Rational String Topology

Yves Félix, Jean-Claude Thomas and Micheline Vigué-Poirrier*

July 9, 2005

Abstract

We use the computational power of rational homotopy theory to provide an explicit cochain model for the loop product and the string bracket of a 1-connected closed manifold M. We prove that the loop homology of M is isomorphic to the Hochschild cohomology of the cochain algebra $C^*(M)$ with coefficients in itself. Some explicit computations of the loop product and the string bracket are given.

AMS Classification : 55P35, 54N45,55N33, 17A65, 81T30, 17B55

Key words : String homology, rational homotopy, Hochschild cohomology, free loop space, loop space homology.

1 Introduction.

Let $C_*(X)$ (respectively $H_*(X)$) be the singular chains (respectively the singular homology) of a space X with coefficients in $\mathbb{Q}$. For simplicity we identify $H_*(X \times X)$ with $H_*(X) \otimes H_*(X)$ and the singular cohomology $H^*(X)$ with the graded dual of the homology i.e $H^*(X) := H(C^*(X)) = (H_*(X))^\vee$.

Let M be a 1-connected closed oriented m-manifold and let LM (respectively M^I and ΩM) be the space of free loops (respectively of paths and of based loops). M. Chas and D. Sullivan [**?**] have constructed a product, called the *loop product*

$$H_*(LM) \otimes H_*(LM) \to H_{*-m}(LM)\,, \quad x \otimes y \mapsto x \bullet y$$

so that $\mathbb{H}_*(LM) := H_{*+m}(LM)$ is a commutative graded algebra, and a morphism, called *the I-homomorphism*

$$I : \mathbb{H}_*(LM) \to H_*(\Omega M)$$

which relates the loop product structure on $\mathbb{H}_*(LM)$ with the Pontryagin algebra $H_*(\Omega M)$. The loop product induces the *string bracket* on the S^1- equivariant homology of LM, $\mathcal{H}_* = H^{S^1}_{*+m}(LM)$,

$$\mathcal{H}_* \otimes \mathcal{H}_* \to \mathcal{H}_{*-2}\,, \quad a \otimes b \mapsto [a, b]$$

such that $(\mathcal{H}, [-,-])$ is a graded Lie algebra of degree 2. Few things are known about this bracket. For surfaces of genus larger than zero, Chas and Sullivan recover formulae proved in the context of symplectic geometry.

The purpose of this paper is to provide us with explicit computational tools for the dual of these two operations in cohomology and for the I-homomorphism.

*The authors are partly supported by INTAS program 03 51 3251

First we will describe the dual of the loop product (Theorem A) and the dual of the string bracket (Theorem B) in terms of Sullivan models, [?]. It results from Theorem A that the loop product structure on $H_*(LM)$ is invariant under orientation preserving maps which are quasi-isomorphisms. In Theorem C, we adapt the technics of Theorem A to a special type of Sullivan models, the cochain of a differential graded Lie algebra with coefficients in a differential graded Lie module. The main point in this paper is Theorem D which allows to describe the dual of the loop product in terms of chains (instead of cochains) of a differential graded Lie algebra. This translation is performed by mean of "the cap-homomorphism". This homomorphism realizes, in some sense, the Poincaré duality at the level of free loop spaces. The other interest for considering the cap-homomorphism is that it allows us, in Theorem E, to identify the loop product in $\mathbb{H}_*(LM)$ with the Gerstenhaber product on the Hochschild cohomology $HH^*(C^*(M); C^*(M))$. Here $C^*(M)$ denotes the cochain algebra of singular cochains on M.

Theorem E. *Let M be a simply connected closed oriented manifold. There exists a natural isomorphism of graded algebras*

$$J : \mathbb{H}_*(LM) \xrightarrow{\cong} HH^*(C^*(M), C^*(M)) .$$

Such an identification has been proved by Merkulov, [?] for the field of real numbers using iterated path integrals. Finally we show

Theorem F. *Let J be the map in theorem E. There exists a canonical isomorphism of graded algebras, $\bar{J}$, making commutative the diagram*

$$\begin{array}{ccc} H_*(LM) & \xrightarrow{J\,\cong} & HH^*(C^*M, C^*M) \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle HH^*(C^*M,\varepsilon)} \\ H_*(\Omega M) & \xrightarrow{\bar{J}} & HH^*(C^*M) \end{array}$$

where $\varepsilon : C^*M \to \mathbb{Q}$ denotes the usual augmentation.

Theorem E and Theorem F are complementary results of our previous paper, [?]. In particular, Theorems 2 and 3 in [?] apply verbatim.

The paper is organized as follows.

2. Preliminaries on differential homological algebra.
3. The dual of the loop product.
4. Theorem A. Dual of the loop product in terms of Sullivan models.
5. Theorem B. Dual of the string bracket in terms of Sullivan models
6. Theorem C. Dual of the loop product in terms of cochains on a differential graded Lie algebra.
7. Theorem D. Dual of the loop product in terms of chains on a differential graded Lie algebra.
8. Theorem E. Loop product and Gerstenhaber product.
9. Theorem F. I-homomorphism and Hochschild homology.

2 Preliminaries on differential homological algebra.

All the graded vector spaces, algebras, coalgebras and Lie algebras V are defined over $\mathbb{Q}$ and are supposed of finite type, i.e. dim $V_n < \infty$ for all n.

2.1 Graded vector spaces

If $V = \{V_i\}_{i\in\mathbb{Z}}$ is a (lower) graded $\mathbb{Q}$-vector space (when we need upper graded vector space we put $V_i = V^{-i}$ as usual) then $V^\vee$ denotes the graded dual vector space and sV denotes the suspension of V:

$$V^\vee = \mathrm{Hom}(V,\mathbb{Q})\,, \quad (sV)_n = V_{n-1}\,,\ (sV)^n = V^{n+1}\,,$$

Since we work with graded objets, we will make a special attention to signs. Recall that if $P = \{P_i\}$ and $N = \{N_i\}$ are differential graded vector spaces with $\partial_P : P_i \to P_{i-1}$ and $\partial_N : N_i \to N_{i-1}$ then

- $P \otimes N$ is a differential graded vector space :

$$(P\otimes N)_r = \oplus_{p+q=r} P_p \otimes N_q\,, \quad \partial_{P\otimes N} = \partial_P \otimes id_N + id_P \otimes \partial_N\,,,$$

- $\mathrm{Hom}(P,N)$ is a differential graded vector space :

$$Hom_n(P,N) = \prod_{k-l=n} \mathrm{Hom}(P_l, N_k)\,, \quad D_{Hom(P,N)} f = \partial_N \circ f - (-1)^{|f|} f \circ \partial_P\,.$$

The same formulae hold if $P = \{P^i\}$ and $N = \{N^i\}$ are differential graded vector spaces with $d_P : P^i \to P^{i+1}$ and $d_N : N^i \to N^{i+1}$. Later on we will omit subscripts on the differentials.

2.2 Differential graded algebras, coalgebras and Lie algebra

For precise definitions we refer to ([**?**]- §3 c,d and §21). Let us precise that TV denotes the tensor algebra on V, while we denote by $T^c(V)$ the free supplemented coalgebra generated by V. If C is a differential graded coalgebra with coproduct Φ and A is a differential graded algebra with product μ, then the cup product, $f \cup g = \mu \circ (f \otimes g) \circ \Phi$, gives to the differential graded vector space $\mathrm{Hom}(C, A)$ a structure of differential graded algebra.

2.3 Sullivan rational homotopy theory

We refer systematically the reader to [**?**]-§12 for notation, terminology and results concerning Sullivan models. Nonetheless, we precise here that if $V = \{V^i\}_{i\geq 0}$ is a graded $\mathbb{Q}$-vector space we denote by $\bigwedge V$ the free graded commutative algebra generated by V. Any path connected space X admits a Sullivan model,

$$\rho_X : \mathfrak{M}_X := (\bigwedge V, d) \xrightarrow{\simeq} A_{PL}(X)$$

where A_{PL} denotes the contravariant functor of piecewise linear differential forms, and ρ_X is a quasi-isomorphism. ([**?**]-§10 and §12). If X and Y are two path-connected spaces then any continuous map $f : X \to Y$ admits a *Sullivan representative* $\mathfrak{M}_f$. Here after we will make the following identifications:

$$H^*(X) = H(A_{PL}(X)) = H(\mathfrak{M}_X)\,,\ H^*(f) = H(A_{PL}(f)) = H(\mathfrak{M}_f)\,.$$

2.4 Semifree modules.

Let A be a differential graded algebra. A differential graded A-module P is called a *semifree module* if P is equipped with a filtration $P = \cup_{n\geq 0} P(n)$, satisfying $P(0) = 0$, $P(n) \subset P(n+1)$ and such that $P(n)/P(n-1)$ is free on a basis of cycles ([**?**]-§6).

For any A-module N, there exists a semifree module P and a quasi-isomorphism $\varphi : P \to N$. The module P is called a semifree resolution of N.

3 The dual of the loop product.

3.1 A convenient definition of the dual of the loop product.

While M. Chas and D. Sullivan, [**?**], have defined the loop product by using "transversal geometric chains" it is convenient for our purpose to define directly the *dual of the loop product* in the following way. First we replace the space LM (respectively M^I and ΩM) by a Hilbert manifold, ([**?**] or [**?**]-Proposition 2.3.1). Secondly we consider the commutative diagram

$$(1)\quad \begin{array}{ccccc} LM^{\times 2} & \stackrel{i}{\longleftarrow} & LM \times_M LM & \stackrel{Comp}{\longrightarrow} & LM \\ p_0 \times p_0 \downarrow & & p_0 \downarrow & & \downarrow p_0 \\ M \times M & \stackrel{\Delta}{\longleftarrow} & M & = & M \end{array}$$

where

- $Comp$ denotes composition of free loops,
- p_0 is the evaluation at 0 and is thus a locally trivial fibre bundle, [**?**],
- the left hand square is a pullback diagram of locally trivial fibrations,
- the inclusion i is the smooth embedding of the Hilbert manifold of composable loops into the product of the two Hilbert manifolds $LM \times LM$.

The embeddings Δ and i have both codimension m. Thus, using the Thom-Pontryagin theory (see 4.5 for a precise definition), we obtain the Gysin maps:

$$\Delta^! : H^k(M) \to H^{k+m}(M^{\times 2})\,, \qquad i^! : H^k(LM \times_M LM) \to H^{k+m}(LM^{\times 2})\,.$$

Thus diagram (1) yields the following diagram:

$$(2)\quad \begin{array}{ccccc} H^{k+m}(LM^{\times 2}) & \stackrel{i^!}{\longleftarrow} & H^k(LM \times_M LM) & \stackrel{H^k(Comp)}{\longleftarrow} & H^k(LM) \\ H^*(p_0)^{\otimes 2} \uparrow & & H^*(p_0) \uparrow & & \uparrow H^*(p_0) \\ H^{k+m}(M^{\times 2}) & \stackrel{\Delta^!}{\longleftarrow} & H^k(M) & = & H^k(M) \end{array}$$

Following ([**?**], [**?**] or [**?**]), the *dual of the loop product* is defined by composition of maps on the upper line :

$$i^! \circ H^*(Comp) : H^*(LM) \to H^{*+m}(LM^{\times 2})$$

while the map on the lower line is the dual of the intersection product. The commutativity of this diagram expresses that $H^*(p_0)$ is a homomorphism between two cocomutative graded coalgebras.

3.2 Why is it possible to convert this definition in terms of Sullivan models?

First we remark that diagram (1) is the pullback diagram of diagram (1') below along the diagram:

$$\begin{array}{ccccc} M^{\times 4} & \stackrel{\Delta^{2,3}}{\longleftarrow} & M^{\times 3} & \stackrel{\pi^{1,3}}{\rightarrow} & M^{\times 2} \\ \Delta^{\times 2} \uparrow & & \Delta' \uparrow & & \uparrow \Delta \\ M^{\times 2} & \stackrel{\Delta}{\longleftarrow} & M & = & M \end{array}$$

$$(1')\qquad \begin{array}{ccccc} M^{I\times 2} & \xleftarrow{i'} & M^I \times_M M^I & \xrightarrow{Comp'} & M^I \\ (p_0,p_1)^{\times 2}\downarrow & & q\downarrow & & \downarrow (p_0,p_1) \\ M^{\times 4} & \xleftarrow{\Delta^{2,3}} & M^{\times 3} & \xrightarrow{\pi^{1,3}} & M^{\times 2} \end{array}$$

where $\Delta' = (\Delta \otimes id)\circ \Delta$, $\pi^{1,3}(x,y,z) = (x,z)$, $q(\alpha,\beta) = (\alpha(0),\alpha(1) = \beta(0),\beta(1)$ and $\Delta^{2,3}(x,y,z) = (x,y,y,z)$. The other maps are the obvious one's. The second observation consists of remarking that once again the Thom-Pontryagin theory yields the commutative diagram

$$(2')\qquad \begin{array}{ccccc} H^{k+m}(M^{I\times 2}) & \xleftarrow{i'!} & H^k(M^I \times_M M^I) & \xleftarrow{H^*(Comp')} & H^k(M^I) \\ H^*((p_0,p_1)^{\otimes 2}\uparrow & & H^*(q)\uparrow & & \uparrow H^*(p_0,p_1) \\ H^{k+m}(M^{\times 4}) & \xleftarrow{id_M\otimes\Delta^!\otimes id_M} & H^k(M^{\times 3}) & \xleftarrow{H^*(\pi^{1,3})} & H^k(M^{\times 2}) \end{array}$$

with $H^*(\pi^{12})(a\otimes b) = a\otimes 1\otimes b$.

It is then natural to define the *path product* as the composition of the maps on the upper line of the diagram (2').

Since the loop product appears as a "pullback" of the path product it is valuable to determine the path product. This is the purpose of the last observation: There is a commutative diagram

$$(3)\quad \begin{array}{ccccc} M^{I\times 2} & \xleftarrow{i'} & M^I \times_M M^I & \xrightarrow{Comp'} & M^I \\ \sigma\times\sigma\uparrow & & \sigma'\uparrow & & \uparrow\sigma \\ M^{\times 2} & \xleftarrow{\Delta} & M & = & M \end{array}$$

where $\sigma(x)$ is the constant path at $x\in M$ and $\sigma'(x) = (\sigma(x),\sigma(x))$. From the properties of the Thom-Pontryagin construction, diagram (3) converts into a commutative diagram

$$(3')\quad \begin{array}{ccccc} H^{k+m}(M^{I\times 2}) & \xleftarrow{i'!} & H^k(M^I \times_M M^I) & \xleftarrow{Comp'} & H^k(M^I) \\ H^*(\sigma)^{\otimes 2}\uparrow & & H^*(\sigma')\uparrow & & \uparrow H^*(\sigma) \\ H^{k+m}(M^{\times 2}) & \xleftarrow{\Delta^!} & H^k(M) & = & H^k(M) \end{array}\ .$$

$\Delta^!$ is the multiplication by the Euler class of the diagonal embedding, ([?]). This allows us to achieve the description of the loop product in terms of Sullivan model since σ is a homotopy equivalence.

