I\!k)$.
4 - The kernel and the image of I.
5 - Determination of I when $I\!k$ is a field of characteristic zero.
6 - Examples and applications.
7 - Hochschild cohomology and Poincaré duality.

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2. Hochschild cohomology and Gerstenhaber product

In this section we fix some notations and recall the standard definitions of Hochschild cohomology and of Gerstenhaber product.

2.1. Let $\mathbb{k}$ be a principal ideal domain; modules, tensor product, linear map,... are defined over $\mathbb{k}$. For notational simplicity, we avoid to mention $\mathbb{k}$. If V is a lower or upper graded module ($V_i = V^{-i}$) the suspension s is defined by $(sV)_n = V_{n+1}$, $(sV)^n = V^{n-1}$.

2.2. Let (A, d) be a differential graded augmented cochain algebra and (N, d) be a differential graded A-bimodule, $A = \{A^i\}_{i\geq 0}, N = \{N^j\}_{j\in \mathbf{Z}}$ and $\overline{A} = \ker(\varepsilon : A \to \mathbb{k})$. The *two-sided normalized bar construction*,

$$\overline{\mathbb{B}}(N; A; N) = N \otimes T(s\overline{A}) \otimes N\,, \quad \overline{\mathbb{B}}_k(N; A; N) = N \otimes T^k(s\overline{A}) \otimes N\,,$$

is defined as follows: For $k \geq 1$, a generic element $m[a_1|a_2|...|a_k]n \in \overline{\mathbb{B}}_k(N; A; N)$ has degree $|m| + |n| + \sum_{i=1}^k (|sa_i|)$. If $k = 0$, we write $m[\,]n = m \otimes 1 \otimes n \in N \otimes T^0(s\overline{A}) \otimes N$. The differential $d = d_0 + d_1$ is defined by:

$$d_0 : \overline{\mathbb{B}}_k(N; A; N) \to \overline{\mathbb{B}}_k(N; A; N)\,, \quad d_1 : \overline{\mathbb{B}}_k(N; A; N) \to \overline{\mathbb{B}}_{k-1}(N; A; N)\,,$$

with

$$\begin{aligned} d_0(m[a_1|a_2|...|a_k]n) = d(m)[a_1|a_2|...|a_k]n &- \sum_{i=1}^k (-1)^{\epsilon_i} m[a_1|a_2|...|d(a_i)|...|a_k]n \\ &+ (-1)^{\epsilon_{k+1}} m[a_1|a_2|...|a_k]d(n) \end{aligned}$$

$$\begin{aligned} d_1(m[a_1|a_2|...|a_k]n) = (-1)^{|m|} ma_1[a_2|...|a_k]n &+ \sum_{i=2}^k (-1)^{\epsilon_i} m[a_1|a_2|...|a_{i-1}a_i|...|a_k]n \\ &- (-1)^{\epsilon_k} m[a_1|a_2|...|a_{k-1}]a_k n \end{aligned}$$

Here $\epsilon_i = |m| + \sum_{j<i}(|sa_j|)$.

2.3. For any differential graded algebra A, let A^{op} be the opposite graded algebra, $a \cdot^{op} b = (-1)^{|a|\cdot|b|} b \cdot a$, and $A^e = A \otimes A^{op}$ be the enveloping algebra. Any differential graded A-bimodule N is a differential graded A^e-module. Let A and N as in 2.2. The *Hochschild cochain complex* $\mathbf{C}^*(A;N)$ of A with coefficients in N is the differential graded module ([11],[13]):

$$\mathbf{C}^*(A;N) = \mathrm{Hom}_{A^e}(\overline{I\!\!B}(A;A;A), N)\,, \quad \mathbf{C}^n(A,M) = \prod_{p-q=n} \mathrm{Hom}_{A^e}(\overline{I\!\!B}(A,A,A)^p, N^q)\,,$$

equipped with the standard differential D defined by $Df = d\circ f - (-1)^{|f|} f \circ d$. The homology of the complex $\mathbf{C}^*(A;N)$ is called the *Hochschild cohomology* of A with values in N , and is denoted $HH^*(A;N)$.

This definition extends the classical one since:

1. *Lemma ([9]-Lemma 4.3)). — If A is a differential graded algebra such that A is a $I\!k$-free graded module then the multiplication in A extends in a semi-free resolution of A^e-modules*

$$m : \overline{I\!\!B}(A,A,A) \longrightarrow A\,.$$

This means that m is a quasi-isomorphism of differential graded A-bimodules which well behaves with quasi-isomorphisms of differential graded A-bimodules. In particular, we have the following lifting lemma:

2. *Lemma (***Lifting Homotopy Lemma).** — *For any quasi-isomorphism $\varphi : A' \to A$ there exists a unique (up to homotopy in the category of differential graded bimodules) quasi-isomorphism $\hat{m} : \overline{I\!\!B}(A,A,A) \longrightarrow A'$ such that $m \simeq \varphi \circ \hat{m}$.*

2.4. Recall that $\overline{I\!\!B}(A) = \overline{I\!\!B}(I\!k;A;I\!k) := \big(T(s\overline{A}), d\big)$ is a differential graded coalgebra with

$$d([a_1|a_2|\cdots|a_k]) = -\sum_{i=1}^{k}(-1)^{\epsilon_i}[a_1|a_2|...|d(a_i)|...|a_k] + \sum_{i=2}^{k}(-1)^{\epsilon_i}[a_1|a_2|...|a_{i-1}a_i|...|a_k]\,.$$

The canonical isomorphism of graded modules

$$\mathrm{Hom}_{A^e}(\overline{I\!\!B}(A;A;A), N) = \mathrm{Hom}(T(s\overline{A}), N)\,,$$

carries on $\mathrm{Hom}(T(s\overline{A}), N)$ a differential D'. Observe that the differential D' is not the canonical differential D of $\mathrm{Hom}(\overline{I\!\!B}(A), N)$ except when N is the trivial bimodule. If $N = A$, Gerstenhaber ([11]) has proved that the usual cup product on $\mathrm{Hom}(T(s\overline{A}), A)$ makes $(\mathrm{Hom}(T(s\overline{A}), A), D')$ a differential graded algebra such that the induced product on $HH^*(A;A)$, called the *Gerstenhaber product* ([11]) is commutative.