4 Theorem A. The dual of the loop product in terms of Sullivan models

4.1 Relative Sullivan model for path and the loop fibrations

It is convenient to consider cofibrant Sullivan representative of a map $f : X\to Y$ called *relative Sullivan model*, [?]-§14: $\lambda_f : \mathfrak{M}_Y \hookrightarrow (\mathfrak{M}_Y\otimes\bigwedge V, d) = \mathfrak{M}_X$. Our primary example here is the following,

$$(4)\quad \begin{array}{ccc} M^I & \overset{\simeq\sigma}{\leftarrow} & M \\ (p_0,p_1)\searrow & & \swarrow\Delta \\ & M^{\times 2} & \end{array}\ ,\quad (\mathfrak{M}(4))\quad \begin{array}{ccc} & \mathfrak{M}_{M^{\times 2}} = \mathfrak{M}_M^{\otimes 2} & \\ \lambda_{(p_0,p_1)}\swarrow & & \searrow\mathfrak{M}_\Delta = \mu \\ \mathfrak{M}_{M^I} = (\mathfrak{M}_{M^{\times 2}}\otimes\bigwedge sV, d) & \overset{\mathfrak{M}_\sigma\simeq}{\leftarrow} & \mathfrak{M}_M \end{array}$$

where the left hand diagram describes the fibration associated to the diagonal map with σ as in diagram (3) in §3. In the right hand diagram

a) $\mathfrak{M}_M = (\bigwedge V, d)$ is a Sullivan model of M,

b) μ is the product on $\bigwedge V$

c) The differential in $(\bigwedge V)^{\otimes 2}\otimes\bigwedge sV, d) = \mathfrak{M}_{M^I}$, is precisely defined as follows,[?]-§15-Example 1 (We denote by 1 (respectively $\bar{1}$) the unit of $\bigwedge V$ (respectively $\bigwedge sV$))

$$\left\{\begin{array}{ll} d(v\otimes 1\otimes \bar{1}) & = dv\otimes 1\otimes \bar{1}\\ d(1\otimes v\otimes \bar{1}) & = 1\otimes dv\otimes \bar{1}\\ d(1\otimes 1\otimes sv) & = (v\otimes 1 - 1\otimes v)\otimes \bar{1} - \sum_{i=1}^{\infty}\frac{(sd)^i}{i!}(v\otimes 1\otimes \bar{1})\end{array}\right. , v\in V\,, sv\in sV$$

and s being the unique degree -1 derivation of $\bigwedge V\otimes \bigwedge V\otimes \bigwedge sV$ defined by

$$\left\{\begin{array}{l} s(v\otimes 1\otimes \bar{1}) = 1\otimes 1\otimes sv = s(1\otimes v\otimes \bar{1})\\ s(1\otimes 1\otimes sv)=0\end{array}\right. \quad , \qquad v\in V\,, sv\in sV$$

d) $\mathfrak{M}_\sigma = \mu.\bar{\varepsilon} : ((\bigwedge V)^{\otimes 2}\otimes \bigwedge sV, d) = \mathfrak{M}_{M^I}\to \mathfrak{M}_M = (\bigwedge V, d)$ is the quasi-isomorphism when ε is the canonical augmentation of $\bigwedge sV$.

The interest for considering relative Sullivan models is that this allows to convert the pullback of a fibration into a pushout diagram in the category of differential graded commutative algebras, ([?]-Proposition 15.)8: Each pullback diagram

$$(*)\qquad \begin{array}{ccc} f^*E & \xrightarrow{f'} & E\\ p'\downarrow & & \downarrow p\\ B' & \xrightarrow{f} & B\end{array}$$

where p is any fibration converts into the pushout diagram in the category of differential graded commutative algebras,

$$(\mathfrak{M}(*))\qquad \begin{array}{ccc} \mathfrak{M}_B & \xrightarrow{\mathfrak{M}_f} & \mathfrak{M}_{B'}\\ \lambda_p\downarrow & & \downarrow \lambda_{p'}\\ \mathfrak{M}_E = (\mathfrak{M}_B\otimes \bigwedge W, d) & \xrightarrow{\mathfrak{M}_f\otimes id} & (\mathfrak{M}_{B'}\otimes \bigwedge W, d') = \mathfrak{M}_{f^*E}\end{array}$$

and the differential on $\mathfrak{M}_{f^*E}$ is explicitly defined by the canonical isomorphism of graded algebras

$$\mathfrak{M}_{B'}\otimes_{\mathfrak{M}_B}\mathfrak{M}_E = \mathfrak{M}_{B'}\otimes_{\mathfrak{M}_B}(\mathfrak{M}_B\otimes \bigwedge W)\xrightarrow{\cong}\mathfrak{M}_{B'}\otimes \bigwedge W\,.$$

As a first example the left pullback diagram converts into the right pushout diagram,

$$(5)\quad \begin{array}{ccc} LM & \xhookrightarrow{j} & M^I\\ p_0\downarrow & & (p_0,p_1)\downarrow\\ M & \xrightarrow{\Delta} & M\times M\end{array} \rightsquigarrow (\mathfrak{M}(5))\quad \begin{array}{ccc} \mathfrak{M}_m^{\otimes 2} & \xrightarrow{\mathfrak{M}_\delta=\mu} & \mathfrak{M}_M\\ \lambda_{(p_0,p_1)}\downarrow & & \downarrow\lambda_{p_0}\\ \mathfrak{M}_{M^I} = (\mathfrak{M}_m^{\otimes 2}\otimes \bigwedge sV, d) & \xrightarrow{\mathfrak{M}_j} & (\mathfrak{M}_{LM} = (\mathfrak{M}_M\otimes \bigwedge sV, d)\end{array}$$

where, λ_{p_0} is a Sullivan relative model for p_0, [?]-§12-c). The differential on $\mathfrak{M}_{LM} = (\bigwedge V\otimes \bigwedge sV, d)$ is defined by $ds = -sd$ when s denotes the degree -1 derivation which extends the linear isomorphism $V\to sV$ and such that $s\circ s = 0$. In particular $\mathfrak{M}_j = \mu\otimes id$.

In the same way, the following left pullback diagram converts into the right pushout diagram,

$$(6)\quad \begin{array}{ccc} LM^{\times 2} & \xrightarrow{i} & LM\times_M LM\\ \downarrow p_0\times p_0 & & p_0\downarrow\\ M\times M & \xrightarrow{\Delta} & M\end{array} \rightsquigarrow (\mathfrak{M}(6))\quad \begin{array}{ccc} \mathfrak{M}_M^{\otimes 2} & \xrightarrow{\mu} & \mathfrak{M}_M\\ \lambda_{p_0\times p_0}\downarrow & & \downarrow\lambda_{p_0}\\ \mathfrak{M}_{LM}^{\otimes 2}\cong (\mathfrak{M}_M^{\otimes 2}\otimes (\bigwedge sV)^{\otimes 2}, d) & \xrightarrow{\mu\otimes id\otimes id} & (\mathfrak{M}_M\otimes (\bigwedge sV)^{\otimes 2}, d)\end{array}$$

The differential on $\mathfrak{M}_{LM\times_M LM} = (\mathfrak{M}_M\otimes (\bigwedge sV)^{\otimes 2}, d)$ is defined by:

$$\left\{\begin{array}{ll} d(v\otimes \bar{1}\otimes \bar{1}) & = dv\otimes \bar{1}\otimes \bar{1}\\ d(1\otimes sv\otimes \bar{1}) & = -s(dv\otimes \bar{1}\otimes \bar{1})\\ d(1\otimes \bar{1}\otimes sv) & = -s'(dv\otimes \bar{1}\otimes \bar{1})\end{array}\right. \quad , v\in V\,, \overline{v}\in \overline{V}\,.$$

Here s and s' are the unique degree -1 derivation of $\bigwedge V\otimes (\bigwedge sV)^{\otimes 2}$ such that $s\circ s = 0 = s'\circ s'$ and for $v\in V\,, sv\in s$ and

$$\begin{cases} s(v\otimes\bar{1}\otimes\bar{1}) & = 1\otimes sv\otimes\bar{1} \\ s'(v\otimes\bar{1}\otimes\bar{1}) & = 1\otimes\bar{1}\otimes sv \\ s(1\otimes sv\otimes\bar{1}) & = 0 = s'(1\otimes\bar{v}\otimes\bar{1}) \\ s(1\otimes\bar{1}\otimes sv) & = 0 = s'(1\otimes\bar{1}\otimes sv) \end{cases}.$$

These two examples provide us with relative Sullivan models for $LM\times_M LM \overset{p_0}{\to} M$ and $M^I\times_M M^I \overset{(p_0,p_1)}{\to} M$. In particular $\mathfrak{M}_i = \mu\otimes id\otimes id$.

In a similar way, the pullback diagram which appears as the left part in diagram (1') converts into the pushout diagram

$$(\mathfrak{M}(7))\qquad \begin{array}{ccc} \mathfrak{M}_M^{\otimes 4} & \overset{id\otimes\mu\otimes id}{\to} & \mathfrak{M}^{\otimes}_M 3 \\ \lambda_{(p_0,p_1)^{\times 2}}\downarrow & & \downarrow\lambda_q \\ \mathfrak{M}_{M^I}^{\otimes 2}\cong(\mathfrak{M}_M^{\otimes 4}\otimes(\bigwedge sV)^{\otimes 2},d)^{\otimes 2} & \overset{id\otimes\mu\otimes id\otimes id}{\to} & (\mathfrak{M}_M^{\otimes 3}\otimes(\bigwedge sV)^{\otimes 2},d) \end{array}$$

where the differential on $\mathfrak{M}_{M^I\times_M M^I} = (\mathfrak{M}_M^{\otimes 3}\otimes(\bigwedge sV)^{\otimes 2},d)$ is defined for $v\in V$ and $\overline{v}\in\overline{V}$ by:

$$\begin{cases} d(v\otimes 1\otimes 1\otimes\bar{1}\otimes\bar{1}) & = dv\otimes 1\otimes 1\otimes\bar{1}\otimes\bar{1} \\ d(1\otimes v\otimes 1\otimes\bar{1}\otimes\bar{1}) & = 1\otimes dv\otimes 1\otimes\bar{1}\otimes\bar{1} \\ d(1\otimes 1\otimes v\otimes\bar{1}\otimes\bar{1}) & = 1\otimes 1\otimes 1\otimes dv\otimes\bar{1}\otimes\bar{1} \\ d(1\otimes 1\otimes 1\otimes sv\otimes\bar{1}) & = (v\otimes 1 - 1\otimes v)\otimes 1\otimes\bar{1}\otimes\bar{1} - \sum_{i=1}^{\infty}\frac{(sd)^i}{i!}(v\otimes 1\otimes 1\otimes\bar{1}\otimes\bar{1}) \\ d(1\otimes 1\otimes 1\otimes\bar{1}\otimes sv) & = 1\otimes(v\otimes 1 - 1\otimes v)\otimes\bar{1}\otimes\bar{1} - \sum_{i=1}^{\infty}\frac{(s'd)^i}{i!}(1\otimes v\otimes 1\otimes\bar{1}\otimes\bar{1}). \end{cases}$$

Here s and s' are the unique degree -1 derivation of $(\bigwedge V)^{\otimes 3}\otimes(\bigwedge sV)^{\otimes 2}$ such that $s\circ s = 0 = s'\circ s'$ and

$$\begin{cases} s(v\otimes 1\otimes 1\otimes\bar{1}\otimes\bar{1}) & = 1\otimes 1\otimes 1\otimes sv\otimes\bar{1} = s(1\otimes v\otimes 1\otimes\bar{1}\otimes\bar{1}) \\ s(1\otimes 1\otimes v\otimes\bar{1}\otimes\bar{1}) & = 0 = s'(v\otimes 1\otimes 1\otimes\bar{1}\otimes\bar{1}) \\ s'(1\otimes v\otimes 1\otimes\bar{1}\otimes\bar{1}) & = 1\otimes 1\otimes 1\otimes\bar{1}\otimes sv = s(1\otimes 1\otimes v\otimes\bar{1}\otimes\bar{1}) \end{cases}, v\in V\,, sv\in sV\,.$$

In particular $\mathfrak{M}_{i'} = id\otimes\mu\otimes id\otimes id$.

4.2 Sullivan representative of composition of free paths and free loops

Consider the next diagram, which relates the right parts of diagram (1) and (1').

$$(8)\qquad \begin{tikzcd} LM\times_M LM \arrow[rr] \arrow[dd] \arrow[dr, "Comp"] & & M^I\times_M M^I \arrow[dd, "q"] \arrow[dr, "Comp'"] & \\ & LM \arrow[rr] \arrow[dd] & & M^{[0,1]} \arrow[dd, "(p_0,p_1)"] \\ M \arrow[rr, "(\Delta,id)\circ\Delta"] \arrow[dr, "id_M"] & & M\times M\times M \arrow[dr, "\pi^{1,3}"] & \\ & M \arrow[rr, "\Delta"] & & M\times M \end{tikzcd}$$

Lemma 1. *There exists a unique homomorphism of differential graded algebras $\mathfrak{M}_{Comp'}$ such that, in the next diagram, the upper square commutes while the lower square commutes up to homotopy:*

$$
\begin{tikzcd}
\mathfrak{M}_M \arrow[r, equal] & \mathfrak{M}_M \\
\mathfrak{M}_{M^I} \otimes_{\mathfrak{M}_M} \mathfrak{M}_{M^I} \arrow[u, "\mathfrak{M}_{\sigma'}"] & \mathfrak{M}_{M^I} \arrow[l, "\mathfrak{M}_{Comp'}"'] \arrow[u, "\mathfrak{M}_\sigma = \mu.\bar{\varepsilon}"'] \\
\mathfrak{M}_M^{\otimes 3} \arrow[u, "\lambda_q"] & \mathfrak{M}_M^{\otimes 2} \arrow[l, "\mathfrak{M}_{\pi^{13}}"'] \arrow[u, "\lambda_{(p_0,p_1)}"']
\end{tikzcd}
\tag{9}
$$

Moreover, $\mathfrak{M}_{\sigma'} = \mu.\bar{\varepsilon} \otimes \mu.\bar{\varepsilon}$ is a surjective quasi-isomorphism.