3. A chain model for $I : HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M)) \to HH^*(\mathcal{C}^*(M);\mathbb{k})$

In this section we construct, for any field of coefficients $\mathbb{k}$, an explicit model for the Hochschild cohomology algebra at the chain level.

3.1. Recall the Adams Cobar construction ΩC on a coaugmented differential graded coalgebra $C = \mathbb{k}\oplus\overline{C}$. This is the differential graded algebra $(T(s^{-1}\overline{C}),d)$, where $d = d_1+d_2$ is the unique derivation determined by:

$$d_1 s^{-1}c = -s^{-1}dc\,, \text{ and } d_2 s^{-1}c = \sum_i (-1)^{|c_i|} s^{-1}c_i\otimes s^{-1}c_i'\,, \quad c\in\overline{C}\,,$$

where the reduced coproduct of $c\in\overline{C}$ is written $\bar{\Delta}c = \sum_i c_i\otimes c_i'$. For sake of simplicity we put $\langle x_1|x_2|\cdots|x_n\rangle := s^{-1}x_1\otimes\cdots\otimes s^{-1}x_n$.

3.2. Assume $\mathbb{k}$ is a field, and M is a 1-connected compact d-dimensional manifold. Denote by $f : (T(V),d)\to\mathcal{C}^*(M)$ a free minimal model for the singular cochain algebra on M ([9]), i.e. $(T(V),d)$ is a differential graded algebra, f is a quasi-isomorphism of differential graded algebras, and $d(V)\subset T^{\geq 2}(V)$. The differential graded algebra $(T(V),d)$ is uniquely defined, up to isomorphism, by the above properties. Moreover, $V^p\cong H^{p-1}(\Omega M)$, ([9]). Denote by S a complement of the vector space generated by the cocycles of degree d. The differential graded ideal $J = (T(V))^{>d}\oplus S$ is acyclic and the quotient algebra $A = T(V)/J$ is a finite dimensional graded differential algebra.

3.3. Since A is finite dimensional, the graded dual $A^\vee$ is a differential graded coalgebra and we consider the differential graded algebra $\Omega A^\vee = (T(W),d)$, with in particular, $W\cong \mathrm{Hom}(s\overline{A},\mathbb{k})$, and $\Omega A^\vee = \mathrm{Hom}(\overline{\mathbb{B}}A,\mathbb{k}) = (\mathrm{Hom}(T(s\overline{A}),\mathbb{k}),D)$ (2.4). We choose a homogeneous linear basis e_i for $\overline{A}$, and its dual basis w_i for W. This determines the constants of structure α_{ij}^k and ρ_i^j:

$$\langle w_i, se_k\rangle = -(-1)^{|w_i|}\delta_{ik}\,,\quad e_i\cdot e_j = \textstyle\sum_k \alpha_{ij}^k e_k\,,\quad d(e_i) = \sum_j \rho_i^j e_j$$
$$d(w_i) = \textstyle\sum_{jk} a_i^{jk} w_jw_k + \sum_j \beta_i^j w_j\,,\quad a_i^{jk} = (-1)^{|e_j|+|e_je_k|}\alpha_{jk}^i\,,\quad \beta_i^j = (-1)^{|w_j|}\rho_j^i\,.$$

5. *Theorem. — Let $\mathbb{k}$ be a field and M be a 1-connected closed oriented manifold of dimension d. With notation introduced above:*

a) the derivation D uniquely defined on the tensor product of graded algebras $A\otimes T(W)$ by

$$\begin{cases} D(a\otimes 1) = d(a)\otimes 1 + \sum_j (-1)^{|a|+|e_j|}[a,e_j]\otimes w_j\,, & a\in A\,,\\ D(1\otimes b) = 1\otimes d(b) - \sum_j (-1)^{|e_j|} e_j\otimes [w_j,b]\,, & b\in TW\,,\end{cases}$$

is a differential. Here [,] *denotes the Lie bracket in the graded algebras* A *and* $T(W)$.

b) the graded algebras $H_*(A\otimes T(W),D)$ *and* $HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M))$ *are isomorphic.*

Proof. — a) is proved by a direct but laborious computation.
b) is a direct consequence of the definition. □

Observe that this model is dual to those constructed by one of us ([13]).

1. *Proposition.* — *Let* $\Bbbk$ *be a field and* M *be a 1-connected closed oriented manifold of dimension* d. *There is a cohomology spectral sequence of graded algebras such that*

$$E_2 = HH^*(H^*(M), H^*(M)) \Rightarrow HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M))\,.$$

Proof. — The spectral sequence is obtained by filtering the complex $(\mathrm{Hom}\,(T(s\overline{A},A),D'))$ by the differential ideals $\mathrm{Hom}\,(T^{\leq p}(s\overline{A}),A)$ (2.4). Since $H^*(A) = H^*(M)$, it follows that $E_1 = \mathrm{Hom}(\overline{I\!B}(H^*(M)),\Bbbk)\otimes H^*(M)$ and $E_2 = H^*(\mathrm{Hom}\,(\overline{I\!B}(H^*(M)),\Bbbk)\otimes H^*(M),D)\,.$ □

1. *Example.* — If M is a formal space, (for instance M is a simply connected compact Kähler manifold for $\Bbbk = \mathbf{Q}$ ([7]) one can choose $A = H^*(M)$ and thus the algebras $HH^*(H^*(M);H^*(M))$ and $HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M))$ are isomorphic graded vector spaces. If we put $H_* = H_*(M)$ the algebra $HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M))$ is isomorphic to the graded algebra $H(A\otimes T(s\overline{H}_*),D)$ with $D(a\otimes 1)=0,\ a\in A$ and $D(1\otimes b) = -\sum_J(-i)^{|e_j|}e_j\otimes[w_j,b],\ b\in\overline{H}_*$.

3.4. *The commutative case.* Suppose that the algebra $\mathcal{C}^*(M)$ is connected by a sequence of quasi-isomorphisms to a commutative differential graded algebra (A,d). This is the case if either $\Bbbk$ is of characteristic zero, or else if $\Bbbk$ is a field of characteristic $p>d$ ([2], Proposition 8.7). We can also suppose that A is finite dimensional, $A^0=\Bbbk,\ A^1=0,\ A^{>d}=0$ and $A^d=\Bbbk\omega$. Then formulas of 3.3-Theorem 5 simplify as:

$$\begin{cases} D(a\otimes 1) = d(a)\otimes 1\,,\\ D(1\otimes b) = 1\otimes d(b) - \sum_j(-1)^{e_j}e_j\otimes[w_j,b]\,.\end{cases}$$

3.5. We can now interpret the intersection morphism in terms of our model:

6. *Theorem. — Let $\Bbbk$ be a field and M be a 1-connected closed oriented manifold of dimension d. There is a commutative diagram of algebras*

$$\begin{array}{ccc} HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M)) & \xrightarrow{\cong} & H_*(A\otimes T(W),D) \\ {\scriptstyle \theta\circ I}\downarrow & & \downarrow {\scriptstyle H(\varepsilon_A\otimes 1)} \\ H_*(\Omega M) & \xrightarrow{\cong} & H_*(T(W),d)\,. \end{array}$$

Proof. — Recall that Hochschild cohomology $HH^*(A;N)$ is covariant in N and contravariant in A. Moreover, if $f:A\to B$ is a quasi-isomorphism of differential graded algebras and $g:N\to N'$ is a quasi-isomorphism of A-bimodules, we have isomorphisms

$$HH^*(B;N)\xrightarrow{\cong} HH^*(A;N)\xrightarrow{\cong} HH^*(A;N')\,.$$

We obtain therefore the following commutative diagram

$$\begin{array}{ccccc} HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M)) & \xrightarrow{\cong} & HH^*(A;A) & \xrightarrow{\cong} & H_*(A\otimes T(W),D) \\ \downarrow {\scriptstyle HH^*(\mathcal{C}^*(M),\varepsilon)} & & \downarrow {\scriptstyle HH^*(A,\varepsilon_A)} & & \downarrow {\scriptstyle H(\varepsilon_A\otimes 1)} \\ HH^*(\mathcal{C}^*(M);\Bbbk) & \xrightarrow{\cong} & HH^*(A;\Bbbk) & \xrightarrow{\cong} & H_*(T(W),d) \end{array}\,.$$

□

4. The kernel and the image of I

4.1. If J is an ideal of an algebra A, we put $J^1=J$ and $J^{n+1}=JJ^n\,,n\geq 1$. In the case J is nilpotent, we define

$$\mathrm{Nil}\,(J)=\sup\{n\mid J^n\neq 0\}\,.$$

7. *Theorem. — Let $\Bbbk$ be a field and M be a simply connected closed oriented d-dimensional manifold.*

a) The kernel of the intersection morphism I is nilpotent and $Nil(Ker\,I)\leq d/2$.

b) The image of $\theta\circ I$ is contained in the center of $H_*(\Omega M)$.

Proof. — a) By 3.5-Theorem 6, the kernel of I is generated by the classes of cocycles in $\overline{A}\otimes T(W)$. Since $A^1=0$ and $A^{>d}=0$, the nilpotency of the kernel of I is less than or equal to $d/2$.

b) Let e_i and w_i be the elements defined in 3.3 and $[\alpha]$ be an element in the image of $H(\epsilon_A\otimes id)$. Then α is a cocycle in $T(W)$ and there exist elements α_i in $T(W)$ such that $\bar{\alpha} = 1\otimes\alpha+\sum_i e_i\otimes\alpha_i$ is a cycle in $A\otimes T(W)$. A short calculation shows that the component of e_i in $d(\bar{\alpha})$ is

$$(-1)^{|e_i|}\left(d(\alpha_i)-[w_i,\alpha]+\sum_j \beta_i^j\alpha_j+\sum_{j,k} a_i^{j,k}(-1)^{|u||w_k|}\alpha_j w_k+\sum_{j,k} a_i^{kj}(-1)^{|w_k|}w_k\alpha_j\right).$$

Since this component must be 0, by Lemma 3 below there exists a surjective morphism

$$H(T(W),d)\otimes \Bbbk[u]\to H(T(W),d)$$

that maps u to $[\alpha]$. This implies that $[\alpha]$ is in the center of $H(T(W),d)\cong H_*(\Omega M)$. □

3. *Lemma. — Assume* $\Bbbk$ *is a field. Let* α *be a cycle in* $(T(W),d)$ *and let* u *be a variable in the same degree. Then with the notation of 3.3:*

1. There exists a surjective quasi-isomorphism

$$\varphi:(T(w_i,u,w_i'),D)\to(T(W),d)\otimes(\Bbbk[u],0)\,,\qquad |w_i'|=|u|+|w_i|+1\,,$$

such that $\varphi(u)=u$*,* $\varphi(w_i)=w_i$ *and* $\varphi(w_i')=0$*, and with* D *defined by*

$$D(w_i')=[w_i,u]-\sum_j\beta_i^j w_j'-\sum_{j,k}a_i^{j,k}(-1)^{|u||w_k|}w_j'w_k-\sum_{j,k}a_i^{kj}(-1)^{|w_k|}w_k w_j'\,.$$