Proof. Let us denote

a) $\overline{V} = sV$, $\hat{V} = V$,

b) $(\bigwedge \overline{V} \otimes \bigwedge \hat{V}, d)$ the acyclic Sullivan algebra with differential d defined by $d(\bar{v}) = \hat{v}$, $d(\hat{v}) = 0$

c) $\pi : (\bigwedge V \otimes \bigwedge \overline{V} \otimes \bigwedge \hat{V}, d) \to (\bigwedge V, d)$ the quasi-isomorphism such that $\pi(v) = v$, $\pi(\bar{v}) = 0 = \pi(\hat{v})$.

d) s the derivation on $(\bigwedge V, d) \otimes (\bigwedge \overline{V} \otimes \bigwedge \hat{V}, d)$ defined by $s(v) = \bar{v}$, $s(\bar{v}) = 0 = s(\hat{v})$.

Following, [?]-§15-Example 1, there is an isomorphism

$$\varphi : \mathfrak{M}_{M^I} \to (\bigwedge V, d) \otimes (\bigwedge \overline{V} \otimes \bigwedge \hat{V}, d)$$

defined by

$$\begin{cases} \varphi(v \otimes 1 \otimes \bar{1}) & = v \\ \varphi(1 \otimes v \otimes \bar{1}) & = v + \hat{v} + \sum_{n=1}^{\infty} \frac{(sd)^n}{n!}(v) \\ \varphi(1 \otimes 1 \otimes sv) & = \bar{v}\,. \end{cases}$$

This isomorphism gives the relative Sullivan model of (p_0, p_1) considered in Diagram (4). The composite

$$
\begin{array}{ccc}
\mathfrak{M}_{M^I} \otimes_{\mathfrak{M}_M} \mathfrak{M}_{M^I} & \overset{\varphi \otimes_{\mathfrak{M}_M} \varphi}{\to} & (\bigwedge V, d) \otimes (\bigwedge \overline{V} \otimes \bigwedge \hat{V}, d) \otimes_{\bigwedge V} (\bigwedge V, d) \otimes (\bigwedge \overline{V} \otimes \bigwedge \hat{V}, d) \\
 & & \downarrow id \otimes_{\bigwedge V} \pi \\
 & & (\bigwedge V, d) \otimes (\bigwedge \overline{V} \otimes \bigwedge \hat{V}, d) \otimes_{\bigwedge V} (\bigwedge V, d) \\
 & & \downarrow \pi \otimes_{\bigwedge V} id \\
(\bigwedge V, d) & \overset{id \cdot \pi \cdot \pi}{\leftarrow} & (\bigwedge V, d) \otimes_{\bigwedge V} (\bigwedge V, d)
\end{array}
$$

is precisely $\mathfrak{M}_{\sigma'}$. From the lifting lemma, [?]-Lemma 12.4, we deduce that there exists a unique homomorphism $\mathfrak{M}_{Comp'}$ such that $\mathfrak{M}_{Comp'} \circ \mathfrak{M}_{\sigma'} = \mathfrak{M}_\sigma$. It follows s from ([?]-Proposition 14.6) that the lower square of Diagram (9) is commutative up to homotopy. □

Therefore, using diagram $\mathfrak{M}(4)$ in 4.1, we obtain the commutative diagram

$$
\begin{array}{rcccc}
\mathfrak{M}_{M^I} & \overset{\mathfrak{M}_{Comp'}}{\to} & \mathfrak{M}_{M^I} \otimes_{\mathfrak{M}_M} \mathfrak{M}_{M^i} & \cong & (\mathfrak{M}_M^{\otimes 3} \otimes (\bigwedge sV)^2, d) = \mathfrak{M}_{M^I \times_M M^I} \\
\mathfrak{M}_j = \mu \otimes id \downarrow & & \downarrow \mathfrak{M}_j \otimes_{\mathfrak{M}_M} \mathfrak{M}_j & & \downarrow \mu \circ (\mu \otimes id) \otimes id \\
\mathfrak{M}_{LM} & \overset{\mathfrak{M}_{Comp}}{\to} & \mathfrak{M}_{LM} \otimes_{\mathfrak{M}_M} \mathfrak{M}_{LM} & \cong & (\mathfrak{M}_M \otimes (\bigwedge sV)^{\otimes 2}, d) = \mathfrak{M}_{LM \times_M LM}\,.
\end{array}
$$

This gives the Proposition

Proposition 1. *The homomorphism $\mathfrak{M}_{Comp}$ is a Sullivan representative of $Comp : LM \times_M LM \to LM$.*

4.3 A $\mathfrak{M}_{LM}^{\otimes 2}$-linear representative for $i^! : H^*(LM \times_M LM) \to H^*(LM)^{\otimes 2}$

First recall, ([?]-VI-Theorem 12.)4, that the Euler class of the diagonal embedding $\Delta : M \to M \times M$ (also called the diagonal class) is the cohomology class

$$e_\Delta = \sum_l (-1)^{|\beta_j|} \hat{\beta}_l \otimes \beta_l \in H^m(M \times M) = (H^*(M) \otimes H^*(M))^m$$

where $\{\beta_l\}$ denotes a homogeneous linear basis of $H^*(M)$ and $\{\hat{\beta}_l\}$ its Poincaré dual basi ($\langle \beta_i \cup \hat{\beta}_j, [M] \rangle = \delta_i 1^j$). Let b_l (resp. $\hat{b}_l$) be a cocycle of $\mathfrak{M}_M$ representing $\beta_l \in H^*(\mathfrak{M}_M) = H^*(M)$ (respectively $\hat{\beta}_l \in H^*(\mathfrak{M}_M) = H^*(M)$). Then

$$\mathfrak{M}_\Delta^! : \mathfrak{M}_M = (\bigwedge V, d) \to \mathfrak{M}_M^{\otimes 2} = (\bigwedge V, d)^{\otimes 2}, \quad x \mapsto (x \otimes 1) \sum_l (-1)^{|b_l|} \hat{b}_l \otimes b_l$$

is a $\mathfrak{M}_M^{\otimes 2}$-linear map of degree m which satisfies $\mathfrak{M}_\Delta^!(dx) = (-1)^m d \circ (\mathfrak{M}_\Delta^! x))$, $x \in \mathfrak{M}_M$ and $\Delta^! = H(\mathfrak{M}_\Delta^!)$. Here $\mathfrak{M}_M$ is viewed as a $\mathfrak{M}_M^{\otimes 2}$-module via the multiplication $\mu : \mathfrak{M}_M^{\otimes 2} \to \mathfrak{M}_M$.

The crucial point in our construction of $\mathfrak{M}_i^!$ is the following result

Proposition 2. *The $\mathfrak{M}_M^{\otimes 2}$-linear map $\mathfrak{M}_\Delta^!$ extends to a $\mathfrak{M}_M^{\otimes 2}$- linear map of degree m*

$$f : \mathfrak{M}_{M^I} = (\mathfrak{M}_M^{\otimes 2} \otimes \bigwedge sV, d) \to \mathfrak{M}_M^{\otimes 2}$$

such that $f(d(x)) = (-1)^m d \circ f(x)$. Moreover, f is unique up to a homotopy of differential graded $\mathfrak{M}_M^{\otimes 2}$-modules.

Proof. The homomorphism defined in (4.1), $\mathfrak{M}_\sigma : \mathfrak{M}_{M^I} = (\mathfrak{M}_M^{\otimes 2} \otimes \bigwedge sV, d) \to \mathfrak{M}_M$, is a $\mathfrak{M}_M^{\otimes 2}$-semifree resolution (2.4). Thus (see [?]-Appendix) we have the Moore spectral sequence:

$$\mathrm{Ext}_{H \otimes H}^{p,q}(H, H \otimes H) \Rightarrow \mathrm{Ext}_{\mathfrak{M}_M^{\otimes 2}}^{p+q}(\mathfrak{M}_M, \mathfrak{M}_M^{\otimes 2}) := H^*(\mathrm{Hom}_{\mathfrak{M}_M^{\otimes 2}}(\mathfrak{M}_{M^I}, \mathfrak{M}_M)),$$

with $H = H(\bigwedge V, d)$. Since $H \otimes H$ is a Poincaré duality algebra of formal dimension $2m$ then, by [?]-Theorem 3.1,

$$\mathrm{Ext}_{H \otimes H}^{p,q}(\mathbb{Q}, H \otimes H) \cong \begin{cases} \{0\} & \text{if } p + q \neq 2m \\ \mathbb{Q} & \text{if } p + q = 2m\,. \end{cases}$$

An induction on the dimension of E proves that $\mathrm{Ext}_{H \otimes H}^{p,q}(E, H \otimes H) = 0$ if $0 \leq p + q < 2m - d$ for any finite dimensional $H \otimes H$-module E concentrated in degree $\leq d$. In particular,

$$(C_1) \qquad \mathrm{Ext}_{H \otimes H}^{p,q}(H, H \otimes H) = 0 \text{ if } 0 \leq p + q < m\,.$$

On the other hand since M is simply connected then $H^{m-1} = 0$ and from the long exact sequence associated to the short exact sequence $0 \to H^m \to H \to H/H^m \to 0$ we deduce that

$$(C_2) \qquad \mathrm{Ext}_{H \otimes H}^{p,q}(H, H \otimes H) = \begin{cases} \{0\} & \text{if } p + q = m + 1 \\ \mathbb{Q}\alpha & \text{some } (p,q) \text{ such that } p + q = m\,. \end{cases}$$

Conditions C_1 and C_2 and the convergence of the Moore spectral sequence imply that

$$\mathrm{Ext}_{\mathfrak{M}_M^{\otimes 2}}^m(\mathfrak{M}_M, \mathfrak{M}_M^{\otimes 2}) \cong \mathbb{Q}.$$

The multiplication by e_Δ defines a generator of $\mathrm{Ext}^m_{H\otimes H}(H, H\otimes H)$ which survives as an element in $\mathrm{Ext}^m_{\mathfrak{M}_M^{\otimes 2}}(\mathfrak{M}_M, \mathfrak{M}_M^{\otimes 2}) \cong \mathbb{Q}$. Any cocycle in $\mathrm{Hom}_{\mathfrak{M}_M^{\otimes}2}(\mathfrak{M}_{M^I}, \mathfrak{M}_M)$ can be viewed as a map f satisfying conclusion of Proposition 2. If f and f' are two such cocycles then $f - f' = Dh$ and h is the required homotopy between f and f'.

□

Consider the quasi-isomorphism $\mathfrak{M}_\sigma = \mu.\bar{\varepsilon} : \mathfrak{M}_{M^I} = (\mathfrak{M}_M^{\otimes} \bigwedge sV, d) \to \mathfrak{M}_M$ defined in (4.2)-Diagram (9). It extends in a quasi-isomorphism

$$\mathfrak{M}_\sigma \otimes id : \mathfrak{M}_{M^I} \otimes_{\mathfrak{M}_M} \mathfrak{M}_{LM\times_M LM} \to \mathfrak{M}_M \otimes_{\mathfrak{M}_M} \mathfrak{M}_{LM\times_M LM} = \mathfrak{M}_{LM\times_M LM}$$

On the other hand the $\mathfrak{M}_M^{\otimes 2}$-linear map, f defined in Proposition 2, extends in a linear map of degree m

$$\mathfrak{M}_i^! = f \otimes id : \mathfrak{M}_{M^I} \otimes_M \mathfrak{M}_{LM\times_M LM} \to \mathfrak{M}_M^{\otimes 2} \otimes_{\mathfrak{M}_M} \mathfrak{M}_{LM\times_M LM} = \mathfrak{M}_{LM}^{\otimes 2}\,.$$

The proof of the next result is postponed to subsection 4.5,

Proposition 3. *The map $\mathfrak{M}_i^!$ and $\mathfrak{M}_\sigma$ are $\mathfrak{M}_{LM}^{\otimes 2}$-linear and*

$$H(\mathfrak{M}_i^!) \circ H(\mathfrak{M}_\sigma)^{-1} = i^! : H^k(LM \times_M LM) \to H^{*+m}(LM^{\times 2})\,.$$

4.4 Statement of Theorem A and Examples.

¿From Diagram (2), Proposition 1 and Proposition 3 one deduces Theorem A:

Theorem A. *The dual of the loop product*

$$H^*(LM) \overset{H^*(Comp)}{\to} H^*(LM\times_M LM) \overset{i^!}{\to} H^{*+m}(LM^{\times 2})$$

is induced on homology by the maps

$$\begin{array}{cccc} \mathfrak{M}_{LM} & & & \\ \mathfrak{M}_{Comp}\downarrow & & & \\ \mathfrak{M}_{LM\times_M LM} = \mathfrak{M}_M \otimes_{\mathfrak{M}_M} \mathfrak{M}_{LM\times_M LM} & \overset{\mathfrak{M}_\sigma\otimes id}{\longleftarrow} & \mathfrak{M}_{M^I} \otimes_{\mathfrak{M}_M} \mathfrak{M}_{LM\times_M LM} \simeq \mathfrak{M}_{LM\times_M LM} \\ & & \mathfrak{M}_i^! \downarrow \\ & & \mathfrak{M}_{LM} \otimes \mathfrak{M}_{LM} \end{array}$$

The example of formal spaces. Let M be a formal space, i.e. a space M whose minimal model $\mathfrak{M}_M = (\bigwedge V, d)$ is quasi-isomorphic to $(H^*(M), 0) =: H$. Examples of formal spaces are given by connected compact Kähler manifolds ([**?**]) and quotients of compact connected Lie groups by closed subgroups of the same rank. When M is a formal space the dual of the loop product is induced by the composition

$$\begin{array}{ccc} (H\otimes \bigwedge sV, D) & & \\ H(\mathfrak{M}_{Comp})\downarrow & & \\ (H\otimes (\bigwedge sV)^{\otimes 2}, d) \quad) & \overset{\mu.\bar{\varepsilon}\otimes id}{\longrightarrow} & (H^{\otimes 2}\otimes \bigwedge sV)\otimes_H (H\otimes (\bigwedge sV)^{\otimes 2}) \quad . \\ & & \mathfrak{M}_i^! \downarrow \\ & & (H^{\otimes 2} \otimes (\bigwedge sV)^{\otimes 2}, d) \end{array}$$

Let us describe explicitly the particular case $M = \mathbb{C}P^n$. The minimal model of $\mathbb{C}P^n$ is given by $(\bigwedge(x,y), d)$, $d(y) = x^{n+1}$, $|x| = 2$, $|y| = 2n+1$. Thus the relative Sullivan model of the free loop space is $(\bigwedge(x, \bar{x}, y, \bar{y}), d)\,, \quad d(\bar{x}) = 0\,, \quad d(\bar{y}) = -(n+1)x^n\bar{x}$. Since we have a quasi-isomorphism $(\bigwedge(x,\bar{x},y,\bar{y}),d) \to (\bigwedge(x,\bar{x},\bar{y})/(x^{n+1}),d) =: H$, the space $\mathbb{C}P^n$ is formal. A linear basis of

$$H^*(LM) = H(\bigwedge(x,\bar{x},\bar{y})/(y^{n+1}),d) \cong \mathbb{Q}\cdot 1 \oplus \left(\bigwedge^{\geq 1}(x,\bar{x})/(x^{n+1},x^n\bar{x}) \otimes \bigwedge \bar{y}\right)$$

is composed by the elements 1, $x^p\bar{y}^{[q]}$, and $x^r\bar{x}\bar{y}^{[s]}$ $p = 1,\dots,n$, $q \geq 0$, $s \geq 0$, $r = 0,\dots,n-1$, with $\bar{y}^{[s]} = \bar{y}^s/s!$.