2. There exists a morphism of differential graded algebras

$$\rho:(T(w_i,u,w_i'),D)\to(T(W),d)$$

such that $\rho(u)=\alpha$ *and* $\rho(w_i)=w_i$ *if and only if there are elements* $\alpha_i\in T(W)$ *satisfying*

$$d(\alpha_i)=[w_i,\alpha]-\sum_j\beta_i^j\alpha_j-\sum_{j,k}a_i^{j,k}(-1)^{|u||w_k|}\alpha_j w_k-\sum_{j,k}a_i^{kj}(-1)^{|w_k|}w_k\alpha_j\,.$$

Proof. — We define $D(w_i')$ by the above formula. Proving that $D^2=0$ is an easy and standard computation. The morphism

$$\varphi:(T(w_i,u,w_i'),D)\to(T(w_i),d)\otimes(\Bbbk[u],0)$$

defined by $\varphi(w_i)=w_i$, $\varphi(u)=u$ and $\varphi(w_i')=0$ is a surjective homomorphism of differential graded algebras. To prove that φ is a quasi-isomorphism, we filter each differential graded algebra by putting u in filtration degree 0 and the other variables in filtration degree one. We are then reduced to prove that

$$\bar{\varphi}:(T(w_i,u,w_i'),D)\to(T(w_i),0)\otimes(\Bbbk[u],0)\,,\qquad d(w_i)=0,d(w_i')=[w_i,u]\,,$$

is a quasi-isomorphism. Denote by K the kernel of $\bar{\varphi}$ and consider the short exact sequence of complexes

$$0 \to (K \otimes E, D) \to (T(w_i, u, w_i') \otimes E, D) \xrightarrow{\bar{\varphi}\otimes 1} ((T(w_i) \otimes \Bbbk[u]) \otimes E, D) \to 0\,,$$

where E is the linear span of the elements $1, sw_i, su$ and sw_i', and where D is defined by

$$D(sw_i) = w_i\otimes 1\,, D(su) = u\otimes 1\,, D(sw_i') = w_i' - (-1)^{|w_i|} w_i\otimes su + (-1)^{|u||w_i'|+|u|} u\otimes sw_i\,.$$

By construction, $(T(w_i, u, w_i')\otimes E, D)$ and $(T(w_i)\otimes \Bbbk[u]\otimes E, D)$ are contractible and therefore quasi-isomorphic. Now a non-zero cocycle of lowest degree in K remains a non-trivial cocycle in the complex $(K\otimes E, D)$. Therefore $H_*(K) = 0$ and φ is a quasi-isomorphism. Part 2. of Lemma 3 follows directly from the expression of D. □

5. Determination of I when $\Bbbk$ is a field of characteristic zero

In this section $\Bbbk$ is a field of characteristic zero.

5.1. By 4.1-Theorem 7, the image of I is contained in the center of $H_*(\Omega M)$. On the other hand, by the Milnor-Moore theorem (e.g [10]-Theorem 21.5), $H_*(\Omega M)$ is the universal enveloping algebra of the homotopy Lie algebra $L_M = \pi_*(\Omega M) \otimes \Bbbk$ ([10]-p.294).

Let L be any graded algebra. The center, $Z(UL)$, of the universal enveloping algebra UL contains the universal enveloping algebra of the center of the Lie algebra, $UZ(L)$. However the inclusion can be strict. Consider for instance the Lie algebra $L = \mathbf{L}(a,b)/([b,b],[a,[a,b]])$, with $|a| = 2$ and $|b| = 1$. The element $(ab-ba)b$ is in the center of UL, but not in $UZ(L)$. We denote by $R(L)$ the sum of all solvable ideals in L, ([10]-p.495).

8. *Theorem. — If $L = \{L_i\}_{i\geq 1}$ is a graded Lie algebra over a field of characteristic zero satisfying $\dim L_i < \infty$ then $Z(UL) \subset UR(L)$.*

Proof. — It is well known that in characteristic zero, UL decomposes into a direct sum

$$UL = \bigoplus_{k\geq 0} \Gamma^k(L)$$

where the $\Gamma^k(L)$ are sub-vector spaces that are stable for the adjoint representation of L on UL: $\Gamma^0(L) = \Bbbk$, $\Gamma^1(L) = L$, and $\Gamma^n(L)$ is the sub-vector space generated by the elements $\varphi(x_1,\dots,x_n)=\sum_{\sigma\in\Sigma_n}\varepsilon_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)},\ \ x_i\in L$. The coproduct Δ of UL respects the decomposition, i.e.

$$\Delta:\Gamma^n(L)\to\bigoplus_{p+q=n}\Gamma^p(L)\otimes\Gamma^q(L)\,.$$

If we denote by Δ_p the component of Δ in $\Gamma^p(L)\otimes\Gamma^{n-p}(L)$ then

$$\Delta_p(\varphi(x_1,\dots,x_n))=\sum_{\tau\in Sh_p}\varepsilon_\tau\binom{n}{p}\varphi(x_{\tau(1)},\dots,x_{\tau(p)})\otimes\varphi(x_{\tau(p+1)},\dots,x_{\tau(n)})\,,$$

where Sh_p denotes the set of p-shuffles of the set $\{1,2,\dots,n\}$. This implies that the composition $\Gamma^n(L)\xrightarrow{\Delta_p}\Gamma^p(L)\otimes\Gamma^{n-p}(L)\xrightarrow{\text{multiplication}}UL$ is the multiplication by $\binom{n}{p}$. We then consider the composite

$$c:\Gamma^n(L)\xrightarrow{\Delta_1}L\otimes\Gamma^{n-1}(L)\xrightarrow{1\otimes\Delta_1}L\otimes L\otimes\Gamma^{n-2}(L)\to\cdots\to L^{\otimes^n}\,.$$

Let $\alpha\in UL$ be an element in the center of UL, $\alpha=\sum_{i=1}^n\alpha_i$ with $\alpha_i\in\Gamma^i(L)$. Since $\Gamma^i(L)$ is stable by adjunction, each α_i is in the center of UL. Therefore we can assume that $\alpha\in\Gamma^n(L)$. We write $c(\alpha)$ as a sum of monomials $x_{i_1}\otimes\ldots\otimes x_{i_n}$. Since $\mathrm{mult}\circ c:\Gamma^n(L)\to UL$ is the multiplication by $n!$, the element α belongs to the Lie algebra generated by the x_{i_j}. Suppose that in the decomposition of $c(\alpha)$ the number of monomials is minimal, then for each r, $1\le r\le n$, the elements $x_{i_1}\otimes\ldots\otimes x_{i_{r-1}}\otimes x_{i_{r+1}}\ldots\otimes x_{i_n}$ are linearly independent. Since $[\alpha,x]=0,\ x\in L$, we obtain the equation:

$$0=\sum_{k=1}^n\left(\sum_i(-1)^{|x|\cdot(|x_{i_1}|+\ldots+|x_{i_{k-1}}|)}x_{i_1}\otimes\ldots\otimes[x,x_{i_k}]\otimes\ldots\otimes x_{i_n}\right)\,.$$

Let us assume that the x_{i_k} are ordered by increasing degrees then the elements x_{i_k} with maximal degree belong to $Z(L)$. The above equation shows also that $[x_{i_k},x]$ belongs to the subvector space generated by the elements x_{i_l} with higher degree. A decreasing induction on the degree shows that all the x_{i_k} belong to $R(L)$. □

5.2. Denote by X_0 the 0-localization of a simply connected space X. The Lusternik-Schnirelmann category of X_0, $\operatorname{cat}X_0$, is less than or equal to the Lusternik-Schnirelmann of X, cat X. Moreover the invariant $\operatorname{cat}X_0$ is easier to compute than $\operatorname{cat}X$, ([10]-§-27).

9. *Theorem. — Let M be a simply connected oriented closed manifold and $\Bbbk$ is a field of characteristic zero. Then*

a) The kernel of I is a nilpotent ideal and Nil(Ker (I)) $\le$ cat M_0 .

b) $(\mathit{Im}\ \theta \circ I) \cap (\pi_*(\Omega M) \otimes \Bbbk) = G_*(M) \otimes \Bbbk$.

c) $\sum_{i=0}^{n}$ *dim* $(\mathit{Im}\ \theta \circ I \cap H_i(\Omega M; \Bbbk)) \leq Cn^k$, *some constant* $C > 0$ *and* $k \leq$ *cat* M_0.

Proof. — a) By ([10]-Theorems 29.1 and 28.5), $\mathcal{C}^*(M; \mathbf{Q})$ is connected by a sequence of quasi-isomorphisms to a connected finite dimensional commutative differential graded algebra (A, d) satisfying Nil $(\overline{A}) \leq n$ for $n > \operatorname{cat} M_0$. Thus we conclude as in 4.1-proof of theorem 7.

b) The differential graded algebra $\Omega(A^\vee) = (T(W), d)$ is the universal enveloping algebra on the graded Lie algebra $\mathcal{L}_M = (\mathbf{L}(W), d)$, and the differential graded algebra $(T(W \oplus \Bbbk u \oplus W'), D)$ is the universal enveloping algebra of the differential graded Lie algebra $\mathcal{L}^1_M = (\mathbf{L}(W \oplus ku \oplus W'), D)$, (e.g [10]-p.289), with

$$\begin{cases} d(w_i) = \sum_j \beta_i^j w_j + \sum_{j,k} \frac{1}{2} a_i^{jk} [w_j, w_k]\,, \\ D(w_i') = [w_i, u] - \sum_j \beta_i^j w_j' - \sum_{j,k} a_i^{kj} (-1)^{|w_k|} [w_k, w_j']\,. \end{cases}$$

By construction $\mathcal{L}_M$ is a free Lie model for M and $\mathcal{L}^1_M$ is a free Lie model for $M \times S^n$ with $n = |u| + 1$, ([10]-§24). Moreover there exists a bijection between homotopy classes of maps:

$$[X \times S^n, X] \cong [(\mathbf{L}(W \oplus \Bbbk u \oplus W'), D), (\mathbf{L}(W), d)]\,.$$

Therefore a homomorphism $\varphi : (\mathbf{L}(W \oplus \Bbbk u \oplus W'), D) \to (\mathbf{L}(W), d)$ such that $\varphi(u) = \alpha$ and $\varphi(w) = w$, $w \in W$, corresponds to a map $f : M \times S^n \to M$ which extends $id_M \vee g : M \times S^n \to M$, such that $[g] = \alpha$ modulo the identifications $\pi_n(M) \otimes \Bbbk \cong \pi_{n-1}(\Omega M) \otimes \Bbbk \cong H_{n-1}(\mathbf{L}(W), d)$. This means exactly that $\operatorname{Image} I \cap (\pi_*(\Omega M) \otimes \Bbbk) = G_*(M) \otimes \Bbbk$. c) By, Theorems 36.4, 36.5 and 35.10 of [10] we know that if $L = \pi_*(\Omega M) \otimes \Bbbk$ then $R(L)$ is finite dimensional and $\dim R(L)_{\text{even}} \leq \operatorname{cat} M_0$. We conclude using the graded Poincaré-Birkhoff-Witt theorem ([10]-Theorem 21.1): $Z(UL) \subset UR(L) \cong \Lambda(R(L)_{\text{odd}}) \otimes \Bbbk[(R(L)_{\text{even}}]$. □

6. Examples and applications

In this section we assume that $\Bbbk$ is a field.