A Sullivan representative of the composition free loops $Comp : LM \times_M LM \to LM$ is given by

$$\mathfrak{M}_{Comp}(\bar{x}) = \bar{x} + \bar{x}'\,, \qquad \mathfrak{M}_{Comp}(\bar{y}) = \bar{y} + \bar{y}' - \frac{n(n+1)}{2}x^{n-1}\bar{x}\bar{x}'\,.$$

The dual of the loop product is induced by the map

$$\theta : (H^*(M)\otimes \bigwedge(\bar{x},\bar{y}),d) \to (H^*(M)\otimes \bigwedge(\bar{x},\bar{y}),d) \otimes (H^*(M)\otimes \bigwedge(\bar{x},\bar{y}),d),$$

$$\begin{cases} \theta(\alpha\otimes\bar{y}^{[s]}) = & \displaystyle\sum_{p=0}^{n}\sum_{j=0}^{s} \alpha x^p\bar{y}^{[j]} \otimes x^{n-p}\bar{y}^{[s-j]} \\ & \displaystyle -\frac{n(n+1)}{2}\sum_{p=0}^{n}\sum_{j=0}^{s-1} \alpha x^{n-1+p}\bar{x}\bar{y}^{[j]} \otimes x^{n-p}\bar{x}\bar{y}^{[s-j]}\,, \\ \theta(\alpha\otimes\bar{x}\otimes\bar{y}^{[s]}) = & (1\otimes\bar{x}+\bar{x}\otimes 1)\cdot\left(\displaystyle\sum_{p=0}^{n}\sum_{j=0}^{s} \alpha x^p\bar{y}^{[j]}\otimes x^{n-p}y^{[s-j]}\right)\,, \end{cases}$$

with $\alpha \in H^*(M)$. The dual basis

$$1\,, a_{p,q}\,, b_{r,s}\,, \quad p = 1,\dots,n,\ q\geq 0,\ s\geq 0,\ r = 0,\dots,n-1\,,$$

with $|a_{p,q}| = 2p+2qn$, $|b_{r,s}| = 2r+1+2sn$, is a linear basis of $H_*(LM)$. Thus the loop product $H_*(LM\times LM) \to H_{*-m}(LM)\,,\quad x\otimes y \mapsto x\bullet y$ is described by the formulae

$$a_{p,q}\bullet a_{r,s} = a_{p+r-n,q+s}\,, \quad a_{p,q}\bullet b_{r,s} = b_{p+r-n,q+s}\,, \quad (a_{n-1,0})^n = 1\,, \quad 1\bullet a_{n,1} = 0\,.$$

This shows that

$$\mathbb{H}_*(L(\mathbb{C}P^n);\mathbb{Q}) \cong \bigwedge(a,b,t)/(a^{n+1},a^nb,a^nt)\,,$$

with $|a| = -2$, $|b| = -1$ and $|t| = 2n$, $a = a_{n-1,0}$, $b = b_{n-1,0}$, $t = a_{n,1}$. (Compare with [**?**].)

4.5 Thom-Pontryagin technology and Proof of Proposition 3.

Let M and N be (smooth Banach and without boundary) connected manifolds and $f : M \to N$ be a (smooth) closed embedding, ([**?**]-II.§2). Then we have the exact sequence of fiber bundles

$$0 \to TM \stackrel{Tf}{\to} TN_{|_M} \to \nu_f \to 0$$

where TM and TN are the tangent bundles and ν_f is the *normal fiber bundle of* f. By definition of an immersion, ([**?**]-II Proposition 2.3), this exact sequence splits. Hereafter we will identify ν_f with a factor of $TN_{|_M}$. When the fiber of ν_f is of finite dimension k the embedding has codimension k. Consider the associated disk and sphere bundles, ν^D, ν^S. The *Thom class* of the oriented normal fiber bundle pair (ν^D,ν^S), denoted τ_f, is called the *Thom class of the oriented embedding* f, [**?**]. The exponential map $D\subset TN \to N$

restricted to ν_f is a local isomorphism on the zero cross section of the bundle $TN \to N$. Since f is a closed embedding and since N admits a partition of unity then, by ([?]-IV-Theorem 5.1), there exists, an open neighborhood Z of the zero section of ν_f, an open neighborhood U of $f(M)$ in N, and an isomorphism $\theta : Z \to U$ which identifies the zero section of ν_f with $f(M)$.

Since f has finite codimension k we identify Z with ν_f^D and the isomorphism $\nu_f^D \stackrel{\theta}{\to} U := \text{tube}\, f$ restricts to an isomorphism $\nu_f^S \cong \theta(\nu_f^S) := \partial\text{tube}\, f$. It results from, ([?]-VII-Corollary 4.2), that $\nu_f \cong \nu_f^D$. The above discussion is summarized in the next commutative diagram

$$\begin{array}{ccc}
H^*(N, N-f(M)) & \stackrel{j^{(N,N-f(N))}}{\longrightarrow} & H^*(N) \\
\cong\downarrow \text{Excision} & & \downarrow \\
H^*(\text{tube}\, f, \partial\text{tube}\, f) & \stackrel{j^{(\text{tube}\, f, \partial\text{tube}\, f)}}{\longrightarrow} & H^*(\text{tube}\, f) \\
|| & & || \\
H^*(\nu_f^D, \nu_f^S) & \stackrel{j^{(\nu_f^S, \nu_f^S)}}{\longrightarrow} & H^*(\nu_f^D) \\
-\cup\tau_f \uparrow\cong & & \uparrow \pi^D \circ f^{-1}_{|f(M)} \\
H^{*-k}(M) & \stackrel{-\cup e^{\pi^S}}{\longrightarrow} & H^*(M)
\end{array}$$

The composition

$$TP^* : H^*(\nu_f^D, \nu_f^S) = H^*(\text{tube } f, \partial\text{tube} f) \stackrel{Excision}{\cong} H^*(N, N-f(M)) \stackrel{j^{N,N-f(M)}}{\to} H^*(N)$$

is called *the cohomology Thom-Pontryagin collapse map*. The $H^*(N)$-linear map

$$f^! : H^*(M) \stackrel{H^*(\pi^D \circ f)(-)\cup\tau_f}{\to} H^{*+k}(\nu_f^D, \nu_f^S) \stackrel{TP^*}{\to} H^{*+k}(N)\,.$$

is called the *cohomology Gysin map* and $f^!(1) = e_f \in H^k(N)$ is called the *Euler class* of the embedding f.

It result directly from the "unicity property" of the tubular neighborhood [?]-IV-Theorem 6.2, that the Gysin maps are natural with respect to smooth maps which preserve orientations. $\square$

End of the proof of Proposition 3. We apply the previous considerations to our situation described in the next commutative diagrams

$$\begin{array}{ccc}
LM \times_M LM & \stackrel{j}{\to} & M^I \times_M M^I \\
p_0 \downarrow & & p_0 \times p_1 \downarrow \\
M & \stackrel{\Delta}{\to} & M \times M
\end{array}\quad , \qquad
\begin{array}{ccc}
M^{I\times 2} & \stackrel{i'}{\longleftarrow} & M^I \times_M M^I \\
\sigma' \times \sigma' \uparrow & & \uparrow \sigma' \\
M^{\times 2} & \stackrel{\Delta}{\leftarrow} & M
\end{array}$$

where $i = i' \circ j$, the left hand square is a pullback. The righthand square characterizes $\Delta^!$. $\square$

5 Theorem B. String bracket in terms of Sullivan models

5.1 Proof and statement of Theorem B.

The *string homology* (respectively the *string cohomology*) is the desuspended equivariant homology of the free loop space (respectively the graded dual of the desuspended equivariant homology)

$$\mathcal{H}_* = H^{S^1}_{*+m}(LM) = H_{*+m}(LM \times_{S^1} ES^1) \text{ (respectively } \mathcal{H}^* = (\mathcal{H}_*)^\vee)\,.$$

Let $\xi \in H^2(LM \times_{S^1} ES^1)$ be the characteristic class of the circle bundle

$$S^1 \to LM \times ES^1 \xrightarrow{p} LM \times_{S^1} ES^1$$

and consider the associated Gysin sequence

$$(9) \qquad \cdots \to \mathbb{H}_n \xrightarrow{H(p)} \mathcal{H}_n \xrightarrow{\xi\cap -} \mathcal{H}_{n-2} \xrightarrow{\mathbb{M}} \mathbb{H}_{n-1} \to \cdots .$$

The *string bracket* on $\mathcal{H}_*$ is the bilinear map, [?],

$$[-,-] : \mathcal{H}_* \otimes \mathcal{H}_* \to \mathcal{H}_{*-2}\,, \quad a \otimes b \mapsto [a,b] = (-1)^{|a|} H_*(p)(\mathbb{M}(a) \bullet \mathbb{M}(b))\,,$$

where $\bullet$ denotes the loop product on $\mathbb{H}_*(LM)$.

Let $\mathfrak{M}_M \overset{\lambda_{p_0}}{\hookrightarrow} \mathfrak{M}_{LM} = (\mathfrak{M}_M \otimes \bigwedge sV, d)$ be a Sullivan model of $LM \xrightarrow{p_0} M$, (see Diagram (5)). Then a Sullivan representative for the inclusion $LM \to LM \times_{S^1} ES^1$ is given by the $\mathfrak{M}_{LM}$-linear map, [?],

$$\pi : (\mathfrak{M}_{LM} \otimes \bigwedge u, D) \to \mathfrak{M}_{LM}\,, \quad 1 \otimes u \mapsto 0\,.$$

where $(\mathfrak{M}_{LM} \otimes \bigwedge u, D)$ is a Sullivan model for the equivariant free loop space $LM \times_{S^1} ES^1$ whose differential D given by $D(u) = 0\,,\quad |u| = 2\,,\quad D(v) = d(v) + us(v)\,, D(sv) = -s(dv)$. This proves, that the dual of the Gysin sequence (9) is the homology long exact sequence associated to the short exact sequence of cochain complexes

$$0 \to (\mathfrak{M}_{LM} \otimes \bigwedge u, D) \xrightarrow{\mu_u} (\mathfrak{M}_{LM} \otimes \bigwedge u, D) \xrightarrow{\pi} \mathfrak{M}_{LM} \to 0\,,$$

The connecting map $\tilde{s}$ is induced by the derivation s considered in the definition of $\mathfrak{M}_{LM}$ (4.1) and μ_u denotes the product by the cocycle u. Then Theorem B follows from Theorem A and the definition of the string bracket:

Theorem B. *The dual of the string bracket*

$$\mathcal{B}^\vee : \mathcal{H}^* \to (\mathcal{H}^* \otimes \mathcal{H}^*)^{*+2}$$

is induced on homology by the composition of homomorphisms of complexes:

$$\begin{array}{ccc}
\mathfrak{M}_{LM} & \overset{\pi}{\leftarrow} & (\mathfrak{M}_{LM} \otimes \bigwedge u, D) \\
\mathfrak{M}_{Comp} \downarrow & & \\
\mathfrak{M}_{LM \times_M LM} = \mathfrak{M}_M \otimes_{\mathfrak{M}_M} \mathfrak{M}_{LM \times_M LM} & \overset{\mu.\tilde{s}\otimes id}{\longleftarrow} & \mathfrak{M}_{M^I} \otimes_{\mathfrak{M}_M} (\mathfrak{M}_{LM \times_M LM}) \simeq \mathfrak{M}_{LM \times_M LM} \\
 & & \mathfrak{M}_i^! \downarrow \\
(\mathfrak{M}_{LM} \otimes \bigwedge u, D) \otimes (\mathfrak{M}_{LM} \otimes \bigwedge u, D) & \overset{\tilde{s}\otimes\tilde{s}}{\leftarrow} & \mathfrak{M}_{LM} \otimes \mathfrak{M}_{LM}\,.
\end{array}$$

5.2 Examples.

1) Let us assume that M is such that $H^*(M) = \begin{cases} \bigwedge u & \text{if } |u| \text{ is odd} \\ \bigwedge u/(u^{n+1}) & \text{if } |u| \text{ is even}\,. \end{cases}$

From [?], we have the following facts:
If $H^*(M) = \Lambda u$ with $|u| = 2p+1$ then $\mathcal{H}_{2i} = 0$ for all i.
If $H^*(M) = \Lambda u/(u^{n+1})$ with $|u| = 2p$, then $\dim_{\mathbb{Q}} \mathcal{H}_{2i} = 1$ for all i.
Furthermore, the space M is formal and it is shown in [?] that the map $\xi \cap -$ is an isomorphism.
This proves, in the two cases, the nullity of the maps:

$$\mathbb{E} = H_*(p) : \mathbb{H}_{2i} \longrightarrow \mathcal{H}_{2i} \text{ and } \mathbb{M} : \mathcal{H}_{2i} \longrightarrow \mathbb{H}_{2i+1}\,.$$

Let $a \in \mathcal{H}_{2i-1}$ and $b \in \mathcal{H}_{2b-1}$, for some $(i,j) \in \mathbb{Z}^2$, then $\mathbb{M}(a) \bullet \mathbb{M}(b) \in \mathbb{H}_{2(i+j+1)}$, so we have $[a,b] = -\mathbb{E}(\mathbb{M}(a) \bullet \mathbb{M}(b)) = 0$. Thus, for such manifolds M the string bracket is trivial.