6.1. *The spheres* S^n. Since the differential graded algebra $\mathcal{C}^*(S^n)$ is quasi-isomorphic to $(H^*(S^n), 0) = (\wedge u/u^2, 0)$, $|u| = n$, by 3.3-Example 1, $HH^*(\mathcal{C}^*(S^n); \mathcal{C}^*(S^n))$ is isomorphic as an algebra to

$$H^*(\wedge u \otimes T(v), D)\,,\ |v| = n - 1\,,\ |u| = -n\,,\ D(u) = 0\,,\ D(v) = u \otimes [v, v]\,.$$

When n is odd, $D=0$, $HH^*(\mathcal{C}^*(S^n);\mathcal{C}^*(S^n))\cong \wedge u\otimes T(v)$ and $I=\varepsilon\otimes 1:\wedge u\otimes T(v)\to T(v)$. When n is even, $D(v^{2n})=0$, $D(v^{2n+1})=2u\otimes v^{2n+2}$. Therefore a set of generators is given by the elements $c=1\otimes v^2\,, b=u\otimes v\,, a=u\otimes 1$, $|a|=-n\,, |b|=-1\,, |c|=2n-2$ and,

$$HH^*(\mathcal{C}^*(S^n);\mathcal{C}^*(S^n))\cong \wedge(b)\otimes Ik[a,c]/(2ac,a^2,ab)(\text{see also [5]}\,.$$

The homomorphism $\theta\circ I : HH^*(\mathcal{C}^*(S^n);\mathcal{C}^*(S^n))\to H_*(\Omega S^n)=T(v)$ is given by: $I(c)=v^2$, $I(a)=I(b)=0$.

6.2. *An example where I is the trivial homomorphism.* Let M be the connected sum $M=(S^3\times S^3\times S^3)\#(S^3\times S^3\times S^3)$. The wedge $N=(S^3\times S^3\times S^3)\vee(S^3\times S^3\times S^3)$ is then obtained by attaching a 9-dimensional cell to M along the homotopy class determined by the collar between the two components of M. Recall that

$$\pi_*(\Omega N)\otimes\mathbf{Q}\cong Ab(a,b,c)\coprod Ab(e,f,g)\,,$$

where $Ab(u,v,w)$ means the abelian Lie algebra generated by u, v and w considered in degree 2. The inclusion $i:M\to N$ induces a surjective map $\pi_*(\Omega M)\otimes\mathbf{Q}\to\pi_*(\Omega N)\otimes\mathbf{Q}$, This means that the attachment of the cell is inert in the sense of [10]-p.503. Therefore, ([10]-Theorem 38.5),

$$\pi_*(\Omega M)\otimes\mathbf{Q}\cong Ab(a,b,c)\coprod Ab(e,f,g)\coprod \mathbf{L}(x)$$

with $|x|=7$. In particular $R(L)$ is zero, and by 4.1-Theorems 7 and 5.1-Theorem 8, when Ik is of characteristic zero, the homomorphism I is trivial.

6.3. *Lie groups.* Let Ik be a field of characteristic zero and G be a connected Lie group. Since G has the rational homotopy type of a product of odd dimensional spheres, we obtain

$$HH^*(\mathcal{C}^*(G);\mathcal{C}^*(G))\cong\wedge(u_1,\dots,u_n)\otimes\mathbf{T}(v_1,\dots,v_n)\,,$$

and I_G is onto. This example generalizes in:

10. *Theorem. — Let Ik be a field of characteristic zero and M be a simply connected closed oriented d-dimensional manifold. The morphism $\theta\circ I : HH^*(\mathcal{C}^*(M);\mathcal{C}^*(M))\to H_*(\Omega M)$ is surjective if and only if M has the rational homotopy type of a product of odd dimensional spheres.*

Proof. — When M has the rational homotopy type of the product of odd dimensional spheres, then I is clearly surjective. Conversely, if I is surjective, then $\pi_*(\Omega M)\otimes \mathbf{Q} = G_*(M)\otimes \mathbf{Q}$. Thus, $\pi_*(M)\otimes \mathbf{Q} = G_{\text{odd}}\otimes \mathbf{Q}$, ([10], Proposition 29.8). Let $\{f_i : S^{n_i}\to M,\ i=1,\cdots,r\}$ represent a given linear basis of $\pi_*(M)\otimes \mathbf{Q}$, and let $\varphi_i : S^{n_i}\times M \to M$ be maps that restrict to $f_i\vee id_M$ on $S^{n_i}\vee M$. Then the composition

$$S^{n_1}\times\ldots\times S^{n_r} \hookrightarrow S^{n_1}\times\ldots\times S^{n_r}\times M \stackrel{1\times\varphi_r}{\longrightarrow} S^{n_1}\times\ldots\times S^{n_{r-1}}\times M \stackrel{1\times\varphi_{r-1}}{\longrightarrow} \ldots \stackrel{1\times\varphi_1}{\longrightarrow} M$$

induces an isomorphism on the homotopy groups. Therefore, M has the rational homotopy type of a product of odd dimensional spheres. □

7. Hochschild cohomology and Poincaré duality

When two A-bimodules M and N are quasi-isomorphic as bimodules, then the Hochschild cohomologies $HH^*(A;M)$ and $HH^*(A;N)$ are isomorphic. In this section we relate the Hochschild cohomology of the singular cochains algebra on X with coefficients in itself and with coefficients in the singular chains on X when X is a Poincaré duality space. The usual cap product with the fundamental class is not a bimodule morphism. However the vector spaces $HH^n(\mathcal{C}^*(M);\mathcal{C}_*(M))$ and $HH^{n-d}(\mathcal{C}^*(M);\mathcal{C}^*(M))$ are isomorphic.

7.1. Let V be a graded module, then $V^\vee$ denotes the graded dual, $V^\vee = \mathrm{Hom}_{I\!k}(V, I\!k)$, and $\langle -;-\rangle : V^\vee\otimes V \to I\!k$ denotes the duality pairing. We denote by $\lambda_V : V\to V^{\vee\vee}$ the natural inclusion defined by $\langle\lambda_V(v),\xi\rangle = (-1)^{|\xi|}\langle \xi, v\rangle$.

7.2. Let X be topological space. The $\mathcal{C}^*(X)$-bimodule structures on $\mathcal{C}_*(X)$ and $\mathcal{C}^*(X)^\vee$ are explicitly defined by:

$$\begin{aligned} f\cdot c\cdot g &:= (-1)^{|c|(|f|+|g|)+|f|+|f|\,|g|}(g\otimes id\otimes f)(\Delta_X\otimes id)\circ\Delta_X(c)\,, \quad c\in\mathcal{C}_*(X)\,,\\ \langle f\cdot\alpha\cdot g; h\rangle &:= (-1)^{|f|}\langle \alpha; g\cup h\cup f\rangle\,, \quad f,g,h\in\mathcal{C}^*(X), \alpha\in\mathcal{C}^*(X)^\vee\,. \end{aligned}$$

Remark that the associativity properties of AW and of Δ_X imply directly that $\mathcal{C}_*(X)$ is a graded $\mathcal{C}^*(X)$-bimodule.

Let $1\in\mathcal{C}^0(X)$ be the 0-cochain which value is 1 on the points of X. The usual cap product is then defined by

$$\mathcal{C}^p(X)\otimes\mathcal{C}_k(X)\longrightarrow\mathcal{C}_{k-p}(X)\,,\quad f\otimes c\mapsto f\cap c = f\cdot c\cdot 1 = \sum_i (-1)^{|c_i|\cdot|f|}c_i f(c_i')\,.$$

The cap product with a cycle $x \in \mathcal{C}_k(X)$ is a well defined homomorphism of differential graded modules, but is not a "degree k homomorphism" of $\mathcal{C}^*(X)$-bimodules. However,

11. *Theorem. — Let X be a path connected space and $c \in \mathcal{C}_*(X)$ be a cycle of degree $k > 0$. Then there exists a (degree k) morphism of $\mathcal{C}^*(X)$-bimodules*

$$\gamma_c : \overline{I\!B}(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X)) \to \mathcal{C}_*(X)$$

such that

- $\gamma_c(1[\,]1) = c$,
- $H(\gamma_c) \circ H(m)^{-1} : H^*(X) \to H_*(X)$ *is the cap product by $[c]$, m is the quasi-isomorphism of $\mathcal{C}^*(X)$-modules defined in 2.3-Lemma 1.*

Recall that γ_c is a degree k morphism of $\mathcal{C}^*(X)$-bimodules means that the following two properties are satisfied:

a) $d \circ \gamma_c = (-1)^k \gamma_c \circ d$,

b) $\gamma_c(f \cdot \alpha \cdot g) = (-1)^{|f|\,k} f \cdot \gamma_c(\alpha) \cdot g$,

for $f, g \in \mathcal{C}^*(X)$ and $\alpha \in \overline{I\!B}(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X))$.

Proof. — For simplicity we denote by A^e the enveloping algebra of $A = \mathcal{C}^*(X)$ and by B the differential graded $\mathcal{C}^*(X)$-bimodule $\overline{I\!B}(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X))$.

Recall the loop space fibration $\mathrm{ev} : X^{S^1} \to X$, $\gamma \mapsto \gamma(0) = \gamma(1)$ with the canonical section $\sigma : X \to X^{S^1}, x \mapsto$ the constant loop at x. Jones defined a quasi-isomorphism of differential graded modules ([4]-Theorem 8),

$$J_* : B \otimes_{A^e} A \to \mathcal{C}^*(X^{S^1})$$

making commutative the following diagram of differential graded modules

$$\begin{array}{ccc} B \otimes_{A^e} A & \xrightarrow{J_*} & \mathcal{C}^*(X^{S^1}) \\ {\scriptstyle i}\nwarrow & & \nearrow{\scriptstyle \mathcal{C}^*(\mathrm{ev})} \\ & \mathcal{C}^*(X) & \end{array}$$

where $i : \mathcal{C}^*(X) \to B \otimes_{A^e} \mathcal{C}^*(X)$, $\quad f \mapsto 1[\,]1 \otimes f$, denotes the canonical inclusion. Let ρ be the composite $\mathcal{C}^*(\sigma) \circ J_*$ then ρ is a retraction of i: $\rho \circ i = id$.

Let $u \in \mathcal{C}^k(X)^\vee$, $k > 0$, be a cycle. Using the canonical isomorphism of differential graded modules

$$\Psi : \mathrm{Hom}(B \otimes_{A^e} A, I\!k) \to \mathrm{Hom}_{A^e}(B, A^\vee)\,, \quad (\Psi(\theta)(\alpha))\,(f) = \theta(\alpha \otimes f)\,,$$

we define the map

$$\theta_u : \overline{I\!B}(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X)) \to (\mathcal{C}^*(X))^\vee\,, \quad \theta_u = \Psi(u \circ \rho)\,.$$

The element θ_u is a k-cycle in $\mathrm{Hom}_{A^e}(B, A^\vee)$ and for any $f \in A$, $\theta_u(1[\,]1)(f) = u \circ \rho(1[\,]1 \otimes f) = u \circ \rho \circ i(f) = u(f)$.

Since the linear map

$$\lambda : \mathcal{C}_*(X) \to \mathcal{C}^*(X)^\vee$$

is a morphism of differential graded $\mathcal{C}^*(X)$-bimodules, for a cycle $c \in \mathcal{C}_k(X)$, we have a morphism

$$\theta_{\lambda(c)} : \overline{I\!B}(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X)) \to (\mathcal{C}^*(X))^\vee$$

with $\theta_{\lambda(c)}(1[\,]1) = \lambda(c)$.