2) If $M := S^{2k+1} \times S^{2k+1}, k \geq 1$, then models for M, LM and $LM \times_{S^1} ES^1$ are then given by

$$
\begin{aligned}
&M \,:\, (\wedge(x,y), 0)\,,\\
&LM \,:\, (\wedge(x,y,\bar{x},\bar{y}), 0)\,,\\
&LM \times_{S^1} ES^1 \,:\, (\wedge(x,y,\bar{x},\bar{y},u), D)\,, Du = D(\bar{x}) = D(\bar{y}) = 0\,, Dy = u\bar{y}\,, Dx = u\bar{x}\,.
\end{aligned}
$$

A set of cocycles representing a basis of the vector space $\tilde{H}^*_{S^1}(LM) := H^*_{S^1}(LM)/\mathbb{Q}u$ is composed of the elements ([**?**])

$$
e_{a,b} = \bar{x}^a\bar{y}^b\,, \quad (a,b) \in \mathbb{N}^2 - (0,0)\,, \qquad f_{a,b} = (y\bar{x} - x\bar{y})\bar{x}^a\bar{y}^b\,, \quad (a,b) \in \mathbb{N}^2\,.
$$

The vector space $\tilde{H}^*(LM)$ has a basis given by the following elements

$$
\left\{ \begin{array}{l} e_{a,b} \\ f_{a,b} \end{array} \right. , \qquad \left\{ \begin{array}{l} e'_{a,b} = xy\bar{x}^a\bar{y}^b,\ (a,b) \in \mathbb{N}^2, \\ f'_{a,b} = x\bar{x}^a\bar{y}^b,\ (a,b) \in \mathbb{N}^2\,, \\ f''_b = y\bar{y}^b,\ b \in \mathbb{N}. \end{array} \right.
$$

¿From the above description of the Gysin sequence we deduce $H^*(p)(e_{a,b}) = e_{a,b}$, $H^*(p)(f_{a,b}) = f_{a,b}$, $M'(f'_{a,b}) = e_{a+1,b}$, $M'(e'_{a,b}) = f_{a,b}$, $M'(f''_b) = e_{0,b+1}$, $M'(e_{a,b}) = 0$, and $M'(f_{a,b}) = 0$.

To fix signs, denote by xy the fundamental class of M. A straightforward computation shows that

$$
\begin{aligned}
&\mathcal{B}^\vee(u^r) = 0,\\
&\mathcal{B}^\vee(\bar{x}^p\bar{y}^q) = \sum_{r=0}^{p}\sum_{s=0}^{q} \binom{p}{r} \binom{q}{s} \left(\bar{x}^r\bar{y}^{s+1} \otimes \bar{x}^{p-r+1}\bar{y}^{q-s} - \bar{x}^{r+1}\bar{y}^s \otimes \bar{x}^{p-r}\bar{y}^{q-s+1}\right)\\
&\mathcal{B}^\vee(f_{p,q}) = (f_{0,0} \otimes 1 + 1 \otimes f_{0,0})(\bar{x} \otimes \bar{y} - \bar{y} \otimes \bar{x}) \cdot \mathcal{B}^\vee(\bar{x}^p\bar{y}^q).
\end{aligned}
$$

To describe the string bracket in $\mathcal{H}_*(M)$ we choose dual basis t_r, $a_{p,q}$ and $b_{p,q}$ of u^r, $\bar{x}^p\bar{y}^q$ and $f_{p,q}$. In that basis the string bracket satisfies

$$
\begin{aligned}
&[b_{k,t}, a_{l,m}] = \binom{k+l}{k} \binom{m+t}{t} \frac{km-lt}{(k+l)(t+m)}\ b_{k+l-1,t+m-1}\,,\\
&[a_{k,t}, a_{l,m}] = \binom{k+l}{k} \binom{m+t}{t} \frac{lt-km}{(k+l)(t+m)}\ a_{k+l-1,t+m-1}\,,\\
&[b_{r,s}, b_{m,n}] = 0.
\end{aligned}
$$

In particular the string Lie algebra $\mathcal{H}_*(M)$ is not nilpotent, since for instance $[a_{1,1}, a_{r,s}] = (r-s)\,a_{r,s}$.

6 Theorem C. Dual of the loop product in terms of cochains on a differential graded Lie algebra

6.1 Chains and cochains on a differential graded Lie algebra

Let (L, ∂) be a differential graded Lie algebra with $\partial : L_i \to L_{i-1}$ and (P, ∂) (respectively (Q, ∂)) be a left (respectively a right) differential L-module. The *two-sided chain complex*

$\mathcal{C}_*(P;L,Q)$ is defined as follows.

$$\mathcal{C}_k(P;L;Q) = (P \otimes \bigwedge^k sL \otimes Q, \partial_0 + \partial_1)\,.$$

A generic element is written $p \otimes sx_1 \wedge ... \wedge sx_k \otimes q$ with degree $|p| + |q| + \sum_{i=1}^k(|sx_i|)$. The differential $\partial = \partial_0 + \partial_1$ is defined by:

$$\mathcal{C}_k(P;L;Q) \stackrel{\partial_0}{\to} \mathcal{C}_k(P;L;Q)\,, \qquad \mathcal{C}_k(P;L;Q) \stackrel{\partial_1}{\to} \mathcal{C}_{k-1}(P;L;Q)$$

$$\begin{aligned}
\partial_0(p \otimes c \otimes q) \;&= \partial p \otimes sx_1 \wedge ... \wedge sx_k \otimes q \\
&\quad -p \otimes \sum_{i=1}^k (-1)^{|p|+\epsilon_i}\; sx_1 \wedge \cdots \wedge s\partial x_i \wedge \cdots \wedge sx_k \otimes q \\
&\qquad +(-1)^{|p|+\epsilon_k} p \otimes c \otimes \partial q \\
\partial_1(p \otimes sx_1 \wedge \;&\cdots \wedge sx_k \otimes q) = \\
&\sum_{i=1}^k (-1)^{|p|+|sx_i|\epsilon_i} p \cdot x_i \otimes sx_1 \wedge \cdots \widehat{sx_i} \cdots \wedge sx_k \otimes q \\
&\quad +p \otimes \sum_{1 \le i < j \le k} (-1)^{|p|+\epsilon_{ij}}\; s[x_i,x_j] \wedge \cdots \widehat{sx_i} \cdots \widehat{sx_j} \cdots \wedge sx_k \otimes \partial q \\
&\quad +\sum_{i=1}^k (-1)^{|p|+\epsilon_{k+1}+|x_i|+|sx_i|\epsilon_{k,i}} p \otimes sx_1 \wedge \cdots \widehat{sx_i} \cdots \wedge sx_k \otimes x_i \cdot q
\end{aligned}$$

where $\epsilon_{i,j} = \epsilon_{i+1} + \epsilon_j$ for $i < j$ and $\epsilon_i = \sum_{k<i} |sx_k|$.

The *chain coalgebra* of (L,∂) is the graded differential coalgebra $\mathcal{C}_*(L) = (\bigwedge sL, \partial_0 + \partial_1) := \mathcal{C}_*(\mathbb{Q};L;\mathbb{Q})$ with coproduct

$$sx_1 \wedge \cdots \wedge sx_k \mapsto \sum_{j=0}^k \sum_{\sigma \in Sh(j)} \varepsilon_\sigma (sx_{\sigma(1)} \wedge \cdots \wedge sx_{\sigma(j)}) \otimes (sx_{\sigma(j+1)} \wedge \cdots \wedge sx_{\sigma(k)})\,,$$

where ε_σ is the graded signature and $Sh(j)$ denotes the set of $(j, k-j)$-shuffles.

Let us recall that if Q is a left L-module then it is also a right L-module for the action defined by $a \cdot x := -(-1)^{|a|\cdot|x|} x \cdot a\,, \quad a \in Q\,.x \in UL$. For any left L-differential module Q, (respectively for any right differential module P) one put

$$\mathcal{C}_*(L;Q) = \mathcal{C}_*(\mathbb{Q};L,Q) \text{ (respectively } \mathcal{C}_*(P;L) = \mathcal{C}_*(P;L,\mathbb{Q})\text{)}\,.$$

Thus, as graded vector spaces, $\mathcal{C}_*(L;Q) = \bigwedge sL \otimes Q$ and $\mathcal{C}_*(P;L) = P \otimes \bigwedge sL$. The *cochain complex* of L with coefficients in a right L-module P is defined by:

$$\mathcal{C}^*(L;P) = \mathrm{Hom}_{UL}(\mathcal{C}_*(L;UL),P)\,.$$

When $Q = UL$ with the action induced by left multiplication then

$$\mathcal{C}_*(L;UL) = (\bigwedge sL \otimes UL, \partial)$$

is a left $\mathcal{C}_*L$-comodule and a right UL-module and both structures are compatible. Moreover the inclusion $\mathbb{Q} \hookrightarrow \mathcal{C}_*(L;UL)$ is a quasi-isomorphism and $\mathcal{C}_*(L;UL)$ is a semifree L-module, [**?**]-Proposition 22.3. Thus,

$$H(\mathcal{C}_*(L;P)) = \mathrm{Tor}^L(\mathbb{Q},P) \text{ and } H(\mathcal{C}^*(L;P)) = Ext_L(\mathbb{Q},P)\,.$$

6.2 Cochains with coefficients in the adjoint module $UL-a$ and free loop space

Let (L,∂) be as in (6.1) and consider the *adjoint modules* UL_a and $UL_a^\vee$ with actions defined by:

$$\begin{cases} l\cdot x & = [l,x] \text{ on } UL_a \\ (f\cdot l)(x) & = f([l,x]) \text{ on } UL_a^\vee \end{cases} \quad , \quad l\in L, x\in L, f\in UL^\vee .$$

Let us precise also that since UL is a Hopf algebra then for any right (respectively left) differential L-modules (P,∂) (respectively (Q,∂)) the *left diagonal L-module* $P\otimes Q$ is defined by:

$$x\cdot(a\otimes b)=\sum_i(-1)^{|x_i'||a|}(x_i\cdot a)\otimes(x_i'\cdot b)\,,\quad x\in UL\,,a\in P\,,b\in Q$$

when $x\mapsto\sum_i x_i\otimes x_i'$ denotes the coproduct in UL.

Now let (L,∂) be such that, ([?]- Lemma 23.1 and (24.b)) the differential graded algebra $\mathcal{C}^*(L)$ is quasi-isomorphic to a minimal model of M :

$$\mathcal{C}^*(L)=(\bigwedge V,d)\simeq\mathfrak{M}_M\,.$$

The differential graded Lie algebra is uniquely defined by this condition. We have $V=s(L^\vee)$, $H(L,\partial)=\pi_*(\Omega M)$ and $H_*(\Omega M)=H(UL,\partial)$ as Hopf algebras, ([?]-Theorem 21.15). Moreover, by the Poincaré-Birkoff-Witt theorem, [?]-Proposition 21.2, the natural linear isomorphism $\gamma:\bigwedge L\to UL\,,\quad x_1\wedge...\wedge x_k\mapsto\frac{1}{k!}\sum_\sigma\epsilon_\sigma x_{\sigma(1)}...x_{\sigma(k)}$ is a isomorphism of graded coalgebras. Therefore it results from a) b) and c) that we have the following isomorphisms of graded algebras

$$UL^\vee\cong\bigwedge sV\,.$$

Lemma 4. *Considering UL_a as a right L-module then the graded linear isomorphisms*

$$\mathcal{C}^*(L;UL_a^\vee)\cong\mathcal{C}^*(L;UL^\vee)\cong\mathcal{C}^*(L)\otimes UL^\vee\cong\mathcal{C}^*(L)\otimes\bigwedge sV$$

define a structure of graded algebra on $\mathcal{C}^*(L;UL_a^\vee)$ which is compatible with the differential of $\mathcal{C}^*(L;UL_a^\vee)$. Moreover the natural inclusion

$$\mathcal{C}^*(L)\hookrightarrow\mathcal{C}^*(L;UL_a^\vee)\simeq(\mathcal{C}^*(L)\otimes\bigwedge sV,d)\,,\quad f\mapsto f\otimes 1_{UL^\vee}$$

is a relative Sullivan model of $p_0:LM\to M$.

Proof. Let us consider the differential graded Lie algebra L^S defined as follows:

$$L_n^S=L_n\oplus\overline{L}_n\,,\quad\overline{L}_n=L_{n+1}\,,\quad d\overline{x}=-\overline{dx}\,,\quad(-1)^{|a|}[a,\overline{b}]=\overline{[a,b]}\,,\quad[\overline{a},\overline{b}]=0\,.$$

Then, the inclusion

$$\mathcal{C}^*(L)\hookrightarrow\mathcal{C}^*(L^S)=(\bigwedge(V\oplus sV),d)=(\mathcal{C}^*(L)\otimes\bigwedge sV,d)\text{ with }d(sv)=-sd(v)$$

is a relative Sullivan model of $p_0:LM\to M$ (see 4.1-Diagram 5). The remaining of the proof follows from the following sequence of isomorphisms of differential graded algebras

$$\begin{aligned}\mathcal{C}^*(L^S):=\mathrm{Hom}(\mathcal{C}_*(L;(UL)_a),\mathbb{Q})&\cong\mathrm{Hom}(\mathcal{C}_*(L;UL)\otimes_{UL}(UL)_a,\mathbb{Q})\\&\cong\mathrm{Hom}_{UL}(\mathcal{C}_*(L;UL),(UL)_a^\vee)=:\mathcal{C}^*(L;(UL)_a^\vee)\,.\end{aligned}$$

□

6.3 Statement and proof of theorem C

The multiplication $\nu : UL \otimes UL \to UL$ which induces the Pontryagin product on $H_*(\Omega M)$ is not a homomorphism of graded algebras. Nonetheless, $\nu : UL_a \otimes UL_a \to UL_a$ is a homomorphism of UL-modules when $UL_a \otimes UL_a$ is the diagonal left L-module. Thus the dual of the product on UL, $\nu^\vee : UL_a^\vee \to UL_a^\vee \otimes UL_a^\vee$ is a homomorphism of differential graded right UL-modules when $UL_a^\vee \otimes UL_a^\vee$ denotes the right diagonal module.

Proposition 4. *The homomorphism of complexes*

$$\mathcal{C}^*(L;\nu^\vee) : \mathcal{C}^*(L, UL_a^\vee) \to \mathcal{C}^*(L, UL_a^\vee \otimes UL_a^\vee) \simeq (\mathcal{C}^*(L) \otimes (\bigwedge sV)^{\otimes 2}, d)$$

is a homomorphism of commutative differential graded algebras. This is a Sullivan model for the composition of free loops $LM \times_M LM \stackrel{Comp}{\to} LM$.

Proof. The contractible chain complexes, defined in 6.1), $\mathcal{C}_*(L;UL)$ and $\mathcal{C}_*(UL;L)$ are differential graded coalgebras via the graded linear isomorphisms $\mathcal{C}_*(L;UL) \cong \mathcal{C}_*(L) \otimes UL$ and $\mathcal{C}_*(UL;L) \cong UL \otimes \mathcal{C}_*(L)$. Therefore, the coproduct $\mu^\vee$ on $\mathcal{C}_*L$ induces a quasi-isomorphism of differential graded coalgebras

$$\Phi : \mathcal{C}_*(L) \to \mathcal{C}_*(L;UL) \otimes_{UL} \mathcal{C}_*(UL;L) \cong (\mathcal{C}_*(L) \otimes UL \otimes \mathcal{C}_*(L), \partial)$$

defined by $\Phi(x) = \sum_i x_i \otimes 1 \otimes x_i'$ when $\mu^\vee(x) = \sum_i x_i \otimes x_i'$ and the dual map

$$\Phi^\vee : (\mathcal{C}^*(L) \otimes UL^\vee \otimes \mathcal{C}^*(L), d) := (\mathcal{C}_*(L;UL) \otimes_{UL} \mathcal{C}_*(UL;L))^\vee \to \mathcal{C}^*(L) \tag{9}$$

is a quasi-isomorphism of differential graded algebras. This implies that the natural injection

$$\mathcal{C}^*(L) \otimes \mathcal{C}^*(L) \stackrel{\lambda}{\to} (\mathcal{C}^*(L) \otimes UL^\vee \otimes \mathcal{C}^*(L), d) \cong (\mathcal{C}^*(L)^{\otimes 2} \otimes \bigwedge sV, d)\,, \quad x \otimes y \mapsto x \otimes 1 \otimes y$$

is a relative Sullivan model for the product $\mu : \mathcal{C}^*L \otimes \mathcal{C}^*L \to \mathcal{C}^*L$ and thus, by (4.1)-Diagram $\mathfrak{M}(4)$, it is a Sullivan relative model for the path fibration $M^I \stackrel{(p_0,p_1)}{\to} M^{\times 2}$. This fact allows us to follow the construction performed in (4.2) with $\mathfrak{M}_{M^I} = (\mathcal{C}^*(L) \otimes (UL)^\vee \otimes \mathcal{C}^*(L), d)$ and $\mathfrak{M}_{\sigma'} = \Phi^\vee$.

First observe that the cochain model of the vertical face on the right in Diagram (8)

$$\begin{array}{ccc} M^I \times_M M^I & \stackrel{Comp'}{\to} & M^I \\ q\downarrow & & \downarrow (p_0,p_1) \\ M^{\times 3} & \stackrel{\pi^{1,3}}{\to} & M^{\times 2} \end{array}$$

is the following pushout diagram

$$\begin{array}{ccc} E & \stackrel{id\otimes\nabla\otimes id}{\longleftarrow} & \mathcal{C}(L) \otimes UL^\vee \otimes \mathcal{C}(L) \\ id \otimes \varepsilon^\vee \otimes id \otimes \varepsilon^\vee \otimes id \uparrow & & \uparrow id \otimes \varepsilon^\vee \\ \mathcal{C}^*(L)^{\otimes 3} & \stackrel{id\otimes\tau\otimes id}{\to} & \mathcal{C}^*(L)^{\otimes 2} \end{array}$$

where $\varepsilon : UL \to \mathbb{Q}$ is the augmentation, $\tau : \mathbb{Q} \hookrightarrow \mathcal{C}(L)$ is the natural inclusion and ∇ is the composition

$$UL^\vee \stackrel{\nu^\vee}{\to} (UL^\vee)^{\otimes 2} \simeq UL^\vee \otimes \mathbb{Q} \otimes UL^\vee \stackrel{id\otimes\tau\otimes id}{\to} UL^\vee \otimes \mathcal{C}^*(L) \otimes UL^\vee\,.$$

Precise here that

$$
\begin{aligned}
E &= \mathcal{C}^*(L) \otimes UL^\vee \otimes \mathcal{C}^*(L) \otimes UL^\vee \mathcal{C}^*(L) \\
&\simeq (\mathcal{C}_*(L; UL) \otimes_{UL} \mathcal{C}_*(UL; L; UL) \otimes_{UL} \mathcal{C}_*(UL; L))^\vee
\end{aligned}
$$

Secondly we use the machinery developed in (4.2) to translate Diagram (8) in terms of Sullivan models. We deduce the explicit model of the map comp by considering the pushout diagram

$$
\begin{tikzcd}
(\mathcal{C}^*(L) \otimes ((UL)^\vee)^{\otimes 2}) & & & E \arrow[lll, "id\otimes\varepsilon^\vee\otimes id"'] & \\
& (\mathcal{C}^*(L)\otimes(UL)^\vee, d)) \arrow[ul, "\mathfrak{M}_{Comp}"] & & & (\mathcal{C}^*(L)\otimes UL^\vee \otimes \mathcal{C}(L), d) = \mathfrak{M}_{MI} \arrow[lll] \arrow[ul, "1\otimes\nabla\otimes 1 = \mathfrak{M}_{Comp'}"'] \\
\mathcal{C}^*(L) \arrow[uu] & & & \mathcal{C}^*(L)^{\otimes 3} \arrow[lll, "(\mu\otimes id)\otimes\mu"'] \arrow[uu] & \\
& \mathcal{C}^*(L) \arrow[ul, "id_{\mathcal{C}^*L}"] \arrow[uu] & & & \mathcal{C}^*(L)^{\otimes 2} \arrow[lll, "\mu"'] \arrow[ul, "1\otimes\tau\otimes 1"'] \arrow[uu]
\end{tikzcd}
$$

The map $\mathfrak{M}_{Comp} : (\mathcal{C}^*(L) \otimes UL^\vee, d) \to (\mathcal{C}^*(L) \otimes (UL^\vee)^{\otimes 2}, d)$ is precisely $\mathcal{C}(L; \nu^\vee)$.

$\square$

Lemma 6. *Let R be a differential graded algebra, S be a left semifree R-module, and let $f, g : N \to P$ be homomorphisms of right R-modules. If $H(f) = H(g)$ then the maps $f \otimes 1, g \otimes 1 : N \otimes_R (R \otimes V) \to P \otimes_R (R \otimes V)$ induce the same map in homology.*

Proof Recall from (2.4) that $S = \sum_{n\geq 0} R \otimes W(n)$ with $dW(n) \subset R \otimes \sum_{k<n} W(k)$ and where each $W(n)$ is a graded vector space. The subcomplex $F^p(N \otimes_R S) = \sum_{k\leq p} N \otimes (R \otimes W(k))$ (respectively $F^p(P \otimes_R S) = \sum_{k\leq p} P \otimes (R \otimes W(k))$) form a filtration that induces a spectral sequence converging to $H(N \otimes_R S)$ (respectively $H(P \otimes_R S)$). Since $E_1(f \otimes id) = H(f) \otimes id = H(g) \otimes id = E_1(g \otimes id)$ we deduce that $H(f \otimes id) = H(g \otimes id)$

$\square$

Remark. Denote by L_1^S and L_2^S two copies of L^S, and $L^T = L_1^S \oplus_L L_2^S = L \oplus \bar{L}_1 \oplus \bar{L}_2$. We denote by $\pi : L^T \to L^S$ the projection obtained by mapping identically each L_i^S to L^S. We observe that $\mathcal{C}_*(L^T)$ is isomorphic to $\mathcal{C}_*(L; UL_a^\vee \otimes UL_a^\vee)$ and that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{C}^*(L; UL_a^\vee) & \stackrel{\mathcal{C}^*(L;\nu^\vee)}{\longrightarrow} & \mathcal{C}^*(L; UL_a^\vee \otimes UL_a^\vee) \\
\| & & \| \\
\mathrm{Hom}(\mathcal{C}_*(L; UL_a), \mathbb{Q}) & \stackrel{\mathrm{Hom}(\mathcal{C}_*(L;\nu),\mathbb{Q})}{\longrightarrow} & \mathrm{Hom}(\mathcal{C}_*(L; UL_a \otimes UL_a), \mathbb{Q}) \\
\| & & \| \\
\mathcal{C}^*(L^S) & \stackrel{\mathcal{C}^*(\pi)}{\longrightarrow} & \mathcal{C}^*(L^T)
\end{array}
$$

This shows that $\pi : L^T \to L^S$ is a "Lie representative" for the composition of free loops $LM \times_M LM \to LM$. Therefore,

Theorem C. *If $\mathcal{C}^*(L)$ is a Sullivan model of M then the dual of the loop product*

$$
H^*(LM) \stackrel{H^*(Comp)}{\to} H^*(LM \times_M LM) \stackrel{i^!}{\to} H^{*+m}(LM^{\times 2})
$$

is induced on homology by the maps

$$\begin{array}{ccc}
\mathcal{C}^*(L;UL_a^\vee) & \stackrel{\mathcal{C}^*(L;\nu^\vee)}{\to} & \mathcal{C}^*(L;UL_a^\vee\otimes UL_a^\vee)=\mathcal{C}^*(L)\otimes_{\mathcal{C}^*(L)}\mathcal{C}^*(L;UL_a^\vee\otimes UL_a^\vee)\\
 & & \Phi^\vee\otimes id\uparrow\\
\mathcal{C}^*(L;UL_a^\vee)\otimes\mathcal{C}^*(L;UL_a^\vee) & \stackrel{\mathfrak{M}_i^!}{\longleftarrow} & (\mathcal{C}^*(L)\otimes(UL)^\vee\otimes\mathcal{C}^*(L),d)\otimes_{\mathcal{C}(^*(L)}\mathcal{C}^*(L;UL_a^\vee\otimes UL_a^\vee)\\
 & & ||\\
 & & (\mathcal{C}^*(L;UL_a^\vee)\otimes\mathcal{C}^*(L;UL_a^\vee)\otimes\bigwedge sV,d)\,,
\end{array}$$

with $\Phi^\vee$ defined in (9).

6.4 Example.

Recall that a coformal space M is a space that admits a Sullivan minimal model with a purely quadratic differential.In this case $\mathcal{C}^*(L)=(\bigwedge V,d)$, $L=\pi_*(\Omega M)\otimes\mathbb{Q}$ and $H^*(\Omega M)=\bigwedge sV$. Thus, if M is a coformal manifold with minimal model $(\bigwedge V,d)$, then a model for the path composition $LM\times_M LM\to LM$ is given by

$$id\otimes\nu^\vee:(\bigwedge V\otimes\bigwedge\bar V,d)\to(\bigwedge V\otimes(\bigwedge\bar V)^{\otimes 2},d)\,,$$

where ν is the Pontryagin product on $H_*(\Omega M)$ and $(\bigwedge V\otimes\bigwedge\bar V,d)$ is the model of the free loop space, defined in (4.1).

As a particular case let us consider the 11-dimensional manifold M obtained by taking the pullback of the tangent sphere bundle to S^6 along the map $f:S^3\times S^3\to S^6$ that collapses the 3-skeleton into a point.

$$\begin{array}{ccc}
M & \to & \tau_S S^6\\
\downarrow & & \downarrow\\
S^3\times S^3 & \stackrel{f}{\to} & S^6\,.
\end{array}$$

The minimal model of M is

$$(\bigwedge(x,y,z),d)\,,\quad dx=dy=0\,,dz=xy\,,\qquad |x|=|y|=3\,,|z|=5\,.$$

Thus M is a coformal space.
A model for the path composition $M^{[0,1]}\times_M M^{[0,1]}\to M^{[0,1]}$ is given by

$$\varphi:(\bigwedge(x,y,z,x',y',z',\bar x,\bar y,\bar z),d)\to(\bigwedge(x,y,z,x',y',z',x'',y'',z'',\bar x,\bar y,\bar z,\bar x',\bar y',\bar z'),d)\,,$$

with $d(\bar x)=x-x'$, $d(\bar y)=y-y'$, $d(\bar z)=z-z'-\frac12\bar x(y+y')+\frac12(x+x')\bar y$, $d(\bar x')=x'-x''$, $d(\bar y')=y'-y''$, $d(\bar z')=z'-z''-\frac12\bar x'(y'+y'')+\frac12(x'+x'')\bar y'$, $\varphi(\bar x)=\bar x+\bar x'$, $\varphi(\bar y)=\bar y+\bar y'$, $\varphi(\bar z)=\bar z+\bar z'+\frac12\bar x\bar y'-\frac12\bar x'\bar y$.

The induced model for the path composition $LM\times_M LM\to LM$ is then given by

$$\varphi:(\bigwedge(x,y,z,\bar x,\bar y,\bar z),d)\to(\bigwedge(x,y,z,\bar x,\bar y,\bar z,\bar x',\bar y',\bar z'),d)\,,$$

with $d(\bar x)=0$, $d(\bar y)=0$, $d(\bar z)=-\bar xy+x\bar y$, $d(\bar x')=0$, $d(\bar y')=0$, $d(\bar z')=-\bar x'y+x\bar y'$, $\varphi(\bar x)=\bar x+\bar x'$, $\varphi(\bar y)=\bar y+\bar y'$, $\varphi(\bar z)=\bar z+\bar z'+\frac12\bar x\bar y'-\frac12\bar x'\bar y$.