Since $\overline{I\!B}(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X))$ is semifree, we deduce from the lifting homotopy property (2.3-Lemma 2) a morphism of $\mathcal{C}^*(X)$-bimodules

$$\gamma_c : \overline{I\!B}(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X)) \to \mathcal{C}_*(X)$$

making commutative, up to homotopy, the diagram

$$\begin{array}{ccc}
\overline{I\!B}(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X)) & \xrightarrow{\theta_{\lambda(c)}} & \mathcal{C}^*(X)^\vee \\
\| & & \uparrow \lambda \\
\overline{I\!B}(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X)) & \xrightarrow{\gamma_c} & \mathcal{C}_*(X)
\end{array}$$

and such that $\gamma_c(1[\,]1) = c$. The equality $H(\gamma_c) \circ H(m)^{-1} = - \cap [c]$ comes from the commutativity of the diagram

$$\begin{array}{ccc}
\overline{I\!B}_0(\mathcal{C}^*(X), \mathcal{C}^*(X), \mathcal{C}^*(X)) & \xrightarrow{\theta_{\lambda(c)}} & \mathcal{C}^*(X)^\vee \\
m \downarrow & & \uparrow \lambda \\
\mathcal{C}^*(X) & \xrightarrow{-\cap c} & \mathcal{C}_*(X)
\end{array}$$

i.e., for any $f, g, h \in \mathcal{C}^*(X)$, we have $\langle \theta_{\lambda(c)}(f[\,]g), h \rangle = \langle \lambda \circ (-\cap c) \circ m(f[\,]g), h \rangle$.
□

As a special case, we deduce:

12. *Theorem. — Let M be a 1-connected $I\!k$-Poincaré duality space of formal dimension d. Then there are quasi-isomorphisms of $\mathcal{C}^*(M)$-bimodules*

$$\mathcal{C}^*(M) \xleftarrow{m} \overline{I\!B}(\mathcal{C}^*(M), \mathcal{C}^*(M), \mathcal{C}^*(M)) \xrightarrow{\gamma} \mathcal{C}_*(M)$$

where m is defined in 2.3-lemma 1 and $\gamma = \gamma_{[M]}$ with $[M] \in H_d(M)$ a fundamental class of M. In particular, the composite, $H(m) \circ H(\gamma)^{-1}$ is the Poincaré isomorphism $\mathcal{P} : H_*(M) \to H^{d-*}(M)$.

Applying Hochschild cohomology, we obtain:

13. *Theorem. — Let M be a 1-connected $\mathbb{k}$-Poincaré duality space of formal dimension d then there exist natural linear isomorphisms*

$$D : HH^n(\mathcal{C}^*(M); \mathcal{C}_*(M)) \xrightarrow{\cong} HH^{n-d}(\mathcal{C}^*(M); \mathcal{C}^*(M))\,.$$

Proof. — Let $\varphi : N \to N'$ be a homomorphism of differential graded A-bimodules and assume that A is a $\mathbb{k}$-module. Then we deduce from 2.3-lemma 1 (see [9] for more details) that φ induces an isomorphism of graded modules

$$HH^*(A; N) \to HH^*(A; N')\,.$$

Theorem 13 follows directly from Theorem 12 when one observes that the suspended map $s^d\gamma$ is a quasi-isomorphism of differential graded $\mathcal{C}^*(X)$-bimodules. □

REFERENCES

1. J.F. ADAMS, On the cobar construction, *Proc. Nat. Acad. Sci.*, **42,** (1956), 409-412.
2. D. ANICK, Hopf algebras up to homotopy, *J. Amer. Math. Soc.*, **2,** (1989), 417-453.
3. M. CHAS AND D. SULLIVAN, String topology, *Annals of Mathematics* (to appear) GT/9911159.
4. R. COHEN AND J. JONES, A homotopy theoretic realization of string topology, *Mathematische Annalen*, **324,** (2002), 773-798.
5. R. COHEN, J. D.S. JONES AND J. YAN, The loop homology algebra of spheres and projective spaces, in : *Categorical Decomposition Techniques in Algebraic Topology*, Progress in Mathematics **215**, Birkhauser Verlag, Basel-Boston-Berlin (2004), 77-92.
6. R. COHEN, Multiplicative properties of Atiyah duality, in preparation.
7. P. DELIGNE, P. GRIFFITHS, J. MORGAN AND D. SULLIVAN, Real homotopy theory of Kähler manifolds, *Invent. Math.*, **29,** (1975), 245-274.
8. Y. FÉLIX, S. HALPERIN AND J.-C. THOMAS, Adams's cobar construction, *Trans. Amer. Math. Soc.*, **329,** (1992), 531-549.
9. Y. FÉLIX, S. HALPERIN AND J.-C. THOMAS, Differential graded algebras in topology, in: *Handbook of Algebraic Topology, Chapter 16* Elsevier, North-Holland-Amsterdam (1995), 829-865.
10. Y. FÉLIX, S. HALPERIN AND J.-C. THOMAS, *Rational Homotopy Theory*, Graduate Texts in Mathematics-205, Springer-Verlag, New-York (2000).
11. M. GERSTENHABER, The cohomology structure of an associative ring, *Ann. of Math.*, **78,** (1963), 267-288.
12. J. D.S. JONES, Cyclic homology and equivariant homology, *Invent. Math.*, **87,** (1987), 403-423.
13. M. VIGUÉ-POIRRIER, Homologie de Hochschild et homologie cyclique des algèbres différentielles graduées, in: *Asterisque:International Conference on Homotopy Theory (Marseille-Luminy-1988)*, **191**, (1990), 255-267.

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