7 Dual of the loop product in terms of chains on a differential graded Lie algebra

7.1 The cap-homomorphism.

Let (L, ∂) be a differential graded Lie algebra, N be a differential graded L-module and $\mathcal{C}_*(L;N)$, $\mathcal{C}^*(L;N)$ as in (6.1). Let $c = \sum_i sx_{i_1} \wedge \cdots \wedge sx_{i_q} \in \mathcal{C}_q(L)$ be a cycle of degree m we define the *cap product by c*:

$$cap_c : \mathcal{C}^{q-r}(L;N) = \mathrm{Hom}_{UL}(\mathcal{C}_{q-r}(L;UL), N) \to \mathcal{C}_r(L;N)\,, \quad f \mapsto f \cap c$$

where $f \cap c =$

$$(-1)^m \sum_i \sum_{\sigma \in \Sigma_q} (-1)^{|f|\cdot(|sx_{\sigma(i_1)}\wedge\cdots\wedge sx_{\sigma(i_r)}|)}\, \varepsilon_\sigma \; sx_{\sigma(i_1)} \wedge \cdots \wedge sx_{\sigma(i_r)} \otimes f\big(sx_{\sigma(i_{r+1})} \cdots sx_{\sigma(i_q)}\big)\,.$$

A standard computation proves that:

Lemma 5. *The morphism cap_c is a homomorphism of complexes which is natural with respect to homomorphisms of differential graded L-modules.*

7.2 Statement of theorem D

Since the dual of the multiplication $\nu^\vee : UL_a^\vee \to UL_a^\vee \otimes UL_a^\vee$ is a morphism of UL-modules, Lemma 5 implies:

Proposition 5. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{C}_*(L;UL_a^\vee) & \stackrel{\mathcal{C}_*(L;\nu^\vee)}{\longrightarrow} & \mathcal{C}_*(L;UL_a^\vee \otimes UL_a^\vee) \\ {\scriptstyle cap_c}\uparrow & & {\scriptstyle cap_c}\uparrow \\ \mathcal{C}^*(L;UL_a^\vee) & \stackrel{\mathcal{C}^*(L;\nu^\vee)}{\longrightarrow} & \mathcal{C}^*(L;UL_a^\vee \otimes UL_a^\vee) \end{array}$$

The next result, whose proof is postponed in subsection 7.3 furnishes a representative of the Gysin map $i^! : H^*(LM \otimes_M LM) \to H^{*+m}(LM^{\times 2})$ at the level of chains.

Let (L, ∂) be such that $\mathcal{C}^*(L) \simeq \mathfrak{M}_M$. The algebra $H^*(\mathcal{C}^*(L)) = H^*(M)$ is a Poincaré duality algebra with a fundamental class $[M] \in H_m(M)$. We denote by u a cycle in $\mathcal{C}_*(L)$ representing $[M]$. Then, an easy spectral sequence argument shows that $cap_u : \mathcal{C}^*(L;N) \to \mathcal{C}_*(L;N)$ is a quasi-isomorphism.

Proposition 6. *With the notation of (6.2), the following diagram commutes on homology*

$$\begin{array}{ccc} \mathcal{C}_*(L;UL_a^\vee \otimes UL_a^\vee) & \stackrel{\mathcal{C}_*(\delta;id)}{\longrightarrow} & \mathcal{C}_*(L;UL_a^\vee) \otimes \mathcal{C}_*(L;UL_a^\vee) \\ \uparrow {\scriptstyle (cap_u)} & & \\ \mathcal{C}^*(L;UL_a^\vee \otimes UL_a^\vee) & & \uparrow {\scriptstyle (cap_u)\otimes(cap_u)} \\ {\scriptstyle (\Phi')^\vee \otimes id} \uparrow \cong & & \\ (\mathcal{C}^*(L;UL_a^\vee) \otimes \mathcal{C}^*(L;UL_a^\vee) \otimes \wedge \bar{V}, d) & \stackrel{\mathfrak{M}_i^!}{\longrightarrow} & \mathcal{C}^*(L;UL_a^\vee) \otimes \mathcal{C}^*(L;UL_a^\vee) \end{array}$$

when $\delta : L \to L \oplus L\,, x \mapsto (x,x)$.

It results directly from Proposition 5 and Proposition 6 that the coproduct

$$\mathcal{C}_*(L;UL_a^\vee) \stackrel{\mathcal{C}_*(L;\nu^\vee)}{\longrightarrow} \mathcal{C}_*(L;UL_a^\vee \otimes UL_a^\vee) \stackrel{\mathcal{C}_*(\delta;id)}{\longrightarrow} \mathcal{C}_*(L;UL_a^\vee) \otimes \mathcal{C}_*(L;UL_a^\vee)\,.$$

induces on $H_*(LM)$ a coproduct

$$H_*(LM) \to H_*(LM) \otimes H_*(LM)$$

which is identified via the linear map $H(cap_u)$ to the dual of the loop product

$$H^*(LM) \to (H^*(LM) \otimes H^*(LM))_{*+m}\,.$$

Now, observe that this coproduct on $\mathcal{C}_*(L; (UL)_a^\vee)$ coincides with the coproduct obtained by the tensor product of the the coalgebra $\mathcal{C}^*(L)$ with the coalgebra $(UL^\vee, \nu^\vee)$. Thus we have proved the following result:

Theorem D *The homology of the differential graded coalgebra $\mathcal{C}_*(L; UL_a^\vee)$ is isomorphic, via the isomorphism induces by cap_u, to the coalgebra $\mathbb{H}^*(LM) := H^{*-m}(LM)$ equipped with the dual of the loop product.*

It worthwhile to observe here that the there exists an isomorphism of differential graded algebras between the dual algebra $\mathrm{Hom}(\mathcal{C}_*(L; UL_a^\vee), \mathbb{Q})$ and the differential graded algebra $\mathcal{C}^*(L; UL_a)$. The structure of graded algebra on $\mathcal{C}^*(L; UL_a)$ is defined by the tensor product of the two graded algebras: $\mathcal{C}^*(L) \otimes (UL, \nu)$.

7.3 Proof of Proposition 6

We will use the following general result about modules over differential graded algebras.

Lemma 6. *Let R be a differential graded algebra, S be a left semifree R-module, and let $f, g : N \to P$ be homomorphisms of right R-modules. If $H(f) = H(g)$ then the maps $f \otimes 1, g \otimes 1 : N \otimes_R (R \otimes V) \to P \otimes_R (R \otimes V)$ induce the same map in homology.*

Proof Recall from (2.4) that $S = \sum_{n \geq 0} R \otimes W(n)$ with $dW(n) \subset R \otimes \sum_{k<n} W(k)$ and where each $W(n)$ is a graded vector space. The subcomplex $F^p(N \otimes_R S) = \sum_{k \leq p} N \otimes (R \otimes W(k))$ (respectively $F^p(P \otimes_R S) = \sum_{k \leq p} P \otimes (R \otimes W(k))$) form a filtration that induces a spectral sequence converging to $H(N \otimes_R S)$ (respectively $H(P \otimes_R S)$). Since $E_1(f \otimes id) = H(f) \otimes id = H(g) \otimes id = E_1(g \otimes id)$ we deduce that $H(f \otimes id) = H(g \otimes id)$ $\square$

The next result makes precise the relation between the map $\mathcal{C}_*(\delta)$ and the Gysin map of the diagonal embedding (see 4.3)

Lemma 7. *With the previous notations the following diagram is commutative*

$$\begin{array}{ccc} H_*(\mathcal{C}_*(L)) & \xrightarrow{H_*(\delta)} & H_*(\mathcal{C}_*(L \oplus L)) \\ \uparrow (cap_{[u]}) & & \uparrow (cap_{[u]}) \otimes (cap_{[u]}) \\ H^*(L; \mathbb{Q}) & \xrightarrow{\Delta^!} & H^*(L; \mathbb{Q}) \otimes H^*(L; \mathbb{Q}) \end{array}$$

Proof By direct computation using definitions introduced in (4.3).

$\square$

End of the proof of Proposition 6. We denote $R = \mathcal{C}^*L \otimes \mathcal{C}^*L$. We remark that $R \otimes V = \mathcal{C}^*(L; (UL)_a^\vee) \otimes \mathcal{C}^*(L; (UL)_a^\vee)$ is a semifree R-module. Remark also that the diagram in Proposition 6 is the tensor product of a diagram of R-module by $R \otimes V$ over R. By Lemma 7, this diagram commutes in homology when $V = \mathbb{Q}$. It commutes then in homology by Lemma 6. This achieves the proof.

$\square$

8 Loop product and Gerstenhaber product

8.1 Bar construction and Chain of a differential graded Lie algebra

Let (A,d) be a differential graded supplemented algebra, $A=\mathbb{Q}\oplus\overline{A}$ and (P,d) (resp. (Q,d)) be a right (respectively left) differential graded A-module. The *(normalized) two-sided bar constructions*, $\mathbb{B}(P;A;Q)$ is defined as follows:

$$\mathbb{B}_k(P;A;Q)=P\otimes T^k s\overline{A}\otimes Q\,.$$

where $T^kV=V^{\otimes k}$. A generic element is written $m[a_1|a_2|...|a_k]n$ with degree $|m|+|n|+\sum_{i=1}^k(|sa_i|)$. The differential $d=d_0+d_1$ is defined by

$$\mathbb{B}_k(P;A;Q)\stackrel{d_0}{\to}\mathbb{B}_k(P;A;Q)\,,\qquad \mathbb{B}_k(P;A;Q)\stackrel{d_1}{\to}\mathbb{B}_{k-1}(P;A;Q)$$

$$\begin{aligned}d_0(p[a_1|a_2|...|a_k]q)\ &=d(p)[a_1|a_2|...|a_k]q-\sum_{i=1}^k(-1)^{|p|+\epsilon_i}p[a_1|a_2|...|d(a_i)|...|a_k]q\\&+(-1)^{|p|+\epsilon_{k+1}}p[a_1|a_2|...|a_k]d(q)\end{aligned}$$

.

$$\begin{aligned}d_1(p[a_1|a_2|...|a_k]q)\ &=(-1)^{|p|}pa_1[a_2|...|a_k]q+\sum_{i=2}^k(-1)^{|p|+\epsilon_i}p[a_1|a_2|...|a_{i-1}a_i|...|a_k]q\\&-(-1)^{|p|+\epsilon_k}m[a_1|a_2|...|a_{k-1}]a_kq\end{aligned}$$

where $\epsilon_i=\sum_{j<i}(|sa_j|)$.

The *(normalized) bar construction* on A is the differential graded coalgebra

$$\mathbb{B}(A):=\mathbb{B}(k;A;k)=\Big(T^c(s\overline{A}),d\Big)\ .$$

The relation between the bar construction on UL and the chain on L is given by the following result:

Proposition 7. ([?]-Proposition 22.7) *The canonical homomorphism*

$$\mathcal{C}_*(P;L;Q)\stackrel{\Psi}{\to}\mathbb{B}(P,UL,Q)\,,\quad p\otimes sx_1\wedge...\wedge sx_k\otimes q\mapsto p\otimes\left(\sum_{\sigma\in\Sigma_k}\varepsilon_\sigma[x_{\sigma(1)}|\cdots|x_{\sigma(k)}]\right)\otimes q$$

is a quasi-isomorphism.

In particular when $P=N=UL$ is the canonical L-bimodule then $\Phi:\mathcal{C}_*(UL;L;UL)\to\mathbb{B}(UL,UL,UL)$ is a quasi-isomorphism of differential graded UL-bimodules.

On the other hand, when $P=Q=\mathbb{Q}$ is the trivial L-module then $\psi:\mathcal{C}_*(L)\to\mathbb{B}(UL)$ is a quasi-isomorphism of differential graded coalgebras.

8.2 The Hochschild complex and chains with coefficients in an adjoint module

Let $A=\mathbb{Q}\oplus\overline{A}$ be a supplemented algebra, A^{op} be the opposite algebra and $A^e=A\otimes A^{op}$ be the enveloping algebra of A thus A^e-modules are A-bimodules. The *Hochschild cochain complex* of A with coefficients in the A-bimodule P is the differential module

$$CC^*(A;P)=(\mathrm{Hom}(T^c(s\overline{A}),P),D_0+D_1)\cong\mathrm{Hom}_{A^e}(\mathbb{B}(A,A,A),P)\,.$$

where the differential D_0+D_1 on $\mathrm{Hom}(T^c(s\overline{A}),P)$ is defined by : if $f\in \mathrm{Hom}(T^c(sA),M)$,

$$\begin{array}{ll} D_0(f)([a_1|a_2|...|a_k]) & = d_M(f\,([a_1|a_2|...|a_k])) - \sum_{i=1}^k(-1)^{|f|+\epsilon_i}f([a_1|...|d_Aa_i|...|a_k]) \\ D_1(f)([a_1|a_2|...|a_k]) & = -(-1)^{|sa_1|\,|f|}a_1f([a_2|...|a_k]) - \sum_{i=2}^k(-1)^{|f|+\epsilon_i}f([a_1|...|a_{i-1}a_i|...|a_k]) \\ & \quad +(-1)^{|f|+\epsilon_k}f([a_1|a_2|...|a_{k-1}])a_k\,, \end{array} .$$

The *Hochschild cohomology of A with coefficients in M* is

$$HH^*(A;M) = H(CC^*(A;M)) = H((\mathrm{Hom}(T^c(A),M),D_0+D_1))\,.$$

The product defined on $\mathrm{Hom}(T^c(s\overline{UL}),UL)$, see (2.2), commutes with the differential D_0+D_1. This is the *Gerstenhaber product*, [?]. The next result makes precice how the Gerstenhaber product on $CC^*(UL;UL)$ and the differential graded coalgebra $\mathcal{C}^*(L,UL_a^\vee)$, as considered in Theorem D, are related. The proof of this result relies with the "inverse process" of Cartan-Eilenberg, [?].

Proposition 8. *Let L be a connected differential graded Lie algebra, then there exists a canonical isomorphisms of graded algebras*

$$HH^*(UL,UL)\cong Hom\left(H_*(\mathcal{C}_*(L;(UL)_a^\vee)),\mathbb{Q}\right)\ .$$

Proof. We form the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{(UL)^e}(I\!\!B(UL;UL;UL),UL) & \overset{Hom(\Psi,UL)}{\longrightarrow} & \mathrm{Hom}_{(UL)^e}(\mathcal{C}_*(UL;L;UL),UL) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}(I\!\!B(UL),UL) & \overset{Hom(\varphi,UL)}{\longrightarrow} & \mathrm{Hom}(\mathcal{C}_*(L),UL) \end{array}$$

where vertical maps are the canonical isomorphisms of graded vector spaces. These isomorphisms define the graded algebra structure on $\mathrm{Hom}_{(UL)^e}(I\!\!B(UL;UL;UL),UL)$ and on $\mathrm{Hom}_{(UL)^e}(\mathcal{C}_*(UL;L;UL),UL)$. Therefore they can be viewed as isomorphisms of graded algebras (but not as isomorphisms of differential graded algebras). Since $\mathrm{Hom}(\psi,UL)$ is a homomorphism of graded algebras the map $\mathrm{Hom}(\Psi,UL)$ is a quasi-isomorphism of differential graded algebras.

Considering the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{(UL)^e}(\mathcal{C}_*(UL;L;UL),UL) & \overset{\alpha}{\longrightarrow} & \mathrm{Hom}(\mathcal{C}_*(UL;L;UL)\otimes_{(UL)^e}UL^\vee,\mathbb{Q}) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}(\mathcal{C}_*(L),UL) & \longrightarrow & \mathrm{Hom}(\mathcal{C}_*(L)\otimes UL^\vee,\mathbb{Q})\,. \end{array}$$

one proves, in the same manner, that the canonical isomorphism α is an isomorphism of differential graded algebras.

Finally, a straightforward computation shows that the canonical linear isomorphism

$$\mathcal{C}_*(L,(UL)_a^\vee)\overset{\beta}{\to}\mathcal{C}_*(UL;L;UL)\otimes_{(UL)^e}(UL)^\vee$$

is an isomorphism of chain complexes. From the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{C}_*(UL;L;UL)\otimes_{UL^e}UL^\vee,\mathbb{Q}) & \overset{\beta^\vee}{\longrightarrow} & \mathrm{Hom}(\mathcal{C}_*(L;UL_a^\vee),\mathbb{Q}) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}(\mathcal{C}_*(L)\otimes UL^\vee,\mathbb{Q}) & = & \mathrm{Hom}(\mathcal{C}_*(L;UL^\vee),\mathbb{Q}) \end{array}$$

we deduce that $\beta^\vee$ is an isomorphism of differential graded algebras.

The composition $\beta^\vee\circ\alpha\circ Hom(\Psi,UL)$ induces a canonical isomorphism of graded algebras

$$HH^*(UL,UL)\cong \mathrm{Hom}\left(H_*(\mathcal{C}_*(L;(UL)_a^\vee)),\mathbb{Q}\right)\ .$$

$\square$

8.3 Proof and statement of theorem E

Assume that $\mathcal{C}^*(L)$ is a Sullivan model of M. Using the natural isomorphism of graded algebras , ([?]),

$$HH^*(UL, UL) \cong HH^*(\mathcal{C}^*(L), \mathcal{C}^*(L))\,,$$

and the existence of quasi-isomorphisms of differential graded algebras, $\mathcal{C}^*L \overset{\cong}{\leftarrow} A \overset{\cong}{\rightarrow} C^*(M)$, we deduce from Proposition 8 and Theorem D that:

Theorem E. *Let M be a simply connected closed oriented manifold. There exists a natural isomorphism of graded algebras*

$$J : \mathbb{H}_*(LM) \overset{\cong}{\rightarrow} HH^*(C^*(M), C^*(M))\,.$$

9 I-homomorphism and Hochschild homology.

9.1 A convenient definition of the I-homomorphism

For each subspace $Z \hookrightarrow M$, we denote by $L_Z M$ the space of loops in LM that with origine in Z. We have thus the commutative pullback diagram

$$\begin{array}{ccc} L_Z M & \xrightarrow{i} & LM \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & M\,. \end{array}$$

Denote now by D a closed disk around the base point x_0. Since D is contractible, $L_D M$ is homotopy equivalent to $D \times \Omega M$. The I-homomorphism $I : H_*(LM) \to H_{*-m}(\Omega M)$ is the composition

$$\tilde{H}_*(LM) \to H_*(LM, L_{M-x_0}M) \overset{Excision}{\longrightarrow} H_*(L_D M, L_{S^{m-1}}M) \cong H_m(D, S^{m-1}) \otimes H_{*-m}(\Omega M)\,.$$

Therefore the I-homomorphism is precisely the Gysin map of the embedding $\Omega M \hookrightarrow LM$.

Let $(\bigwedge V \otimes \bigwedge sV, d)$ be a relative Sullivan model for the free loop space (4.1)-Diagram5, and let $\omega \in (\bigwedge V)^m$ be a cocycle representing the fundamental class. Then the direct sum $\mathcal{J} = (\bigwedge V)^{>m} \oplus S$, where $(\bigwedge V)^m = \mathbb{Q} \cdot \omega \oplus S$, is an acyclic differential ideal in $\bigwedge V$; the quotient map $q : (\bigwedge V, d) \to (A, d) = (\bigwedge V/\mathcal{J}, d)$ is a quasi-isomorphism. We form the tensor product

$$(A \otimes \bigwedge sV, d) := (A, d) \otimes_{\bigwedge V} (\bigwedge V \otimes \bigwedge sV, d)\,.$$

Lemma 7. *The natural injection $i : (\bigwedge sV, 0) \to (A \otimes \bigwedge sV, d)$ defined by $i(a) = (-1)^m \omega \otimes a$ is a morphism of complexes of degree m inducing in cohomology the dual of the I-homomorphism, $I^\vee : H^*(\Omega M) \to H^{*+m}(LM)$.*

Proof. A model for the injection $L_{M-x_0}M \hookrightarrow LM$ is given by the quotient map $q : (A \otimes \bigwedge sV, d) \to (A/(\omega) \otimes \bigwedge sV, d)$. Therefore a model for the cochain complex $C^*(LM, L_{M-x_0}M)$ is given by Ker $q = (\mathbb{Q} \cdot \omega \otimes \bigwedge \bar{V}, 0)$. This implies that a model for the composition

$$H^*(\Omega M) \overset{\cong}{\longrightarrow} H^{*+m}(LM, L_{M-x_0}M) \to H^{*+m}(LM)$$

is given by the composition $\bigwedge sV \overset{a \mapsto (-1)^m \omega \otimes a}{\longrightarrow} \mathbb{Q} \cdot \omega \otimes \bigwedge \bar{V} \to H^*(A \otimes \bigwedge sV, d)\,.$

□

9.2 Statement and proof of theorem F

Theorem F. *Let J as in theorem E. There exists a canonical isomorphism of graded algebras, $\bar{J}$, making commutative the diagram*

$$\begin{array}{ccc} H_*(LM) & \stackrel{J\cong}{\longrightarrow} & HH^*(C^*M, C^*M) \\ I\downarrow & & \downarrow HH^*(C^*M,\varepsilon) \\ H_*(\Omega M) & \stackrel{\bar{J}}{\longrightarrow} & HH^*(C^*M) \end{array}$$

where $\varepsilon : C^*M \to \mathbb{Q}$ denotes the usual augmentation.

Proof.

Let (L, ∂) be such that $\mathcal{C}^*(L) = \mathfrak{M}_M$. The algebra $\mathcal{C}^*(L; UL_a^\vee) = \mathcal{C}^*(L) \otimes UL^\vee$ is then a relative Sullivan model for the free loop space LM, and we can apply Lemma 6 to this model to have a model for the restriction morphism. Now, since the projection $q : \mathcal{C}^*(L; UL_a^\vee) \to A \otimes UL_a^\vee$ is a quasi-isomorphism, q admits a lifting $q' : UL^\vee \to \mathcal{C}^*(L; UL_a^\vee)$.

Let u be a cycle in $\mathcal{C}(L)$ representing the orientation class $[M]$ such that $\langle \omega, [M] \rangle = 1$, we obtain a commutative diagram

$$\begin{array}{ccc} (UL)^\vee & \stackrel{q'}{\longrightarrow} & \mathcal{C}^*(L; UL_a^\vee) \\ \| & & \downarrow (-1)^m - \cap c \\ (UL)^\vee & \stackrel{e}{\longrightarrow} & \mathcal{C}_*(L; UL_a^\vee)\,, \end{array}$$

where $e(a) = 1 \otimes a$. The dual diagram yields in homology a diagram of graded algebras whose vertical maps are isomorphisms

$$\begin{array}{ccccc} H_*(\Omega M) & \stackrel{I}{\longleftarrow} & H_{*+m}(LM) & & \\ \bar{J}\uparrow & & \uparrow J & & \\ H_*(UL) & \stackrel{e'}{\longleftarrow} & H^*(\mathrm{Hom}(\mathcal{C}_*(L; UL_a^\vee), \mathbb{Q})) & = H^*(\mathcal{C}^*(L; UL_a))\,. & \end{array} \qquad (*)$$

Remark that the two $\mathcal{C}_*(L)$-bicomodules $\mathcal{C}_*(L; UL) \otimes_{UL} \mathcal{C}_*(UL; L)$ and $\mathcal{C}_*(L)$ are quasi-isomorphic. Therefore we have by duality a quasi-isomorphism of $\mathcal{C}^*(L)$-bimodules $\mathcal{C}^*(L) \otimes (UL)^\vee \otimes \mathcal{C}^*L = (\mathcal{C}_*(L; UL) \otimes_{UL} \mathcal{C}_*(UL; L))^\vee \to \mathcal{C}^*L$. We deduce the isomorphism of differential graded vector spaces

$$\begin{array}{ll} CC^*(UL; UL) & = \mathrm{Hom}_{(UL)^e}(\mathcal{C}_*(UL; L; UL), UL) \\ CC^*(\mathcal{C}^*L; \mathcal{C}^*L) & = \mathrm{Hom}_{\mathcal{C}^*(L)^e}((\mathcal{C}_*(L; UL) \otimes_{UL} \mathcal{C}_*(UL; L))^\vee, \mathcal{C}^*L)\,. \end{array}$$

The isomorphisms

$$\begin{array}{ll} \mathrm{Hom}_{(UL)^e}(\mathcal{C}_*(UL; L; UL), UL) & \cong Hom(C_*L, UL) \\ \mathrm{Hom}_{(\mathcal{C}^*L)^e}((\mathcal{C}_*(L; UL) \otimes_{UL} \mathcal{C}_*(UL; L))^\vee, \mathcal{C}^*L) & \cong Hom(UL^\vee, \mathcal{C}^*L) \end{array}$$

induce differentials on the right hand side terms. Now a simple computation (see for instance [?]) shows that the map D that associates to a map in $\mathrm{Hom}(C_*L, UL)$ the dual map in $\mathrm{Hom}(UL^\vee, C^*L)$ is an isomorphism of complexes. This induces the following commutative diagram of complexes

$$\begin{array}{ccc} UL & \stackrel{e'}{\longleftarrow} & \mathrm{Hom}_{(UL)^e}(\mathcal{C}_*(UL; L; UL), UL) \\ \| & & \| \\ UL & \stackrel{e'}{\longleftarrow} & \mathrm{Hom}(\mathcal{C}_*(L), UL) \\ \downarrow\cong & & \cong\downarrow D \\ \mathrm{Hom}((UL)^\vee, \mathbb{Q}) & \stackrel{\mathrm{Hom}((UL)^\vee,\varepsilon)}{\longleftarrow} & \mathrm{Hom}(UL^\vee, \mathcal{C}^*(L)) \\ \| & & \| \\ \mathrm{Hom}_{(\mathcal{C}^*L)^e}((\mathcal{C}_*(L; UL) \otimes_{UL} \mathcal{C}_*(UL; L))^\vee, \mathbb{Q}) & \stackrel{CC^*(\mathcal{C}^*(L),\varepsilon)}{\longleftarrow} & CC^*(\mathcal{C}^*(L), \mathcal{C}^*(L)) \\ \| & & \\ CC^*(\mathcal{C}^*(L), \mathbb{Q}) & & \end{array}$$

where $\varepsilon : C^*L \to \mathbb{Q}$ denotes the canonical augmentation. The result follows now directly from the induced diagram in cohomology, combined with diagram ($*$). $\square$

References

[1] A. Baker and C. Õze, *Complex cobordism of Hilbert manifolds wth some applications to flag varieties of loop groups*, Geometry and Topology, Aarhus (1998) 1-19. Comtemp. Math. **258**, Amer. Math. Soc., Providence, RI, 2000.

[2] G.E. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics **139**, Springer-Verlag, 1993.

[3] J.-L. Brylinski, *Loop spaces, characterisic classes and geometric quantization*, Progress in math, **107** Birkhäuser Boston inc., Boston MA 1993.

[4] D. Burghelea and M.Vigué-Poirrier, *A model for cyclic homology and algebraic K-theory*, Journal of Diff. Geom. **22** (1985), 243-253.

[5] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton, 1993.

[6] M. Chas and D. Sullivan, *String topology*, preprint math.GT/9911159

[7] D. Chataur, *A bordism approach to sphere topology*, preprint, Barcelona (2003).

[8] R.L. Cohen and V. Godin *A Polarized view of string topology*, math.AT/03030003.

[9] R.L. Cohen, J.D.S. Jones, and Jun Yan, *The loop homology algebra of spheres and projective spaces*, in Categorical Decomposition Techniques in Algebraic Topology. Progress in Mathematics **215** - Birkhäuser-Verlag - 2004

[10] P. Deligne, J. Griffiths, J. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Inventiones Math. **29** (1975), 245-274.

[11] Y. Felix, S. Halperin and J.-C. Thomas, *Gorenstein spaces.* Advances in Math. **71** (1988) 92-112.

[12] Y. Felix, S. Halperin and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics 205, Springer-Verlag, 2000.

[13] Y. Felix, L. Menichi and J.-C. Thomas, Gerstenhaber duality in Hochschild cohomology,Journal of Pure and Applied Algebra **199** (2005) 43-59.

[14] Y. Felix, J.-C. Thomas and M. Vigué-Poirrier, *Structure of the loop algebra of a closed manifold*, Publ. Math. IHES **99** (2004) 235-252.

[15] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. **78** (1963), 267-288.

[16] A. Hatcher *Algebraic Topology,* Cambridge University Press 2002.

[17] S. Lang, *Differential and Riemannian manifolds*, Third edition. Graduate Text in Mathematics **160** Springer-Verlag, New York 1995.

[18] S.A. Merkulov, De Rham model for string topology, I.M.R.M. **55** (2004) 2956-2987.

[19] J. Milnor and J.D. Stasheff, *Characteristic classes*, Annals of Math. Studies 76, Princeton University Press, 1974.

[20] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Etudes Sci. Publ. Math. **47** (1977), 269-331.

[21] D. Sullivan, *Open and closed string field theory interpreted in classical algebraic topology*, Preprint MathArchiv QA/0302332. .

[22] M. Vigué-Poirrier, *Homologie cyclique des espaces formels*, Journal Pure and Appl. Alg. **91** (1994), 347-354.

felix@math.ucl.ac.be
Département de mathématique
Université Catholique de Louvain
2, Chemin du Cyclotron
1348 Louvain-La-Neuve, Belgium

jean-claude.thomas@univ-angers.fr
Département de mathématique
Faculté des Sciences
2, Boulevard Lavoisier
49045 Angers, France

vigue@math.univ-paris13.fr
Département de mathématique
Institut Galilée,
Université de Paris-Nord
93430 Villetaneuse, France